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ON THE RELATION BETWEEN CURVATURE, DIAMETER AND VOLUME OF A COMPLETE RIEMANNIAN MANIFOLD

ПРО СПІВВІДНОШЕННЯ МІЖ КРИВИЗНОЮ, ДІАМЕТРОМ ТА ОБ'ЄМОМ ПОВНОГО РІМАНОВОГО МНОГОВИДУ

In this note, we prove that if N is a compact totally geodesic submanifold of a complete Riemannian manifold (M, g), whose sectional curvature K satisfies the relation $K \geq k > 0$, then $d(m, N) \leq \frac{\pi}{2\sqrt{k}}$ for any point $m \in M$. In the case where $\dim M = 2$, a Gaussian curvature K satisfies the relation $K \geq k \geq 0$, and γ has the length l, we get $\mathrm{Vol}\,(M, g) \leq \frac{2l}{\sqrt{k}}$ if $k \neq 0$ and $\mathrm{Vol}\,(M, g) \leq 2l\,\mathrm{diam}\,(M)$ if $k = 0$.

Доведено, що якщо N — компактний цілком геодезичний підмноговид повного ріманового многовиду (M, g) із секційною кривизною K, що задовольняє умову $K \geq k > 0$, то для будь-якої точки $m \in M$ виконується нерівність $d(m, N) \leq \frac{\pi}{2\sqrt{k}}$. У випадку, коли $\dim M = 2$, гауссова кривизна K многовиду задовольняє умову $K \geq k \geq 0$ та γ має довжину l, отримано співвідношення $\mathrm{Vol}\,(M, g) \leq \frac{2l}{\sqrt{k}}$ для $k \neq 0$ та $\mathrm{Vol}(M, g) \leq 2l\,\mathrm{diam}\,(M)$ для $k = 0$.

1. Introduction. As well know, one the most interesting problems in Riemannian geometry is to study a relation between geometrical notions as curvature, diameter and volume. This problem have been studied by many authors for the concrete manifolds. In [1], Y. C. Wong considered this problem for the Grassmann manifolds. W. Klingenberg proved that in a compact simply connected even-dimensional Riemannian manifold with sectional curvature K belonging to $[0, k]$, $k > 0$, the length of any closed geodesic is greater than $\frac{2\pi}{\sqrt{k}}$ (see [2]). The relation between curvature and topology of Riemannian manifolds is exposed in [3, 4].

Generalizing the known results for the sphere S^2 in Euclidian three-space, we obtained the following results.

Theorem 1. *Let $(M, \langle\cdot\rangle)$ be a complete Riemannian manifold, whose sectional curvature K satisfies $K \geq k > 0$. Let N be a compact totally geodesic submanifold of M, then for any $m \in M$ we have $d(m, N) \leq \frac{\pi}{2\sqrt{k}}$.*

Theorem 2. *Let (M, g) be a complete Riemannian manifold of dimension 2, whose sectional curvature K satisfies $K \geq k \geq 0$, k is constant. Let γ be a closed geodesic in M of length of l. Then we have*

$$\mathrm{Vol}\,(M) \leq \begin{cases} \dfrac{2l}{\sqrt{k}} & \text{if } k > 0, \\ 2l\,\mathrm{diam}\,(M) & \text{if } k = 0. \end{cases}$$

The basic notions used in this article are from [5, 6].

2. Proof of Theorem 1. Since N is the compact totally geodesic submanifold of M, then every geodesic of N is also a geodesic of M.

Let m be a point of M, we have to prove that $d(m, N) \leq \frac{\pi}{2\sqrt{k}}$. We assume that $m \notin N$, then $d(m, N) = L > 0$, and $\exists p \in N$ such that $d(m, N) = d(m, p) = L$. By the Hopf – Rinow theorem, there exists the minimal geodesic parametrized by arc length:

$$c: [0, L] \to M, \quad c(0) = m, \quad c(L) = p, \quad \| c'(s) \| = 1.$$

Let γ be a geodesic in N passing through p, then γ is also a geodesic in M. Suppose that γ is parametrized by arc length;

$$\gamma: (-\rho, \rho) \to M, \quad \rho > 0, \quad \gamma(0) = p, \quad \| \gamma'(t) \| = 1.$$

Let $Y_L = \gamma'(0) \in T_pM$, take a parallel vector field $Y(s)$ along c such that $Y(L) = Y_L$. We have $\| Y(s) \| = \| Y_L \| = 1$ for all $s \in [0, L]$.

Set $X(s) = \sin \dfrac{\pi s}{2L} Y(s)$, $X(s)$ is a vector field along c, $X(0) = 0$, $X(L) = Y(L) = \gamma'(0)$.

We now consider the variation H of c as follows:

$$H: [0, L] \times (-\rho, \rho) \to M$$
$$(s, t) \mapsto H(s, t) = \exp_{c(s)} tX(s),$$

H is well defined by the completeness of manifold M. We have $H(0, t) = m$.

Set $c_t(s) = H(s, t)$, $c_0(s) = H(s, 0) = c(s)$, then

$$H(L, t) = \exp_{C(L)} tX(L) = \exp_p t\gamma'(0) = \gamma(t),$$

$$H * \frac{\partial}{\partial t}(s, 0) = X(s).$$

By construction of the variation H, the length function $L(c_t)$ attain a minimum at $t = 0$. Hence,

$$\frac{d}{dt} L(c_t)\big|_{t=0} = 0 \tag{1}$$

and

$$\frac{d^2}{dt^2} (L(c_t))\big|_{t=0} \geq 0. \tag{2}$$

Using the first variation formula together with remark that c is a geodesic, i.e., $\Delta_{c'} c' = 0$, we have

$$(1) \Leftrightarrow \langle X(s), c'(s) \rangle \big|_0^L - \int_0^L \langle X(s), \nabla_{c'} c'(s) \rangle ds = 0 \Leftrightarrow$$

$$\Leftrightarrow \langle X(L), c'(L) \rangle = 0 \Leftrightarrow \langle Y(L), c'(L) \rangle = 0.$$

On the other hand,

$$\frac{d}{ds} \langle Y(s), c'(s) \rangle = \left\langle \frac{D}{ds} Y(s), c'(s) \right\rangle + \left\langle Y(s), \frac{D}{ds} c'(s) \right\rangle,$$

where $\dfrac{D}{ds} Y(s) = \nabla_{\partial/\partial s} Y(s)$.

Since $\dfrac{D}{ds} Y(s) = 0$, $\dfrac{D}{ds} c'(s) = 0$, we have $\dfrac{d}{ds} \langle Y(s), c'(s) \rangle = 0$. Thus,

$$\langle Y(s), c'(s) \rangle = \langle Y(L), c'(L) \rangle = 0 \quad \forall s \in [0, L]$$

and $\{Y(s), c'(s)\}$ is an orthonormal system.

Set $\overline{X}(s, t) = H * \frac{\partial}{\partial t}(s, t) \Rightarrow \overline{X}(s, 0) = X(s)$.

Using the second variation formula, we have

$$(2) \Leftrightarrow \langle \nabla_{\partial/\partial t}\overline{X}(s, 0), c'(s)\rangle\big|_0^L + \int_0^L \left(|\overline{X}'|^2 - \langle R(\overline{X}, c', c'), \overline{X}\rangle - \langle c', \overline{X}'\rangle^2\right) ds \geq 0, \quad (3)$$

where $\overline{X}' = \nabla_{\partial/\partial s}\overline{X}(s, 0)$.

We have

$$\overline{X}(s, 0) = X(s) = \sin\left(\frac{\pi s}{2L}\right) Y(s),$$

$$\overline{X}'(s, 0) = X'(s) = \frac{\pi}{2L}\cos\frac{\pi s}{2L} Y(s) + \sin\frac{\pi s}{2L}\frac{D}{ds} Y(s) = \frac{\pi}{2L}\cos\frac{\pi s}{2L} Y(s),$$

$$\langle c'(s), \overline{X}'(s, 0)\rangle = \frac{\pi}{2L}\cos\left(\frac{\pi s}{2L}\right)\langle c'(s), Y(s)\rangle = 0,$$

$$\overline{X}(0, t) = H * \frac{\partial}{\partial t}(0, t) = 0 \quad (\text{since } H(0, t) = m \;\; \forall t) \Rightarrow \nabla_{\partial/\partial t}\overline{X}(0, 0) = 0,$$

$$\overline{X}(L, t) = H * \frac{\partial}{\partial t}(L, t) = \gamma'(t) \Rightarrow \nabla_{\partial/\partial t}\overline{X}(L, 0) = \frac{D}{dt}\gamma'(t)\big|_{t=0} = 0.$$

Thus,

$$(3) \Leftrightarrow \int_0^L \left[\left(\frac{\pi}{2L}\right)^2 \cos^2\left(\frac{\pi s}{2L}\right)|Y(s)|^2 - \sin^2\left(\frac{\pi s}{2L}\right)\langle R(Y(s), c'(s), c(s)), Y(s)\rangle\right] ds \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \int_0^L \left[\left(\frac{\pi}{2L}\right)^2 \sin^2\left(\frac{\pi s}{2L}\right) - \sin^2\left(\frac{\pi s}{2L}\right) R(Y(s), c'(s))\right] ds \geq 0 \Rightarrow$$

$$\Rightarrow \int_0^L \left[\left(\frac{\pi}{2L}\right)^2 - k\right] \sin^2\frac{\pi s}{2L}\, ds \geq 0 \Leftrightarrow \left(\frac{\pi}{2L}\right)^2 - k \geq 0 \Rightarrow L \leq \frac{\pi}{2\sqrt{k}}.$$

Theorem 1 is proved.

3. Proof of Theorem 2. In order to prove Theorem 2, the following lemma is necessary:

Lemma. *Suppose that* $x : (\alpha, \beta) \to R$ *is a differentiable function defined on* (α, β), *where* $\left(-\frac{\pi}{2} \leq \alpha < 0 < \beta \leq \frac{\pi}{2}\right)$, *and satisfies*

$$x(0) = 0, \quad x' + x^2 \leq -1. \quad (4)$$

Then

$$x(t) \leq -\operatorname{tg}(t), \quad t \in [0, \beta),$$
$$x(t) \geq -\operatorname{tg}(t), \quad t \in (\alpha, 0].$$

Proof. Set $x(t) = -\operatorname{tg}\varphi(t)$, then $\varphi(t) = -\operatorname{arctg} x(t)$, $\varphi(t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$(4) \Leftrightarrow \varphi(0) = 0 \text{ and } -\frac{\varphi'}{\cos^2\varphi} + \frac{\sin^2\varphi}{\cos^2\varphi} \leq -1 \Leftrightarrow \frac{\varphi'}{\cos^2\varphi} \geq \frac{1}{\cos^2\varphi} \Leftrightarrow \varphi' \geq 1.$$

Thus $(\varphi(t) - t)$ is the increasing function

$$\Rightarrow \begin{cases} \varphi(t) \geq t & \forall t \in [0, \beta), \\ \varphi(t) \leq t & \forall t \in (\alpha, 0]. \end{cases}$$

Since the function $\operatorname{tg} t$ is increasing on (α, β), we get

$$x(t) = -\operatorname{tg}\varphi(t) \leq -\operatorname{tg}(t) \qquad \forall t \in [0, \beta),$$
$$x(t) = -\operatorname{tg}\varphi(t) \geq -\operatorname{tg}(t) \qquad \forall t \in (\alpha, 0].$$

The lemma is proved.

Now we prove Theorem 2.

Suppose that γ is a closed geodesic parametrized by arc length:

$$\gamma: I \to M, \quad \text{where} \quad [0, l] \subset I, \quad \gamma(0) = \gamma(l), \quad \gamma'(0) = \gamma'(L), \quad \|\gamma'(t)\| = 1$$

$\forall t \in I$, I is interval on R.

Since γ is a closed geodesic, there exists a unit parallel vector field Y along γ such that $Y(0) = Y(l)$ and $\langle Y(0), \gamma'(0)\rangle = 0$.

For $s \in [0, l)$, put

$$\rho(s) = \sup\{t \geq 0 \,|\, t = d(\exp_{\gamma(s)} tY(s), \gamma)\},$$
$$\varphi(s) = \inf\{t \leq 0 \,|\, -t = d(\exp_{\gamma(s)} tY(s), \gamma)\}. \tag{5}$$

We will prove that $\rho(s)$ is upper semicontinuous and $\varphi(s)$ is lower semicontinuous. We consider $s_n \to s$, set $t_0 = \overline{\lim\limits_{n\to\infty}}\, \rho(s_n)$, and claim that $t_0 \leq \rho(s)$. In fact, since

$$t_0 = \overline{\lim_{n\to\infty}}\, \rho(s_n), \quad \text{we have} \quad \exists \{s_{n_k}\} \subset \{s_n\} \,|\, \rho(s_{n_k}) \to t_0.$$

Put $t_k = \max\left\{0, \rho(s_{n_k}) - \dfrac{1}{k}\right\}$, $t_k \to t_0 \Rightarrow \exp_{\gamma(s_{n_k})} t_k Y(s_{n_k}) \to \exp_{\gamma(s)} t_0 Y(s)$ and

$$d(\exp_{\gamma(s)} t_0 Y(s), \gamma) = \lim_{k\to\infty} (\exp_{\gamma(s_{n_k})} t_k Y(s_{n_k}), \gamma) = \lim_{k\to\infty} t_k = t_0 \Rightarrow \rho(s) \geq t_0.$$

Thus, $\rho(s)$ is upper semicontinuous. Similarly, $\varphi(s)$ is lower semicontinuous. This implies that $\rho(s)$ and $\varphi(s)$ are the measurable functions.

We have

$$A = \{(s, t) \,|\, 0 \leq s < l,\ \varphi(s) < t < \rho(s)\} \quad \text{is measurable set and} \quad \operatorname{mes} B = 0,$$

where $B = \{(s, t) \,|\, 0 \leq s < l,\ \varphi(s) = 0 \text{ or } \rho(s) = t\}$.

Consider the sets

$$C = \{\exp_{\gamma(s)} tY(s) \,|\, (s, t) \in A\},$$
$$D = \{\exp_{\gamma(s)} tY(s) \,|\, (s, t) \in B\}.$$

It is clear that C, D are the images of A and B, respectively, under a continuous mapping. Thus, C and D are measurable and $\operatorname{mes}(D) = 0$.

We will prove that $M = C \cup D$ and $C \cap D = \varnothing$.

In fact, since γ is closed, γ is compact.

This implies that, for arbitrary $p \in M$, $\exists q \in \gamma \,|\, r = d(p, q) = d(p, \gamma)$, $q = \gamma(s)$.

Suppose that c is a minimal geodesic length-parametrized joining p and q:

$$c: [c, r] \to M, \quad c(0) = q, \quad c(r) = p.$$

X is the parallel vector field along c such that $X(0) = \gamma'(s)$. We consider the variation

$$H: [0, r] \times (-\varepsilon, \varepsilon) \to M, \qquad H(u, v) = \exp_{c(u)}\left(v\cos\frac{\pi}{2r}u\right)X(u).$$

Put $c_v(u) = H(u, v)$. It is clear that $c_0(u) = c(u)$, $H(0, v) = \exp_q(v\gamma'(s)) \in \gamma$, so the function $L(v)$ attain a minimum at $v = 0$.

Using the first variation formula, we have

$$\frac{d}{dv}(L(c_v))\Big|_{v=0} = 0 \Leftrightarrow \left\langle \cos\frac{\pi u}{2r}X(u), c'(u)\right\rangle\Big|_0^r - \int_0^r\left\langle\cos\frac{\pi u}{2r}X(u), \nabla_{c'}c'(u)\right\rangle du =$$

$$= 0 \Leftrightarrow \langle X(0), c'(0)\rangle = 0 \Leftrightarrow \langle \gamma'(0), c'(0)\rangle = 0 \Leftrightarrow c'(0) = \pm Y(s).$$

Without loss of generality, we can suppose that $c'(0) = Y(s)$. Hence,

$$p = c(r) = \exp_{c(0)} rc'(0) = \exp_{\gamma(s)} rY(s) \quad \text{and} \quad d(p, \gamma) = r.$$

By the definition of $\varphi(s)$, $\rho(s)$, we have $\varphi(s) \le r \le \rho(s)$, whence $(s; r) \in A \bigcup B$ and $p \in C \bigcup D$. Thus, $M = C \bigcup D$.

In order to prove $C \bigcap D = \varnothing$, we suppose that there exists $p_1 \in C \bigcap D$. Then $\exists s_1 \ne s_2$ such that

$$p_1 = \exp_{\gamma(s_1)} t_1 Y(s_1) = \exp_{\gamma(s_2)} t_2 Y_2(s_2),$$

where $\varphi(s_1) < t_1 < \rho(s_1)$, $t_2 = \varphi(s_2)$ or $t_2 = \rho(s_2)$.

Choose a number t_3 such that

$$t_1 < t_3 < \rho(s_1) \quad \text{if } t_1 \ge 0 \text{ and}$$

$$\varphi(s_1) < t_3 < t_1 \quad \text{if } \ t_1 < 0.$$

Put $q_1 = \exp_{\gamma(s_1)} t_3 Y(s_1)$. We get

$$d(q_1, \gamma) = d(q_1, p_1) + d(p_1, \gamma(s_1)) = d(p_1, q_1) + d(p_1, \gamma) =$$

$$= d(p_1, q_1) + d(p_1, \gamma(s_2)) > d(q_1, \gamma(s_2)).$$

This contradition prove our assertion.

Consider the variation $\overline{H}: I \times R \to M$ defined by $\overline{H}(s, t) = \exp_{\gamma(s)} tY(s)$. Put $H = \overline{H}|_A$. It is clear that $H(A) = C$ by definition of the set C. We now prove that H is injective. In fact, suppose (inversely) that H isn't injective, then $\exists (s_1, t_1) \ne (s_2, t_2)$, $(s_i, t_i) \in A$, $i = 1, 2$, such that $H(s_1, t_1) = H(s_2, t_2) = q$, hence $d(q, \gamma) = |t_1| = |t_2| = t_0$ by (4). There are two cases:

***Case* 1.** If $s_1 = s_2$, $t_2 = -t_1 > 0$, then there exists $t_3: 0 < t_2 < t_3 < \rho(s_1)$. Consider the geodesic $c(t) = \exp_{\gamma(s_1)} tY(s_1)$, we have $c(t_1) = c(t_2)$. By a consequence of the Hopf – Rinow theorem (see [4, p. 100]), the geodesic c is no more minimal on the interval $[0, t_3]$. This is a contradiction.

***Case* 2.** If $s_1 \ne s_2$, we can suppose $t_2 = t_0 > 0$, for all $0 < t_2 < t_3 < \rho(s_2)$ $p = \exp_{\gamma(s_2)} t_3 Y(s_2)$ we have

$$d(p, \gamma) \le d(p, \gamma(s_1)) < d(p, q) + d(q, \gamma(s_1)) =$$

$$= d(p, q) + d(q, \gamma(s_2)) = d(p, \gamma).$$

This is a contradiction.

Thus, H is bijective from A on C. Moreover, H is the homeomorphism.

Let μ be the canonical measure on M. We have

$$\operatorname{Vol}(M) = \int_M \mu = \int_{C \cup D} \mu = \int_C \mu + \int_D \mu = \int_C \mu =$$

$$= \int_{H^{-1}(C)} Gr\left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t}\right)^{1/2} ds\,dt = \int_0^l \left(\int_{\varphi(s)}^{\rho(s)} Gr\left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t}\right)^{1/2} dt \right) ds. \tag{6}$$

Since $H(s, \cdot)$ is a geodesic, then

$$\nabla_{\partial/\partial t} \frac{\tilde{\partial}}{\partial t} = 0 \Rightarrow \left| \frac{\tilde{\partial}}{\partial t}(s, t) \right| = |Y(s)| = 1.$$

Because

$$\frac{d}{dt}\left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle = \left\langle \nabla_{\partial/\partial t} \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle = \left\langle \nabla_{\partial/\partial s} \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial t} \right\rangle = \frac{1}{2} \frac{d}{ds} \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle = 0,$$

we have

$$\left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle (s, t) = \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle (s, 0) = \langle \gamma'(s), Y(s) \rangle = 0.$$

Thus,

$$Gr\left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t}\right) = \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle \left\langle \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial t} \right\rangle - \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle^2 = \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle.$$

Put $f = \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle$. We will estimate the function $f(s, t)$.

From $K \geq k$ we have

$$\left\langle \overline{R}\left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial s}\right), \frac{\tilde{\partial}}{\partial s} \right\rangle \geq k \left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right\rangle = kf,$$

hence

$$-\left\langle \nabla_{\partial/\partial t} \nabla_{\partial/\partial s} \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial s} \right\rangle \geq kf. \tag{7}$$

Furthermore, since

$$\left\langle \nabla_{\partial/\partial t} \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t} \right\rangle = \left\langle \nabla_{\partial/\partial s} \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial t} \right\rangle = \frac{1}{2} \frac{d}{ds} \left\langle \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial t} \right\rangle = 0,$$

we have

$$\nabla_{\partial/\partial t} \frac{\tilde{\partial}}{\partial s} = \overline{x}(s, t) \frac{\tilde{\partial}}{\partial s}. \tag{8}$$

Thus,

$$\nabla_{\partial/\partial t} \nabla_{\partial/\partial s} \frac{\tilde{\partial}}{\partial t} = \nabla_{\partial/\partial t} \nabla_{\partial/\partial t} \frac{\tilde{\partial}}{\partial s} = \overline{x}_t'(s, t) \frac{\tilde{\partial}}{\partial s} + \overline{x}^2(s, t) \frac{\tilde{\partial}}{\partial s} = (\overline{x}_t' + \overline{x}^2) \frac{\tilde{\partial}}{\partial s}. \tag{9}$$

On the other hand, we get from (8):

$$\bar{x}(s,0) = \left\langle \nabla_{\partial/\partial t}\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s}\right\rangle(s,0) = \frac{1}{2}\frac{d}{ds}\left\langle \frac{\tilde{\partial}}{\partial t}, \frac{\tilde{\partial}}{\partial s}\right\rangle(s,0) = 0,$$

$$f'_t = \frac{d}{dt}\left\langle \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s}\right\rangle = 2\left\langle \nabla_{\partial/\partial t}\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s}\right\rangle = 2\bar{x}f. \tag{10}$$

From (7) and (9), we obtain

$$-(\bar{x}'_t + \bar{x}^2)f \geq kf \Leftrightarrow \bar{x}'_t + \bar{x}^2 \leq -k. \tag{11}$$

We now consider two cases.

a) In the case $k > 0$, set

$$x(s,t) = \frac{1}{\sqrt{k}}\,\bar{x}\left(s, \frac{t}{\sqrt{k}}\right),$$

$$x'_t(s,t) = \frac{1}{\sqrt{k}}\,\bar{x}'_t\left(s, \frac{t}{\sqrt{k}}\right).$$

So $kx'_t + kx^2 \leq -k \Leftrightarrow x'_t + x^2 \leq -1$.

Theorem 1 states that

$$-\frac{\pi}{2\sqrt{k}} \leq \varphi(s) \leq 0 \leq \rho(s) \leq \frac{\pi}{2\sqrt{k}} \Rightarrow -\frac{\pi}{2} \leq \sqrt{k}\varphi(s) \leq 0 \leq \sqrt{k}\varphi(s) \leq \frac{\pi}{2}.$$

Using Lemma 1, we get

$$\begin{cases} x(s,t) \leq -\operatorname{tg} t, & s \in [0,l), \quad 0 \leq t < \sqrt{k}\rho(s) \\ x(s,t) \geq -\operatorname{tg} t, & s \in [0,l), \quad \sqrt{k}\varphi(s) < t \leq 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \bar{x}(s,t) \leq -\sqrt{k}\operatorname{tg}(\sqrt{k}t), & s \in [0,l), \quad 0 \leq t < \rho(s), \\ \bar{x}(s,t) \geq -\sqrt{k}\operatorname{tg}(\sqrt{k}t), & s \in [0,l), \quad \varphi(s) < t \leq 0. \end{cases}$$

From (10), we have $\dfrac{f'_t}{f} = 2\bar{x}$.

For $t \geq 0 \Rightarrow \dfrac{f'_t}{f} \leq -\sqrt{k}\,2\operatorname{tg}(\sqrt{k}t) \Leftrightarrow (\ln|f|)'_t \leq (\ln(\cos^2(\sqrt{k}t)))' \Rightarrow$

$$\Rightarrow \left[\ln|f| - \ln(\cos^2(\sqrt{k}t))\right]'_t \leq 0 \Rightarrow \left(\ln\frac{|f(s,t)|}{\cos^2(\sqrt{k}t)}\right)'_t \leq 0 \Rightarrow$$

$$\Rightarrow \ln\frac{|f(s,t)|}{\cos^2(\sqrt{k}t)} \leq \ln|f(s,0)| \Rightarrow$$

$$\Rightarrow \ln\frac{|f(s,t)|}{\cos^2(\sqrt{k}t)} \leq |f(s,0)| = \langle\gamma'(s), \gamma'(s)\rangle = 1 \Rightarrow$$

$$\Rightarrow |f| \leq \cos^2(\sqrt{k}t).$$

For $t \leq 0 \Rightarrow \dfrac{f'_t}{f} \geq -\sqrt{k}\,2\operatorname{tg}(\sqrt{k}t) \Leftrightarrow (\ln|f|)'_t \geq (\ln(\cos^2(\sqrt{k}t)))' \Rightarrow$

$$\Rightarrow \frac{|f|}{\cos^2(\sqrt{k}t)} \leq |f(s,0)| = 1 \Rightarrow |f| \leq \cos^2(\sqrt{k}t).$$

So, for all t such that $\varphi(s) < t < \rho(s)$ and for $s \in [0, l)$, we get $|f(s,t)| \le \cos^2(\sqrt{k}t)$.

Thus

$$\mathrm{Vol}(M) = \int_0^l \left(\int_{\varphi(s)}^{\rho(s)} Gr\left(\frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial t}\right)^{1/2} ds \right) dt = \int_0^l \left(\int_{\varphi(s)}^{\rho(s)} \sqrt{f}\, dt \right) ds \le$$

$$\le \int_0^l \left(\int_{\varphi(s)}^{\rho(s)} \cos\sqrt{k}t\, dt \right) ds \le \int_0^l \left(\int_{-\pi/2\sqrt{k}}^{\pi/2\sqrt{k}} \cos\sqrt{k}t\, dt \right) ds = \frac{2l}{\sqrt{k}}.$$

b) Consider the case $k = 0$.

From (11)

$$\Rightarrow \bar{x}_t' + \bar{x}^2 \le 0 \Rightarrow \bar{x}_t' \le 0 \Rightarrow$$

$$\Rightarrow \begin{cases} \bar{x}(s,t) \le \bar{x}(s,0) = 0, & 0 \le t < \rho(s) \\ \bar{x}(s,t) \ge \bar{x}(s,0) = 0, & \varphi(s) < t \le 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f_t' \le 0, & 0 \le t < \rho(s) \\ f_t' \ge 0, & \varphi(s) < t \le 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(s,t) \le f(s,0) = 1, & 0 \le t < \rho(s) \\ f(s,t) \le f(s,0) = 1, & \varphi(s) < t \le 0 \end{cases} \Rightarrow$$

$$\Rightarrow \mathrm{Vol}(M) = \int_0^l \left(\int_{\varphi(s)}^{\rho(s)} dt \right) ds \le \int_0^l 2l\,\mathrm{diam}(M)\, ds = 2l\,\mathrm{diam}(M).$$

Thus,

$$\mathrm{Vol}(M) = \begin{cases} \dfrac{2l}{\sqrt{k}} & \text{if } k > 0, \\ 2l\,\mathrm{diam}(M) & \text{if } k = 0. \end{cases}$$

Theorem 2 is proved.

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