UDC 517.9 + 531.19

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STOCHASTIC DYNAMICS AND HIERARCHY FOR THE BOLTZMANN EQUATION WITH ARBITRARY DIFFERENTIAL SCATTERING CROSS SECTION*

СТОХАСТИЧНА ДИНАМІКА ТА ІЄРАРХІЯ ДЛЯ РІВНЯННЯ БОЛЬЦМАНА З ДОВІЛЬНИМ ДИФЕРЕНЦІАЛЬНИМ ПЕРЕРІЗОМ РОЗСІЯННЯ

The stochastic dynamics for point particles that corresponds to the Boltzmann equation with arbitrary differential scattering cross section is constructed. We derive the stochastic Boltzmann hierarchy the solutions of which outside the hyperplanes of lower dimension, where the point particles interact, are equal to the product of one-particle correlation functions, provided that the initial correlation functions are products of one-particle correlation functions. A one-particle correlation function satisfies the Boltzmann equation. The M. Kac dynamics in the momentum space is obtained.

Побудовано стохастичну динаміку, що відповідає рівнянню Больцмана з довільним перерізом розсіяння. Вивчено стохастичну ієрархію Больцмана, розв'язки якої поза гіперплощин нижчої розмірності, де точкові частинки взаємодіють, збігаються з добутком одночастинкових кореляційних функцій, якщо початкові кореляційні функції є добутком одночастинкових кореляційних функцій. У свою чергу, одночастинкова кореляційна функція задовольняє рівняння Больцмана. Виведено динаміку М. Каца у просторі імпульсів.

Introduction. In the series of papers [1–4], we have introduced the stochastic dynamics of point particles, which is obtained from the Hamilton dynamics of a system of hard spheres in the Boltzmann–Grad limit. According to this stochastic dynamics, point particles move as free ones until their positions coincide, and then they undergo elastic scattering. The unit vector that determines elastic scattering is a random vector uniformly distributed on the unit sphere. Then particles move as free ones until the next collision.

The present work is a generalization of the results obtained in [1–4] to the case of stochastic dynamics in which the unit vector that determines elastic scattering of point particles is distributed on the unit sphere with distribution density corresponding to an arbitrary differential scattering crosssection. As is customary in classical statistical mechanics [5], the initial state of a system is defined by a distribution function on the phase space. The state of the system at arbitrary time is defined as the result of the action of an evolution operator, i.e., the operator of transition along the trajectory, on the initial distribution function. The distribution function thus defined differs from the distribution function of the free system of particles at arbitrary time only on the hyperplanes of lower dimension where the point particles interact.

From the viewpoint of traditional classical statistical mechanics, the system of point particles moving according to stochastic dynamics should be regarded as a free system. Indeed, in traditional statistical mechanics, averages are calculated via the Lebesgue integral, and the behavior of distribution functions on hyperplanes of lower dimension is not taken into account. In this connection, by analogy with [1–4], a new concept of averages of observables over distribution functions was introduced; this concept takes into account, in a special way, the contribution of the hyperplanes where the particles interact.

* This work was performed during the stay November–December 2002, 2003 of Prof. D. Ya. Petrina as visiting professor at the Department of Mathematics of Politecnico di Milano.

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With the use of averages thus introduced, correlation functions are defined that also take into account, in a special way, the contribution of the hyperplanes where the particles interact.

The hierarchy of equations derived for the sequence of correlation functions has the form of the ordinary Boltzmann hierarchy [6, 7] but takes into account the boundary conditions corresponding to the case where the positions of particles coincide. If the hierarchy is considered in the weak sense, then it contains δ-functions, which differ from zero when the positions of particles coincide. The hierarchy obtained was called the stochastic Boltzmann hierarchy.

We construct solutions of the stochastic Boltzmann hierarchy on a finite time interval for a sequence of correlation functions that belong to the space of functions bounded with respect to coordinates and exponentially decreasing with respect to momenta. We also construct solutions of the hierarchy on an arbitrary time interval for initial correlation functions that belong to the space of functions exponentially decreasing with respect to coordinates and momenta. These solutions are equal to the sum of the certain contributions of the hyperplanes where point particles interact.

It should be noted that the solutions of the equations and the Boltzmann hierarchy are also equal to the sum of the contributions of the hyperplanes where point particles interact. For the first time, this fact was noted in [3] for the stochastic dynamics corresponding to the Boltzmann – Grad limit of a system of hard spheres.

The solutions of the stochastic Boltzmann hierarchy coincide with the solutions of the ordinary Boltzmann hierarchy outside the hyperplanes of lower dimension where point particles interact. As is known, the solutions of the ordinary Boltzmann hierarchy are products of one-particle correlation functions, provided that the initial correlation functions are also products of one-particle correlation functions. A one-particle correlation function is a solution of the nonlinear Boltzmann equation [8, 9]. In other words, the solutions of the ordinary Boltzmann hierarchy satisfy the chaos condition.

It follows from the arguments presented above that the solutions of the stochastic Boltzmann hierarchy also satisfy the chaos condition outside the hyperplanes of lower dimension where point particles interact because they coincide there with the solutions of the ordinary Boltzmann hierarchy.

Note that the stochastic Boltzmann hierarchy is obtained from stochastic dynamics in the same way as the BBGKY hierarchy is obtained from the Hamilton dynamics. The ordinary Boltzmann hierarchy is obtained directly from the Boltzmann equations [3] or from the BBGKY hierarchy for a system of hard spheres in the Boltzmann – Grad limit, where the boundary conditions are not taken into account [6, 7], and it is likely that there is no dynamics corresponding to it.

As is known, M. Kac [10, 11] proposed to use a special Markov process in the momentum space and obtained from the corresponding Kolmogorov equation in the mean-field approximation a hierarchy whose solutions satisfy the chaos condition (see also [12, 13]).

In [4], it was shown that this Markov process in the momentum space can be obtained by certain averaging with respect to coordinates from stochastic dynamics in the phase space that corresponds to a system of hard spheres. The Kolmogorov equation can also be obtained by averaging with respect to coordinates from the Liouville equation for the distribution function in the phase space. The results of M. Kac were thus justified. In the present work, we generalize the results of M. Kac to stochastic dynamics considered.

Note that the Boltzmann equation in the momentum space can be obtained directly from the stochastic Boltzmann hierarchy for initial correlation functions that depend only on momenta and satisfy the chaos condition [4]. Indeed, outside the hyperplanes of lower dimension where point particles interact, the solutions of the ordinary hierarchy and the stochastic hierarchy coincide, whereas the solutions of the ordinary hierarchy do not depend on coordinates and satisfy the chaos conditions, and one-particle correlation function is a solution of the Boltzmann equation. In this case, the mean-field approximation is not used.

Thus, we have generalized and reproduced the aforementioned results concerning the calculation of averages of observables, a new concept of correlation functions, stochastic hierarchy, chaos, and an analog of the M. Kac dynamics in the momentum space for our stochastic dynamics with arbitrary differential scattering crosssection.

In authors' opinion, the results are now presented in a more clear and consistent form than in the previous papers of the authors.

1. Stochastic dynamics of N particles. *1.1. Functional-average.* Consider pointwise particles with unit mass in three-dimensional space R^3, and denote by $x_1 = (q_1, p_1), \ldots, x_N = (q_N, p_N)$ — their phase points, $(x)_N = (x_1, \ldots, x_N) = x$ at initial time $t = 0$.

Define their stochastic dynamics for negative time $-t$, $t > 0$ as follows. Particles move as free ones until $q_i - p_i\tau = q_j - p_j\tau$, $0 \le \tau \le t$, $(i, j) \subset (1, \ldots, N)$. Then these two particles collide, their momenta become

$$\begin{gathered} p_i^* = p_i - \eta_{ij}\eta_{ij} \cdot (p_i - p_j), \qquad p_j^* = p_j + \eta_{ij}\eta_{ij} \cdot (p_i - p_j), \\ \eta_{ij} \subset S_2^+(\eta_{ij} | \eta_{ij} \cdot (p_i - p_j) \ge 0), \quad |\eta_{ij}| = 1, \end{gathered} \tag{1.1}$$

if $\eta_{ij} \subset S_2^-(\eta_{ij} \mid \eta_{ij} \cdot (p_i - p_j) \le 0)$, then $p_i^* = p_i, p_j^* = p_j$, $\eta_{ij} \cdot (p_i - p_j)$ is the scalar product of vectors η_{ij} and $(p_i - p_j)$.

At time $-t$ their phase points are

$$\begin{gathered} x_i(-t) = (q_i - p_i\tau - p_i^*(t - \tau), p_i^*), \\ x_j(-t) = (q_j - p_j\tau - p_j^*(t - \tau), p_j^*), \quad \eta_{ij} \subset S_2^+, \end{gathered} \tag{1.2}$$

if $\eta_{ij} \subset S_2^-$, $x_i(-t) = (q_i - p_i t, p_i)$, $x_j(-t) = (q_j - p_j t, p_j)$.

Particles scatter elastically but the vectors η_{ij} are random ones with density of probability $\dfrac{Q(\eta_{ij} \cdot (p_i - p_j))}{\eta_{ij} \cdot (p_i - p_j)}$ where $Q(\eta_{ij} \cdot (p_i - p_j))$ is known as a crosssection $\eta_{ij} \cdot (p_i - p_j)$.

If $\eta_{ij} \subset S_2^-(\eta_{ij} \,|\, \eta_{ij} \cdot (p_i - p_j) \le 0)$ then particles continue move freely even in case $q_i - p_i\tau = q_j - p_j\tau$.

For positive time $t > 0$ it is necessary to put in (1.2) $(+\tau)$ instead of $(-\tau)$ and S_2^- and S_2^+ instead of S_2^+ and S_2^- respectively. We neglect the case when three or more particles collide at the same point.

The above introduced stochastic dynamics defines the trajectory in phase space $X(-t) = (x(-t))_N = (x_1(-t), \ldots, x_N(-t)) = (x_1(-t, (x)_N), \ldots, x_N(-t, (x)_N)) = X(-t, (x)_N) = X(-t, x)$. Obviously trajectory $X(-t)$ satisfies the group property $X(-t_1 - t_2, x) = X(-t_1, X(-t_2, x)) = X(-t_2, X(-t_1, x))$.

Define the operator $S_N(-t)$ as the operator of shift along the trajectory

$$S(-t)f_N(x_1,\dots,x_N) = f_N(x_1(-t),\dots,x_N(-t)) = \\ = f_N(x_1(-t,(x)_N),\dots,x_N(-t,(x)_N)). \tag{1.3}$$

Let $f_N(x_1,\dots,x_N)$ be real symmetric continuously differentiable normalized function and $\varphi_N(x_1,\dots,x_N)$ be real symmetric test function.

Consider an infinitesimal time $-\Delta t$ and introduce the following functional:

$$(S_N(-\Delta t)f_N,\varphi_N) = \int dx_1\dots dx_N \times$$
$$\times f_N(q_1-p_1\Delta t,p_1,\dots,q_N-p_N\Delta t,p_N)\varphi_N(q_1,p_1,\dots,q_N,p_N) +$$
$$+\sum_{i<j=1}^{N}\int dx_1\dots dx_N\int_0^{\Delta t}d\tau\int_{S_2^+}d\eta_{ij}\frac{Q(\eta_{ij}\cdot(p_i-p_j))}{\eta_{ij}\cdot(p_i-p_j)}\times$$
$$\times\eta_{ij}\cdot(p_i-p_j)\delta(q_i-p_i\tau-q_j+p_j\tau)\times$$
$$\times\Big[f_N(q_1-p_1\Delta t,p_1,\dots,q_i-p_i\tau-p_i^*(\Delta t-\tau),$$
$$p_i^*,\dots,q_j-p_j\tau-p_j^*(\Delta t-\tau),p_j^*,\dots,q_N-p_N\Delta t,p_N)-$$
$$-f_N(q_1-p_1\Delta t,p_1,\dots,q_i-p_i\Delta t,p_i,\dots,q_j-p_j\Delta t,$$
$$p_j,\dots,q_N-p_N\Delta t,p_N)\Big]\varphi_N(q_1,p_1,\dots,q_N,p_N) =$$
$$=\int dx_1\dots dx_N\left[f_N(x_1,\dots,x_N)-\Delta t\sum_{i=1}^{N}p_i\frac{\partial}{\partial q_i}f_N(x_1,\dots,x_N)\right]\times$$
$$\times\varphi_N(x_1,\dots,x_N)-\int dx_1\dots dx_N\Delta t\int_{S_2^+}d\eta_{ij}\delta(q_i-q_j)Q(\eta_{ij}\cdot(p_i-p_j))\times$$
$$\times\Big[f_N(q_1,p_1,\dots,q_i,p_i^*,\dots,q_j,p_j^*,\dots,q_N,p_N)-$$
$$-f_N(q_1,p_1,\dots,q_i,p_i,\dots,q_j,p_j,\dots,q_N,p_N)\Big]\varphi_N(q_1,p_1,\dots,q_N,p_N). \tag{1.4}$$

Here the operator $S_N(-\Delta t)$ is defined according to the stochastic dynamics as follows:

for $q_i-p_i\tau=q_j-p_j\tau$

$$S_N(-\Delta t)f_N(x_1,\dots,x_i,\dots,x_j,\dots,x_N)|_{q_i-p_i\tau=q_j-p_j\tau} =$$
$$= f_N(q_1-p_1\Delta t,p_1,\dots,q_i-p_i\tau-p_i^*(\Delta t-\tau),p_i^*,\dots$$
$$\dots,q_j-p_j\tau-p_j^*(\Delta t-\tau),p_j^*,\dots,q_N-p_N\Delta t,p_N)$$

for $\eta_{ij}\in S_2^+$,

$$S_N(-\Delta t)f_N(x_1,\dots,x_i,\dots,x_j,\dots,x_N)|_{q_i-p_i\tau=q_j-p_j\tau} =$$
$$= f_N(q_1-p_1\Delta t,p_1,\dots,q_i-p_i\Delta t,p_i,\dots,q_j-p_j\Delta t,p_j,\dots,q_N-p_N\Delta t,p_N) \tag{1.5}$$

for $\eta_{ij}\in S_2^-$,

$$S_N(-\Delta t)f_N(x_1,\dots,x_N) = f_N(q_1-p_1\Delta t,p_1,\dots,q_N-p_N\Delta t,p_N)$$

for $q_i-\tau p_i\neq q_j-p_j\tau$, for all $(i,j)\subset(1,\dots,N)$, $0\le\tau\le\Delta t$.

We suppose that the function $\dfrac{Q(\eta_{ij}\cdot(p_i-p_j))}{\eta_{ij}\cdot(p_i-p_j)}$ is normalized, i.e.,

$$\int_{S_2^+} \frac{Q(\eta_{ij} \cdot (p_i - p_j))}{\eta_{ij} \cdot (p_i - p_j)} d\eta_{ij} = 1. \tag{1.6}$$

Now represent functional-average as follows:

$$(S_N(-\Delta t) f_N, \varphi_N) = \int dx_1 \dots dx_N \bar{S}_N(-\Delta t) f_N(x_1, \dots, x_N) \varphi_N(x_1, \dots, x_N). \tag{1.7}$$

The introduced operator $\bar{S}_N(-\Delta t)$ is the usual operator of evolution in the theory of Markov processes and was obtained as result of specific averaging procedure, with respect to random vectors η_{ij}, that takes into account the contribution from hypersurfaces of lower dimension $q_i - p_i\tau = q_j - p_j\tau,\ 0 \leq \tau \leq t,\ (i,j) \subset (1, \dots, N)$ where the stochastic particles interact.

The functional-average (1.7) is defined for arbitrary test function and it defines result of action of the operator $\bar{S}_N(-\Delta t)$ on function $f_N(x_1, \dots, x_N)$, i.e., $\bar{S}_N(-\Delta t) f_N(x_1, \dots, x_N)$, as generalized function

$$\begin{aligned}
\bar{S}_N(-\Delta t) f_N(x_1, \dots, x_N) &= f_N(q_1 - p_1\Delta t, p_1, \dots, q_N - p_N\Delta t, p_N) + \\
&+ \sum_{i<j=1}^{N} \int_0^{\Delta t} d\tau \int_{S_2^+} d\eta_{ij} Q(\eta_{ij} \cdot (p_i - p_j)) \delta(q_i - p_i\tau - q_j + p_j\tau) \times \\
&\times \Big[f_N(q_1 - p_1\Delta t, p_1, \dots, q_i - p_i\tau - p_i^*(\Delta t - \tau), \\
&\quad p_i^*, \dots, q_j - p_j\tau - p_j^*(\Delta t - \tau), p_j^*, \dots, q_N - \\
&\quad - p_N\Delta t, p_N) - f_N(q_1 - p_1\Delta t, p_1, \dots, q_i - p_i\Delta t, p_i, \dots \\
&\quad \dots, q_j - p_j\Delta t, p_j, \dots, q_N - p_N\Delta t, p_N) \Big] = \\
&= f_N(q_1 - p_1\Delta t, p_1, \dots, q_N - p_N\Delta t, p_N) + \\
&+ \sum_{i<j=1}^{N} \int_0^{\Delta t} d\tau \int_{S_2} d\eta_{ij} Q(\eta_{ij} \cdot (p_i - p_j)) \times \\
&\times \delta(q_i - p_i\tau - q_j + p_j\tau) S_N(-\Delta t) f_N(x_1, \dots, x_N) = \\
&= \bar{f}_N(\Delta t, x_1, \dots, x_N).
\end{aligned} \tag{1.8}$$

We use in (1.8) that $Q(\eta_{ij} \cdot (p_i - p_j))|_{\eta_{ij} \in S_2^-} = -Q(\eta_{ij} \cdot (p_i - p_j))|_{\eta_{ij} \in S_2^+}$, and (1.5).

Note that there is no contradiction between definition (1.3), (1.5) of $S_N(-\Delta t) \times \times f_N(x_1, \dots, x_N) = f_N(x_1(-\Delta t), \dots, x_N(-\Delta t))$ and (1.8). Formula (1.8) simply defines the function $\bar{S}_N(-\Delta t) f_N(x_1, \dots, x_N)$ as a generalized function and averages of function $\bar{S}_N(-\Delta t) f_N(x_1, \dots, x_N)$ (1.8) over the observable $\varphi_N(x_1, \dots, x_N)$ should be calculated as the following functional:

$$\begin{aligned}
(\bar{f}_N(\Delta t), \varphi_N) &= (\bar{S}_N(-\Delta t) f_N, \varphi_N) = \\
&= \int dx_1 \dots dx_N \bar{S}_N(-\Delta t) f_N(x_1, \dots, x_N) \varphi_N(x_1, \dots, x_N) = \\
&= (S_N(-\Delta t) f_N, \varphi_N).
\end{aligned} \tag{1.9}$$

Thus numerically the state $S_N(-\Delta t) f_N(x_1, \dots, x_N)$ is given by formulas (1.3), (1.5)

$$S_N(-\Delta t)f_N(x_1,\dots,x_N) = f_N(x_1(-\Delta t),\dots,x_2(-\Delta t)) =$$
$$= f_N(x_1(-\Delta t,(x)_N),\dots,x_1(-\Delta t,(x)_N)).$$

When we calculate the average of $S_N(-\Delta t)f_N(x_1,\dots,x_N)$ over the observable $\varphi_N(x_1,\dots,x_N)$ we use the generalized function $\bar{S}_N(-\Delta t)f_N(x_1,\dots,x_N) = = \bar{f}_N(\Delta t, x_1,\dots,x_N)$ given by formula (1.8) and calculate the average $(\bar{S}_N(-\Delta t)f_N, \varphi_N)$ as functional (1.9) that coincide with (1.4).

Note again that functional (1.4) is the average of the observable $\varphi_N(x_1,\dots,x_N)$ over the state

$$S_N(-\Delta t)f_N(x_1,\dots,x_N) = f_N(x_1(-\Delta t),\dots,x_N(-\Delta t))$$

where φ_N is real symmetric test function and $f_N \geq 0$ is also real symmetric continuously differentiable function normalized such that

$$\int f_N(x_1,\dots,x_N)dx_1\dots dx_N = 1.$$

Stress that in functionals (1.4), (1.7), (1.9) the contributions from hyperplanes of lower dimension $q_i - p_i\tau = q_j - p_j\tau,\ 0 \leq \tau \leq t,\ 1 \leq i < j \leq N$, where stochastic particles interact, are taken into account, they are equal to the second term in the right-hand side of (1.4).

1.2. Infinitesimal operator. From (1.8) it follows that

$$\frac{\partial \bar{S}_N(-t)}{\partial t}\Big|_{t=0} f_N(x_1,\dots,x_N) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(x_1,\dots,x_N) +$$
$$+\sum_{i<j=1}^{N}\int_{S_2^+} d\eta_{ij} Q(\eta_{ij}\cdot(p_i-p_j))\delta(q_i-q_j)\Big[f_N(x_1,\dots,x_i^*,\dots,x_j^*,\dots,x_N) -$$
$$-f_N(x_1,\dots,x_i,\dots,x_j,\dots,x_N)\Big] =$$
$$= \bar{\mathcal{H}}_N f_N(x_1,\dots,x_N),\ x_i^* = (q_i,p_i^*),\ x_j^* = (q_j,p_j^*). \tag{1.10}$$

We define formally the group of operators $\bar{S}_N(-t)$ at arbitrary time $-t$ as follows:

$$\bar{S}_N(-t) = \lim_{n\to\infty}\prod_{i=1}^{n}\bar{S}_N(-\Delta t_i), \qquad \sum_{i=1}^{n}\Delta t_i = t \tag{1.11}$$

where the operator $\bar{S}_N(-\Delta t)$ for infinitesimal Δt is defined according to (1.8) and the infinitesimal generator of the group $\bar{S}_N(-t)$ is defined by (1.10) and it is equal to $\bar{\mathcal{H}}_N$.

Now define the state $\bar{f}_N(t,x_1,\dots,x_N)$ at arbitrary time $t>0$ as follows:

$$\bar{f}_N(t) = \bar{f}_N(t,x_1,\dots,x_N) = \bar{S}_N(-t)f_N(x_1,\dots,x_N) =$$
$$= \lim_{n\to\infty}\prod_{i=1}^{n}\bar{S}_N(-\Delta t_i)f_N(x_1,\dots,x_N), \tag{1.12}$$
$$\sum_{i=1}^{n}\Delta t_i = t, \qquad \bar{f}_N(t+\Delta t) = \bar{S}_N(-\Delta t)\bar{f}_N(t),$$

i.e., $\bar{S}_N(-\Delta t)\bar{f}_N(t)$ is defined by formula (1.8) with $\bar{f}_N(t,x_1,\dots,x_N)$ instead of $f_N(x_1,\dots,x_N)$.

Define the functional average (with infinitesimal Δt) of the state $\bar f_N(t+\Delta t)$

$$\begin{gathered}(\bar f_N(t+\Delta t),\varphi_N)=(\bar S_N(-\Delta t)\bar f_N(t),\varphi_N)=\\ =\int dx_1\dots dx_N\bar f_N(t,q_1-p_1\Delta t,p_1,\dots\\ \dots,q_N-p_N\Delta t,p_N)\varphi_N(q_1,p_1,\dots,q_N,p_N)+\\ +\sum_{i<j=1}^N\int dx_1\dots dx_N\int_0^{\Delta t}d\tau\int_{S_2^+}d\eta_{ij}Q(\eta_{ij}\cdot(p_i-p_j))\delta(q_i-p_i\tau-q_j+p_j\tau)\times\\ \times\Big[\bar f_N(t,q_1-p_1\Delta t,p_1,\dots,q_i-p_i\tau-p_i^*(\Delta t-\tau),\\ p_i^*,\dots,q_j-p_j\tau-p_j^*(\Delta t-\tau),p_j^*,\dots,q_N-p_N\Delta t,p_N)-\\ -\bar f_N(t,q_1-p_1\Delta t,p_1,\dots,q_i-p_i\Delta t,p_i,\dots,q_j-p_j\Delta t,\\ p_j,\dots,q_N-p_N\Delta t,p_N)\varphi_N(x_1,\dots,x_N)\Big]=\\ =\int\bar S_N(-\Delta t)\bar f_N(t,x_1,\dots,x_N)\varphi_N(x_1,\dots,x_N)dx_1\dots dx_N=\\ =\int\bar f_N(t+\Delta t,x_1,\dots,x_N)\varphi_N(x_1,\dots,x_N)dx_1\dots dx_N.\end{gathered}\tag{1.13}$$

If follows from (1.13) that

$$\begin{gathered}\bar S_N(-\Delta t)\bar f_N(t,x_1,\dots,x_N)=\bar f_N(t+\Delta t,x_1,\dots,x_N)=\\ =\bar f_N(t,q_1-p_1\Delta t,p_1,\dots,q_N-p_N\Delta t,p_N)+\\ +\sum_{i<j=1}^N\int_0^{\Delta t}d\tau\int_{S_2^+}d\eta_{ij}Q(\eta_{ij}\cdot(p_i-p_j))\delta(q_i-p_i\tau-q_j+p_j\tau)\times\\ \times\Big[\bar f_N(t,q_1-p_1\Delta t,p_1,\dots,q_i-p_i\tau-p_i^*(\Delta t-\tau),p_i^*,\dots,q_j-\\ -p_j\tau-p_j^*(\Delta t-\tau),p_j^*,\dots,q_N-p_N\Delta t,p_N)-\\ -\bar f_N(t,q_1-p_1\Delta t,p_1,\dots,q_i-p_i\Delta t,p_i,\dots,q_j-\\ -p_j\Delta t,p_j,\dots,q_N-p_N\Delta t,p_N)\Big].\end{gathered}\tag{1.14}$$

From (1.14) we obtain the following differential equation for the state $\bar f_N(t,x_1,\dots,x_N)$:

$$\begin{gathered}\frac{\partial}{\partial t}\bar f_N(t,x_1,\dots,x_N)=-\sum_{i=1}^N p_i\frac{\partial}{\partial q_i}\bar f_N(t,x_1,\dots,x_N)+\\ +\sum_{i<j=1}^N\int_{S_2^+}d\eta_{ij}Q(\eta_{ij}\cdot(p_i-p_j))\delta(q_i-q_j)\times\\ \times\Big[\bar f_N(t,x_1,\dots,x_i^*,\dots,x_j^*,\dots,x_N)-\bar f_N(t,x_1,\dots,x_i,\dots,x_j,\dots,x_N)\Big]=\\ =\bar{\mathcal H}_N\bar f_N(t,x_1,\dots,x_N),\\ x_i^*=(q_i,p_i^*),\quad x_j^*=(q_j,p_j^*)\end{gathered}\tag{1.15}$$

with initial condition

$$\bar{f}_N(t, x_1, \dots, x_N)_{|_{t=0}} = f_N(x_1, \dots, x_N).$$

Equation (1.15) *is Kolmogorov equation for* $\bar{f}_N(t, x_1, \dots, x_N)$. It was derived from functional-average (1.4), (1.13) or from formulas (1.8), (1.14). Stress again that in functional-average (1.4), (1.13) the contribution from hyperplanes $q_i - p_i\tau = q_j - p_j\tau$, $1 \le i < j \le N$, are taken into account. These contribution are expressed in (1.4) and (1.13) by the second terms. Note that equation (1.15) defines the derivative of function $\bar{f}_N(t, x_1, \dots, x_N)$ in sense of generalized functions.

1.3. Infinitesimal operator with fixed random vectors. We can also differentiate function $f_N(x_1(-t), \dots, x_N(-t))$ in sense of point by point convergence, i.e., differentiate $f_N(x_1(-t), \dots, x_N(-t))$ with respect to time along the trajectory $(x_1(-t), \dots$ $\dots, x_N(-t))$ with fixed parameters η_{ij}. Denote the function $f_N(x_1(-t), \dots, x_N(-t))$ with fixed parameters η_{ij} by $\tilde{f}_N(t, x_1, \dots, x_N)$. Repeating words by words our calculation from papers [1 – 3] we obtain equation

$$\frac{\partial \tilde{f}_N}{\partial t}(t, x_1, \dots, x_N) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} \tilde{f}_N(t, x_1, \dots, x_N) +$$
$$+ \sum_{i<j=1}^{N} \Theta(\eta_{ij} \cdot (p_i - p_j))\eta_{ij} \cdot (p_i - p_j)\delta(q_i - q_j) \times$$
$$\times \left[\tilde{f}_N(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_N) - \tilde{f}_N(t, x_1, \dots, x_i, \dots, x_j, \dots, x_N)\right] =$$
$$= \tilde{\mathcal{H}}_N \tilde{f}_N(t, x_1, \dots, x_N), \quad f_N(t, x_1, \dots, x_N)_{|_{t=0}} = f_N(x_1, \dots, x_N) \qquad (1.16)$$

with boundary condition according to which when $q_i = q_j$ then in the first term of (1.16) momenta (p_i, p_j) should be replaced by (p_i^*, p_j^*) with $\eta_{ij} \cdot (p_i - p_j) \ge 0$, $\Theta(\alpha) = 1$, $\alpha > 0$, $\Theta(\alpha) = 0$, $\alpha < 1$.

Now we present a new derivation of equation (1.16). The equation (1.16) and infinitesimal operator $\tilde{\mathcal{H}}_N$ can be obtained from the following functional average:

$$\int dx_1 \dots dx_N f_N(q_1 - p_1\Delta t, p_1, \dots, q_N - p_N\Delta t, p_N)\varphi_N(q_1, p_1, \dots, q_N, p_N) +$$
$$+ \sum_{i<j=1}^{N} \int dx_1 \dots dx_N \int_0^{\Delta t} d\tau (\eta_{ij} \cdot (p_i - p_j)) \times$$
$$\times \Theta(\eta_{ij} \cdot (p_i - p_j))\delta(q_i - p_i\tau - q_j + p_j\tau) \times$$
$$\times \Big[f_N(q_1 - p_1\Delta t, p_1, \dots, q_i - p_i\tau - p_i^*(\Delta t - \tau), p_i^*, \dots, q_j -$$
$$- p_j\tau - p_j^*(\Delta t - \tau), p_j^*, \dots, q_N - p_N\Delta t, p_N) -$$
$$- f_N(q_1 - p_1\Delta t, p_1, \dots, q_i - p_i\Delta t, p_i, \dots, q_j - p_j\Delta t, p_j, \dots, q_N - p_N\Delta t, p_N)\Big] \times$$
$$\times \varphi_N(x_1, \dots, x_N) =$$
$$= \int dx_1 \dots dx_N \tilde{S}_N(-\Delta t) f_N(x_1, \dots, x_N)\varphi_N(x_1, \dots, x_N) = (\tilde{f}_N(\Delta t), \varphi_N). \qquad (1.17)$$

Equality (1.17) holds for an arbitrary test function $\varphi_N(x_1, \dots, x_N)$ and $\tilde{S}_N(-\Delta t) \times$ $\times f_N(x_1, \dots, x_N)$ is determined from (1.17) as follows:

$$\tilde{S}_N(-\Delta t)f_N(x_1,\dots,x_N) =$$
$$= \tilde{f}_N(\Delta t,x_1,\dots,x_N) = f_N(q_1-p_1\Delta t,p_1,\dots,q_N-p_N\Delta t,p_N) +$$
$$+\sum_{i<j=1}^{N}\int_0^{\Delta t} d\tau\Theta(\eta_{ij}\cdot(p_i-p_j))\eta_{ij}\cdot(p_i-p_j)\delta(q_i-p_i\tau-q_j+p_j\tau)\times$$
$$\times\Big[f_N(q_1-p_1\Delta t,p_1,\dots,q_i-p_i\tau-p_i^*(\Delta t-\tau),p_i^*,\dots,q_j-$$
$$-p_j\tau-p_j^*(\Delta t-\tau),p_j^*,\dots,q_N-p_N\Delta t,p_N)-$$
$$-f_N(q_1-p_1\Delta t,p_1,\dots,q_i-p_i\Delta t,p_i,\dots,q_j-p_j\Delta t,p_j,\dots,q_N-p_N\Delta t,p_N)\Big],$$
$$\left.\frac{\partial\tilde{S}_N(-t)}{\partial t}\right|_{t=0} f_N(t,x_1,\dots,x_N) = \tilde{\mathcal{H}}_N f_N(x_1,\dots,x_N). \tag{1.18}$$

We define formally the group of operators $\tilde{S}_N(-t)$ at arbitrary time $-t$ as follows: $\tilde{S}_N(-t) = \lim_{n\to\infty}\prod_{i=1}^{n}\tilde{S}_N(-\Delta t_i)$, $\sum_{i=1}^{n}\Delta t_i = t$, where the operator $\tilde{S}_N(-\Delta t)$ for infinitesimal Δt is defined according to (1.18) and infinitesimal generator of group $\tilde{S}_N(-t)$ is equal to $\tilde{\mathcal{H}}_N$. Using (1.18) and definition of $\tilde{S}_N(-t)$, one can obtain distribution function at arbitrary time t and it is $\tilde{f}_N(t,x_1,\dots,x_N) = \tilde{S}_N(-t)f_N(x_1,\dots$ $\dots,x_N)$. Let as suppose that distribution function is already obtained at time t then $\tilde{f}_N(t+\Delta t,x_1,\dots,x_N)$ is defined through $\tilde{f}_N(t,x_1,\dots,x_N)$ as follows:

$$\tilde{f}_N(t+\Delta t,x_1,\dots,x_N) = \tilde{S}_N(-\Delta t)\tilde{f}_N(t,x_1,\dots,x_N) =$$
$$= \tilde{f}_N(t,q_1-p_1\Delta t,p_1,\dots,q_N-p_N\Delta t,p_N) +$$
$$+\sum_{i<j=1}^{N}\int_0^{\Delta t} d\tau\Theta(\eta_{ij}\cdot(p_i-p_j))\eta_{ij}\cdot(p_i-p_j)\delta(q_i-p_i\tau-q_j+p_j\tau)\times$$
$$\times\Big[\tilde{f}_N(t,q_1-p_1\Delta t,p_1,\dots,q_i-p_i\tau-p_i^*(\Delta t-\tau),p_i^*,\dots,q_j-$$
$$-p_j\tau-p_j^*(\Delta t-\tau),p_j^*,\dots,q_N-p_N\Delta t,p_N)-$$
$$-\tilde{f}_N(t,q_1-p_1\Delta t,p_1,\dots,q_i-p_i\Delta t,p_i,\dots,q_j-$$
$$-p_j\Delta t,p_j,\dots,q_N-p_N\Delta t,p_N)\Big]. \tag{1.19}$$

Or in terms of averages with test functions $\varphi_N(x_1,\dots,x_N)$

$$(\tilde{f}_N(t+\Delta t),\varphi_N) = \int dx_1\dots dx_N\tilde{f}_N(t,q_1-p_1\Delta t,p_1,\dots,q_N-p_N\Delta t,p_N)\times$$
$$\times\varphi_N(x_1,\dots,x_N)+\sum_{i<j=1}^{N}\int dx_1\dots dx_N\times$$
$$\times\Bigg\{\int_0^{\Delta t} d\tau\Theta(\eta_{ij}\cdot(p_i-p_j))\eta_{ij}\cdot(p_i-p_j)\delta(q_i-p_i\tau-q_j+p_j\tau)\times$$
$$\times\Big[\tilde{f}_N(t,q_1-p_1\Delta t,p_1,\dots,q_i-p_i\tau-p_i^*(\Delta t-\tau),p_i^*,\dots,q_j-$$
$$-p_j\tau-p_j^*(\Delta t-\tau),p_j^*,\dots,q_N-p_N\Delta t,p_N)-$$
$$-\tilde{f}_N(t,q_1-p_1\Delta t,p_1,\dots,q_i-$$

$$-p_i\Delta t, p_i, \dots, q_j - p_j\Delta t, p_j, \dots, q_N - p_N\Delta t, p_N)\Big]\Bigg\} \times$$
$$\times \varphi_N(x_1, \dots, x_N). \tag{1.20}$$

Differential equation (1.16) follows from (1.19), (1.20).

1.4. Duality principle. Now we explain in what sense the functions $\tilde{S}_N(-\Delta t) \times \times f_N(x_1, \dots, x_N)$ given by formulae (1.18) is equivalent to the function $S_N(-\Delta t) \times \times f_N(x_1 \dots, x_N)$ given by the formulas (1.3), (1.5)

$$S_N(-\Delta t) f_N(x_1, \dots, x_N) = f_N(x_1(-\Delta t), \dots, x_N(-\Delta t)) =$$
$$= f_N(q_1 - p_1\Delta t, p_1, \dots, q_N - p_N\Delta t, p_N)$$

if $q_i - p_i\tau \neq q_j - p_j\tau$ for all $(i,j) \subset (1, \dots, N)$, and $0 \leq \tau \leq \Delta t$;

$$S_N(-\Delta t) f_N(x_1, \dots, x_N) = f_N(x_1(-\Delta t), \dots, x_N(-\Delta t)) =$$
$$= f_N(q_1 - p_1\Delta t, p_1, \dots, q_i - p_i\tau - p_i^*(\Delta t - \tau), p_i^*, \dots, q_j -$$
$$-p_j\tau - p_j^*(\Delta t - \tau), p_j^*, \dots, q_N - p_N\Delta t, p_N) \tag{1.21}$$

if $q_i - p_i\tau = q_j - p_j\tau$, $0 \leq \tau \leq \Delta t$, $\eta_{ij} \subset S_2^+$

$$S_N(-\Delta t) f_N(x_1, \dots, x_N) = f_N(x_1(-\Delta t), \dots, x_N(-\Delta t)) =$$
$$= f_N(q_1 - p_1\Delta t, p_1, \dots, q_i - p_i\Delta t, p_i, \dots, q_j - p_j\Delta t, p_j, \dots, q_N - p_N\Delta t, p_N) \tag{1.22}$$

if $q_i - p_i\tau = q_j - p_j\tau$, $0 \leq \tau \leq \Delta t$, $\eta_{ij} \subset S_2^-$, $(i,j) \subset (1, \dots, N)$.

Thus, numerically the functions $S_N(-\Delta t) f_N(x_1, \dots, x_N), S_N(-\Delta t) f_N(t, x_1, \dots \dots, x_N)$ is given by formulas (1.21), (1.22) in the all phase space $(x_1, \dots, x_N)$. The expressions $\tilde{S}_N(-\Delta t) f_N(x_1, \dots, x_N)$, $\tilde{S}_N(-\Delta t) \tilde{f}_N(t, x_1, \dots, x_N)$ (1.18), (1.19) determines how to integrate the functions $S_N(-\Delta t) f_N(x_1, \dots, x_N)$, $S_N(-\Delta t) \times \times f_N(t, x_1, \dots, x_N)$ with test function $\varphi_N(x_1, \dots, x_N)$, but with the fixed random vectors η_{ij}, $1 \leq i < j \leq N$, and to take into account the contributions from the hyperplanes $q_i - p_i\tau = q_j - p_j\tau$, $1 \leq i < j \leq N$.

The expressions $\bar{S}_N(-\Delta t) f_N(x_1, \dots, x_N), \bar{S}_N(-\Delta t) \bar{f}_N(t, x_1, \dots, x_N)$ (1.8), (1.14) determines how to integrate these functions with test functions $\varphi_N(x_1, \dots, x_N)$, to average with respect to the random vectors η_{ij}, and to take into account the contribution from the hyperplanes $q_i - p_i\tau = q_j - p_j\tau$.

Now we are able to formulate the *principle of duality for the distribution function* $f_N(t, x_1, \dots, x_N)$. Let us suppose that initial distribution function $f_N(0, x_1, \dots, x_N) \equiv \equiv f_N(x_1, \dots, x_N)$ is symmetric continuously differentiable and normalized one in the phase space. Then the distribution function (1.3) $f_N(t, x_1, \dots, x_N) = S_N(-t) \times \times f_N(x_1, \dots, x_N)$ is well defined continuously differentiable function everywhere outside the hyperplanes of lower dimensions where particles interact. But in the functional-average with some observable $\varphi_N(x_1, \dots, x_N)$, which is real symmetric smooth test function, we consider $S(-t) f_N(x_1, \dots, x_N) = f_N(t, x_1, \dots, x_N)$ as some definite generalized function and calculate the contribution from these hyperplanes of lower dimensions where particles interact. Calculating the functional-averages $(\tilde{f}(t), \varphi_N)$ or $(\bar{f}_N(t), \varphi_N)$ we use $\tilde{f}_N(t)$ or $\bar{f}_N(t)$ instead of $f_N(t)$. We are not able to calculate $(\tilde{f}_N(t), \varphi_N)$ or $(\bar{f}_N(t), \varphi_N)$ directly for arbitrary finite time t, we have explicit formulas (1.4), (1.17) only for infinitesimal Δt. For arbitrary time t we use formula (1.14) and (1.19) for definition $\bar{f}_N(t + \Delta t)$ and $\tilde{f}_N(t + \Delta t)$ through already defined $\bar{f}_N(t)$,

$\bar{f}_N(t)$ and then calculate $(\bar{f}_N(t+\Delta t), \varphi_N)$ according to (1.13), and $(\tilde{f}_N(t+\Delta t), \varphi_N)$ according to (1.20).

Duality principle defines the generalized functions $\bar{f}_N(t, x_1, \dots, x_N) = \bar{S}_N(-t) \times \times f_N(x_1, \dots, x_N)$ *or* $\tilde{f}_N(t, x_1, \dots, x_N) = \tilde{S}_N(-t) f_N(x_1, \dots, x_N)$ *through the usual function* $f_N(t, x_1, \dots, x_N) = S_N(-t) f_N(x_1, \dots, x_N)$ *by formulas* (1.8), (1.14) *or* (1.18), (1.19). *If one has to consider the distribution function as usual function (numerically) then one should take* $f_N(t, x_1, \dots, x_N) = S_N(-t) f_N(x_1, \dots, x_N)$, *if one has to calculate the functional-average* (1.13) *or* (1.20) *with observable* $\varphi_N(x_1, \dots, x_N)$ *then one should take the generalized function* $\bar{f}_N(t, x_1, \dots, x_N) = \bar{S}_N(-t) f_N(x_1, \dots, x_N)$ *or generalized function* $\tilde{f}_N(t, x_1, \dots, x_N) = \tilde{S}_N(-t) f_N(x_1, \dots, x_N)$.

In our paper [1] it has been shown that differential equation for $f_N(t, x_1, \dots, x_N)$ in sense of point by point convergence has the following form:

$$\frac{\partial f_N}{\partial t}(t, x_1, \dots, x_N) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(t, x_1, \dots, x_N) +$$
$$+ \sum_{i<j=1}^{S} \Theta(\eta_{ij} \cdot (p_i - p_j)) \delta(\Delta t - \tau_{ij}) \Big|_{\Delta t = 0} \times$$
$$\times \Big[f_N(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_N) -$$
$$- f_N(t, x_1, \dots, x_i, \dots, x_j, \dots, x_N) \Big], \quad q_i = q_j, \tag{1.23}$$

with above described boundary condition in the Poisson bracket and according to which (p_i, p_j) should be replaced by (p_i^*, p_j^*) if $q_i - q_j = 0$ and $[f_N(t+0, x_1, \dots, x_N) = = f_N(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots x_N)]$, $q_i = q_j$, $\eta_{ij} \cdot (p_i - p_j) > 0$, $[f_N(t+0, x_1, \dots \dots, x_N) = f_N(t, x_1, \dots, x_N)]$, $q_i = q_j$, $(\eta_{ij} \cdot (p_i - p_j) \le 0)$.

Here τ_{ij} is the time of collision. In the coordinate system where the first component of the vector $(q_i - q_j)$ is directed along the vector η_{ij}, the time of collision τ_{ij} is defined as follows:

$$\tau_{ij} = \frac{q_i^1 - q_j^1}{p_i^1 - p_j^1}.$$

Then the (i, j)-th term can be expressed as follows:

$$\Theta(p_i^1 - p_j^1) \delta(q_i^1 - q_j^1)(p_i^1 - p_j^1) \Big[f_N(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_N) -$$
$$- f_N(t, x_1, \dots, x_i, \dots, x_j, \dots, x_N) \Big] \Big|_{q_i^2 = q_j^2, q_i^3 = q_j^3}.$$

This term is different from zero on the first axis $q_i^1 - q_j^1$ (with respect to the vector $q_i - q_j$, i.e., for $q_i^2 - q_j^2 = 0$, $q_i^3 - q_j^3 = 0$) and, regarded as a generalized function in the three-dimensional space, is equal to

$$\Theta(p_i^1 - p_j^1) \delta(q_i^1 - q_j^1) \delta(q_i^2 - q_j^2) \delta(q_i^3 - q_j^3)(p_i^1 - p_j^1) \times$$
$$\times \Big[f_N(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_N) -$$
$$- f_N(t, x_1, \dots, x_i, \dots, x_j, \dots, x_N) \Big] =$$
$$= \Theta(\eta_{ij} \cdot (p_i - p_j)) \delta(q_i - q_j) \eta_{ij} \cdot (p_i - p_j) \times$$
$$\times \Big[f_N(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_N) - f_N(t, x_1, \dots, x_i, \dots, x_j, \dots, x_N) \Big]. \tag{1.24}$$

(For analogous calculations, see [1].) The expression obtained does not depend on the choice of coordinate system because $\delta(q_i - q_j)$ and $\eta_{ij} \cdot (p_i - p_j)$ are invariant under rotation. Substituting (1.24) in (1.23) we obtain (1.16).

Recall that the operator

$$-\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(t, x_1, \ldots, x_N) +$$
$$+ \sum_{i<j=1}^{N} \delta(q_i^1 - q_j^1)\Theta(p_i^1 - p_j^1)(p_i^1 - p_j^1) \times$$
$$\times \Big[f_N(t, x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_N) -$$
$$- f_N(t, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) \Big] \Big|_{q_i^2 = q_j^2, q_i^3 = q_j^3} \qquad (1.25)$$

with one-dimensional δ-functions $\delta(q_i^1 - q_j^1)$ is equivalent to the operator

$$-\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(t, x_1, \ldots, x_N) = \mathcal{H}_N f_N(t, x_1, \ldots, x_N) \qquad (1.26)$$

with the boundary condition according to which at points $q_i = q_j$, $(i, j) \subset (1, \ldots, N)$ momenta p_i and p_j should be replaced by p_i^* and p_j^* if $\eta_{ij} \cdot (p_i - p_j) \geq 0$ and (p_i, p_j) do not change if $\eta_{ij} \cdot (p_i - p_j) \leq 0$.

Thus we have three expressions for the infinitesimal operator $\mathcal{H}_N$ of the group $S_N(-t)$. The first one $\bar{\mathcal{H}}_N$ (1.15) was obtained in a week sense and it shows how to take into account, in the functional average, the hypersupfaces of lower dimensions $q_i = q_j$, $(i, j) \subset (1, \ldots, N)$, where particles interact. In the second one $\tilde{\mathcal{H}}_N$ (1.16) (also calculated in a weak sense), the average with respect to the random vectors η_{ij} was not performed, but the hypersupfaces $q_i = q_j$, $(i, j) \subset (1, \ldots, N)$, were taken into account. In the third ones $\mathcal{H}_N$ (1.23), (1.26) calculated point-by-point, the infinitesimal operator $\mathcal{H}_N$ of the group $S_N(-t)$ is equal to the operator $-\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i}$ with the boundary conditions at $q_i = q_j$, $(i, j) \subset (1, \ldots, N)$.

All these expressions for infinitesimal operator $\mathcal{H}_N$ are equivalent but the first one (1.15) and second one (1.16) show how calculate the average of $\frac{\partial \bar{f}}{\partial t}(t, x_1, \ldots, x_N)$ and $\frac{\partial \tilde{f}_N(t, x_1, \ldots, x_N)}{\partial t}$ with observable $\varphi_N(x_1, \ldots, x_N)$, or they define $\frac{\partial \bar{f}_N(t, x_1, \ldots, x_N)}{\partial t}$ and $\frac{\partial \tilde{f}_N(t, x_1, \ldots, x_N)}{\partial t}$ as generalized functions. The third expression for infinitesimal operator (1.25), (1.26) defines $\frac{\partial f_N(t, x_1, \ldots, x_N)}{\partial t} = \mathcal{H}_N f_N(t, x_1, \ldots, x_N)$ in sense of point-by-point differentiation and defines it as a usual function with jumps at $q_i = q_j$, $(i, j) \subset (1, \ldots, N)$, that is expressed in the boundary conditions.

Thus for the derivative $\frac{\partial f_N(t, x_1, \ldots, x_N)}{\partial t}$ *the principle of duality is also formulated as for* $f_N(t, x_1, \ldots, x_N)$ *and according to which the same* $\frac{\partial f_N(t, x_1, \ldots, x_N)}{\partial t}$ *is considered as a usual function or as special generalized functions in the functional-average.*

2. Hierarchy for correlation functions. ***2.1. Derivation of hierarchy from equation for distribution function.*** Define the following sequence of correlation functions:

$$\bar{F}_s^{(N)}(t,x_1,\ldots,x_s) = N(N-1)\ldots(N-s+1)\times$$
$$\times\int dx_{s+1}\ldots dx_N \bar{f}_N(t,x_1,\ldots,x_s,\ldots,x_{s+1},\ldots,x_N),\quad 1\le s\le N. \qquad (2.1)$$

By using equation (1.15) one can derive the following hierarchy for sequence (2.1):

$$\frac{\partial \bar{F}_s^{(N)}(t,x_1,\ldots,x_s)}{\partial t} = -\sum_{i=1}^{s} p_i\frac{\partial}{\partial q_i}\bar{F}_s^{(N)}(t,x_1,\ldots,x_s) +$$
$$+\sum_{i<j=1}^{s}\int_{S_2^+} d\eta_{ij}Q(\eta_{ij}\cdot(p_i-p_j))\delta(q_i-q_j)\times$$
$$\times\Big[\bar{F}_s(t,x_1,\ldots,x_i^*,\ldots,x_j^*,\ldots,x_s)-\bar{F}_s(t,x_1,\ldots,x_i,\ldots,x_j,\ldots,x_s)\Big]+$$
$$+\sum_{i=1}^{s}\int dx_{s+1}\int_{S_2^+} d\eta_{is+1}Q(\eta_{is+1}\cdot(p_i-p_{s+1}))\delta(q_i-q_{s+1})\times$$
$$\times\Big[\bar{F}_{s+1}^{(N)}(t,x_1,\ldots,x_i^*,\ldots,x_{s+1}^*)-\bar{F}_{s+1}^{(N)}(t,x_1,\ldots,x_i,\ldots,x_{s+1})\Big],\quad 1\le s\le N \qquad (2.2)$$

(see detail in [1, 3]).

Performing formal thermodynamic limit $N\to\infty$ for sequence (2.1) and supposing that one can also perform this limit in hierarchy (2.2) we obtain the limiting hierarchy

$$\frac{\partial \bar{F}_s(t,x_1,\ldots,x_s)}{\partial t} = -\sum_{i=1}^{s} p_i\frac{\partial}{\partial q_i}\bar{F}_s(t,x_1,\ldots,x_s) +$$
$$+\sum_{i<j=1}^{s}\int d\eta_{ij}Q(\eta_{ij}\cdot(p_i-p_j))\delta(q_i-q_j)\Big[\bar{F}_s(t,x_1,\ldots,x_i^*,\ldots,x_j^*,\ldots,x_s)-$$
$$-\bar{F}_s(t,x_1,\ldots,x_i,\ldots,x_j,\ldots,x_s)\Big]+$$
$$+\sum_{i=1}^{s}\int dx_{s+1}\int_{S_2^+} d\eta_{is+1}Q(\eta_{is+1}\cdot(p_i-p_{s+1}))\delta(q_i-q_{s+1})\times$$
$$\times\Big[\bar{F}_{s+1}(t,x_1,\ldots,x_i^*,\ldots,x_{s+1}^*)-\bar{F}_{s+1}(t,x_1,\ldots,x_i,\ldots,x_{s+1})\Big],\quad s\ge 1, \qquad (2.3)$$
$$\bar{F}_s(t,x_1,\ldots,x_s)=\lim_{N\to\infty}\bar{F}_s^{(N)}(t,x_1,\ldots,x_s).$$

Note that correlation function $\bar{F}_s(t,x_1,\ldots,x_s)$ do not depend on any random vectors η_{ij}.

The sequence of correlation functions $\bar{F}_s(t,x_1,\ldots,x_s)$, $s\ge 1$, and the hierarchy (2.3) for it was obtained through the distribution function $\bar{f}_N(t,x_1,\ldots,x_N)$ considered as a definite generalized function.

The first two terms in the right-hand side of hierarchy (2.3) are result of action of the infinitesimal operator $\bar{\mathcal{H}}_s$ (of the group $\bar{S}_s(-t)$), that is useful for functional-average. We want to derive the hierarchy for the sequence of correlation function considered as usual not generalized function. To do this we replace in (2.2) operator $\bar{\mathcal{H}}_s$ by equivalent operator $\mathcal{H}_s = -\sum_{i=1}^{s} p_i \dfrac{\partial}{\partial q_i}$ with known boundary condition. One obtains hierarchy

$$\frac{\partial F_s(t, x_1, \ldots, x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \ldots, x_s) +$$

$$+ \sum_{i=1}^{s} \int dx_{s+1} \int\limits_{S_2^+} d\eta_{is+1} Q(\eta_{is+1} \cdot (p_i - p_{s+1})) \delta(q_i - q_{s+1}) \times$$

$$\times \Big[F_{s+1}(t, x_1, \ldots, x_i^*, \ldots, x_s, x_{s+1}^*) - F_{s+1}(t, x_1, \ldots, x_i, \ldots, x_s, x_{s+1}^*) \Big], \quad s \geq 1, \tag{2.4}$$

with boundary conditions according to which at $q_i = q_j, (ij) \subset (1, \ldots, s)$ momenta (p_i, p_j) in the first term on the right-hand side of (2.4) should be replaced by (p_i^*, p_j^*).

The hierarchy (2.4) can be written in equivalent form

$$\frac{\partial F_s(t, x_1, \ldots, x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \ldots, x_s) +$$

$$+ \sum_{i<j=1}^{s} \Theta(\eta_{ij} \cdot (p_i^1 - p_j^1)) \eta_{ij} \cdot (p_i^1 - p_j^1) \delta(q_i^1 - q_j^1) \times$$

$$\times \Big[F_s(t, x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_s) -$$

$$- F_s(t, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_s) \Big] \Big|_{q_i^2 = q_j^2, q_i^3 = q_j^3} +$$

$$+ \sum_{i=1}^{s} \int dx_{s+1} \int\limits_{S_2^+} d\eta_{is+1} Q(\eta_{is+1} \cdot (p_i - p_{s+1})) \times$$

$$\times \Big[F_{s+1}(t, x_1, \ldots, x_i^*, \ldots, x_s, x_{s+1}^*) - F_{s+1}(t, x_1, \ldots, x_i, \ldots, x_s, x_{s+1}) \Big], \quad s \geq 1, \tag{2.5}$$

without boundary condition.

In (2.4) and (2.5) we used equivalent representation of the infinitesimal operator $\mathcal{H}_s$. Note that correlation functions in (2.4), (2.5) depend on the random vectors $\eta_{ij}, (i, j) \subset \subset (1, \ldots, s)$.

In what follows we will use the hierarchy for sequence of $F_s(t, q_1, \ldots, q_s)$, $s \geq 1$, in invariant form, independent on representation of the infinitesimal operator $\mathcal{H}_s$, namely

$$\frac{\partial F_s(t, x_1, \ldots, x_s)}{\partial t} = \mathcal{H}_s F_s(t, x_1, \ldots, x_s) +$$

$$+ \sum_{i=1}^{s} \int dx_{s+1} \int\limits_{S_2^+} d\eta_{is+1} Q(\eta_{is+1} \cdot (p_i - p_j)) \delta(q_i - q_{s+1}) \times$$

$$\times \Big[F_{s+1}(t, x_1, \ldots, x_i^*, \ldots, x_s, x_{s+1}^*) - F_{s+1}(t, x_1, \ldots, x_i, \ldots, x_s, x_{s+1}) \Big] \tag{2.6}$$

with initial condition

$$F_s(t, x_1, \ldots, x_s)|_{t=0} = F_s(x_1, \ldots, x_s).$$

Hierarchies (2.4), (2.5) can be obtained from equation (1.16) by integrating over $x_{s+1}, \ldots \ldots, x_N$ and averaging over all η_{ij} excluding those η_{ij} with $(i, j) \subset (1, \ldots, s)$.

One obtains hierarchy (2.6) with $\tilde{\mathcal{H}}_s$ instead of $\mathcal{H}_s$ (see details in [1, 3]). Then one uses the duality principle and replaces $\tilde{\mathcal{H}}_s$ by $\mathcal{H}_s$. Note that correlation functions $F_s(t, x_1, \ldots, x_s)$ depend on all random vectors η_{ij} with $(i,j) \subset (1, \ldots, s)$ that correspond to collisions of particles with numbers $(1, \ldots, s)$ and appear on entire interval $[-t, 0]$.

2.2. Derivation of hierarchy from functional average. Now define functional-average for the following observable:

$$\varphi_N(x_1, \ldots, x_N) = \sum_{i_1 < \ldots < i_s} \varphi_s(x_1, \ldots, x_s) \tag{2.7}$$

where summation is carried out over all $(i_1 < \ldots < i_s) \subset (1, \ldots, N)$.

It follows from (1.13), using symmetricity of functions $\bar{f}_N(t + \Delta t, x_1, \ldots, x_N)$ and $\bar{f}_N(t, x_1, \ldots, x_N)$ with respect to the variables $(x_1, \ldots, x_N)$,

$$\begin{gathered}
(\bar{f}_N(t+\Delta t), \varphi_N) = \int \bar{f}_N(t + \Delta t, x_1, \ldots, x_N)\varphi_N(x_1, \ldots, x_N)dx_1 \ldots x_N = \\
= \left[\frac{N!}{s!(N-s)!} \int \bar{f}_N(t + \Delta t, x_1, \ldots, x_s, x_{s+1}, \ldots, x_N)dx_{s+1} \ldots dx_N)\right] \times \\
\times \varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s = \\
= \left[\frac{N!}{s!(N-s)!} \int \bar{f}_N(t, q_i - p_1\Delta t, p_1, \ldots, q_N - p_N\Delta t, p_N)dx_{s+1} \ldots dx_N\right] \times \\
\times \varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s + \\
+ \frac{N!}{s!(N-s)!} \int \Bigg\{ \sum_{i<j=1}^{s} \int_0^{\Delta t} d\tau \int_{S_2^+} d\eta_{ij} Q(\eta_{ij} \cdot (p_i - p_j))\delta(q_i - p_i\tau - q_j + p_j\tau) \times \\
\times \int \Big[\bar{f}_N(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\tau - p_i^*(\Delta t - \tau), p_i^*, \ldots, q_j - \\
- p_j\tau - p_j^*(\Delta t - \tau), p_j^*, \ldots, q_N - p_N\Delta t, p_N) - \\
- \bar{f}_N(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\Delta t, p_i, \ldots, q_j - \\
- p_j\Delta t, p_j, \ldots, q_N - p_N\Delta t, p_N)dx_{s+1} \ldots dx_N\Big] \Bigg\} \varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s + \\
+ \frac{N!(N-s)}{s!(N-s)!} \int \Bigg\{ \sum_{i=1}^{s} \int_0^{\Delta t} d\tau \int_{S_2^+} d\eta_{is+1} Q(\eta_{is+1} \cdot (p_i - p_{s+1})) \times \\
\times \delta(q_i - p_i\tau - q_{s+1} + p_{s+1}\tau) \times \\
\times \int \Big[\bar{f}_N(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\tau - p_i^*(\Delta t - \tau), p_i^*, \ldots, q_s - \\
- p_s\Delta t, p_s, q_{s+1} - p_{s+1}\tau - p_{s+1}^*(\Delta t - \tau), p_{s+1}^*, q_{s+2} - p_{s+2}\Delta t, p_{s+2}, \ldots, q_N - \\
- p_N\Delta t, p_N) - \bar{f}_N(t, q_1 - p_1\Delta t, p_1, \ldots, q_s - p_s\Delta t, p_s, q_{s+1} - \\
- p_{s+1}\Delta t, p_{s+1}, q_{s+2} - p_{s+2}\Delta t, p_{s+2}, \ldots, q_N - p_N\Delta t, p_N)\Big] dx_{s+1} \ldots dx_N \Bigg\} \times \\
\times \varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s.
\end{gathered} \tag{2.8}$$

We suppose that the functional average in the right-hand side of (2.8) are already defined through the correlation functions $\bar{F}_s^{(N)}(t, x_1, \ldots, x_s)$. Then the above obtained formula has the following form:

$$\frac{1}{s!}\int \bar{F}_s^{(N)}(t+\Delta t, x_1, \ldots, x_s)\varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s =$$

$$= \frac{1}{s!}\int \bar{F}_s^{(N)}(t, q_1 - p_1\Delta t, p_1, \ldots, q_s - p_s\Delta t, p_s)\varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s +$$

$$+\frac{1}{s!}\int \Bigg\{ \sum_{i<j=1}^{s} \int_0^{\Delta t} d\tau \int_{S_2^+} d\eta_{ij} Q(\eta_{ij}\cdot(p_i - p_j))\delta(q_i - p_i\tau - q_j + p_j\tau) \times$$

$$\times \Big[\bar{F}_s^{(N)}(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\tau - p_i^*(\Delta t - \tau), p_i^*, \ldots, q_j - p_j\tau -$$

$$-p_j^*(\Delta t - \tau), p_j^*, \ldots, q_s - p_s\Delta t, p_s) -$$

$$-\bar{F}_s^{(N)}(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\Delta t, p_i, \ldots, q_j -$$

$$-p_j\Delta t, p_j, \ldots, q_s - p_s\Delta t, p_s)\Big]\Bigg\}\varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s +$$

$$+\frac{1}{s!}\int dx_{s+1}\Bigg\{ \sum_{i=1}^{s} \int_0^{\Delta t} d\tau \int_{S_2^+} d\eta_{is+1} Q(\eta_{is+1}\cdot(p_i - p_{s+1})) \times$$

$$\times \delta(q_i - p_i\tau - q_{s+1} + p_{s+1}\tau) =$$

$$= \int \big[\bar{F}_{s+1}^{(N)}(t, q_1 - p_1\Delta t, p_1, \ldots, q_i -$$

$$-p_i\tau - p_i^*(\Delta t - \tau), p_i^*, \ldots, q_s - p_s\Delta t, p_s, q_{s+1} - p_{s+1}\tau -$$

$$-p_{s+1}^*(\Delta t - \tau), p_{s+1}^*) - \bar{F}_{s+1}^{(N)}(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\Delta t, p_i, \ldots, q_s -$$

$$-p_s\Delta t, p_s, q_{s+1} - p_{s+1}\Delta t, p_{s+1})\big]\Bigg\}\varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s. \tag{2.9}$$

By differentiating by Δt this recurrent formula one obtains the following equation (in weak sense):

$$\int \frac{\partial \bar{F}_s^{(N)}(t, x_1, \ldots, x_s)}{\partial t}\varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s =$$

$$= \int \Bigg\{ -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i}\bar{F}_s^{(N)}(t, x_1, \ldots, x_s) +$$

$$+ \sum_{i<j=1}^{s} \int_{S_2^+} d\eta_{ij} Q(\eta_{ij}\cdot(p_i - p_j))\delta(q_i - q_j) \times$$

$$\times \Big[\bar{F}_s^{(N)}(t, x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_s) -$$

$$-\bar{F}_s^{(N)}(t, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_s)\Big]\Bigg\}\varphi_s(x_1, \ldots, x_s)dx_1 \ldots dx_s +$$

$$+\int\Bigg\{\sum_{i=1}^{s}\int dx_{s+1}\int\limits_{S_2^+} d\eta_{ij} Q(\eta_{is+1}(p_i-p_{s+1}))\delta(q_i-q_{s+1})\times$$
$$\times\Big[\bar{F}_{s+1}^{(N)}(t,x_1,\ldots,x_i^*,\ldots,x_s,x_{s+1}^*)-$$
$$-\bar{F}_{s+1}^{(N)}(t,x_1,\ldots,x_i,\ldots,x_s,x_{s+1})\Big]\Bigg\}\varphi_s(x_1,\ldots,x_s)dx_1\ldots dx_s, \quad N\le s\ge 1. \tag{2.10}$$

In the obtained hierarchy of equations the derivative of $\bar{F}_s^{(N)}(t,x_1,\ldots,x_s)$ with respect to time is expressed through $\bar{F}_s^{(N)}(t,x_1,\ldots,x_s)$ and $\bar{F}_{s+1}^{(N)}(t,x_1,\ldots,x_s,x_{s+1})$ and contributions from the hypersurfaces of lower dimensions, where particles interact, is taken into account. Obviously hierarchy (2.2) directly follows from (2.10).

According to duality principle the obtained hierarchy is equivalent to the following hierarchy for correlation functions considered as usual functions in every point of phase space $(x_1,\ldots,x_s)$

$$\frac{\partial F_s^{(N)}(t,x_1,\ldots,x_s)}{\partial t}=\mathcal{H}_s F_s^{(N)}(t,x_1,\ldots,x_s)+$$
$$+\sum_{i=1}^{s}\int dx_{s+1}\int\limits_{S_2^+} d\eta_{is+1} Q(\eta_{is+1}\cdot(p_i-p_{s+1}))\delta(q_i-q_{s+1})\times$$
$$\times\Big[F_{s+1}^{(N)}(t,x_1,\ldots,x_i^*,\ldots,x_s,x_{s+1}^*)-F_{s+1}^{(N)}(t,x_1,\ldots,x_i,\ldots,x_s,x_{s+1})\Big], \quad s\ge 1, \tag{2.11}$$

where $\mathcal{H}_s$ is the infinitesimal operator of the group of operators $S_s(-t)$, $-\infty<t<\infty$, calculated in sense of point -by-point convergence. Recall that $\mathcal{H}_s$ can be represented as follows on differentiable functions $f_s(x_1,\ldots,x_s)$

$$\mathcal{H}_s f_s(x_1,\ldots,x_s)=-\sum_{i=1}^{s} p_i\frac{\partial}{\partial q_i} f_s(x_1,\ldots,x_s) \tag{2.12}$$

with the boundary condition according to which at points $q_i=q_j$, $(i,j)\subset(1,\ldots,s)$, momenta p_i and p_j should be replaced by p_i^* and p_j^* in $\mathcal{H}_s$, if $\eta_{ij}\cdot(p_i-p_j)\ge 0$ and (p_i,p_j) do not change if $\eta_{ij}\cdot(p_i-p_j)\le 0$.

One can repeat above performed calculation with functional average for $\tilde{f}_N(t,x_1,\ldots$ $\ldots,x_N)$ (1.20) and obtain hierarchy (2.11) with the operator $\tilde{\mathcal{H}}_s$ instead of $\mathcal{H}_s$ and then, using the duality principle, replace $\tilde{\mathcal{H}}_s$ by $\mathcal{H}_s$.

Performing formally thermodynamic limit $N\to\infty$ in hierarchy (2.11) and supposing that limit correlation functions exist, one obtains the limit hierarchy

$$\frac{\partial F_s(t,x_1,\ldots,x_s)}{\partial t}=\mathcal{H}_s F_s(t,x_1,\ldots,x_s)+$$
$$+\sum_{i=1}^{s}\int dx_{s+1}\int\limits_{S_2^+} d\eta_{is+1} Q(\eta_{is+1}\cdot(p_i-p_{s+1}))\delta(q_i-q_{s+1})\times$$
$$\times\Big[F_{s+1}(t,x_1,\ldots,x_i^*,\ldots,x_{s+1}^*)-F_{s+1}(t,x_1,\ldots,x_i,\ldots,x_{s+1}^*)\Big], \tag{2.13}$$
$$F_s(t,x_1,\ldots,x_s)=\lim_{N\to\infty} F_s^{(N)}(t,x_1,\ldots,x_s), \quad s\ge 1,$$

with boundary condition in $\mathcal{H}_s$ and initial data

$$F_s(t, x_1, \dots, x_s)|_{t=0} = F_s(x_1, \dots, x_s). \tag{2.14}$$

Note that correlation functions depend on random vectors $\eta_{ij}, (i,j) \subset (1, \dots, s)$.

Hierarchy (2.13) *is known as the stochastic hierarchy* [1, 3].

Remark. It is easy to show that the hierarchy (2.13) can be also derived in the framework of great canonical ensemble [1, 3].

3. Solutions of the stochastic hierarchy. *3.1. Abstract form of the stochastic hierarchy.* Denote by $\mathcal{H}$ the direct sum of the infinitesimal operators $\mathcal{H}_s$

$$\mathcal{H} = \sum_{s=1}^{\infty} \oplus \mathcal{H}_s. \tag{3.1}$$

Denote by $\mathcal{A}$ the operator that acts on the sequence of correlation functions

$$F(t) = \big(F_1(t, x_1), \dots, F_s(t, x_1, \dots, x_s), \dots\big) \tag{3.2}$$

as follows:

$$(\mathcal{A}F(t))_s(x_1, \dots, x_s) =$$
$$= \sum_{i=1}^{s} \int dx_{s+1} \int_{S_2^+} d\eta_{is+1} Q(\eta_{is+1} \cdot (p_i - p_{s+1}))\delta(q_i - q_{s+1}) \times$$
$$\times \Big[F_{s+1}(t, x_1, \dots, x_i^*, \dots, x_s, x_{s+1}^*) - F_s(t, x_1, \dots, x_i, \dots, x_s, x_{s+1})\Big], \quad s \geq 1. \tag{3.3}$$

Then hierarchy (2.13) can be represented in the following abstract form:

$$\frac{dF(t)}{dt} = \mathcal{H}F(t) + \mathcal{A}F(t) \tag{3.4}$$

with initial data

$$F(t)|_{t=0} = F(0) = F. \tag{3.5}$$

Denote by $S(\pm t)$ the direct sum of the operators $S_s(\pm t)$

$$S(\pm t) = \sum_{s=1}^{\infty} \oplus S_s(\pm t) \tag{3.6}$$

then solutions of hierarchy (3.4) with initial data (3.5) can be represented by series of iterations

$$F(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n S(-t)S(t_1)\mathcal{A}S(-t_1)\dots S(t_n)\mathcal{A}S(-t_n)F(0) \tag{3.7}$$

or component wise

$$F_s(t, (x)_s) = \sum_{n=0}^{\infty} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n S_s(-t, (x)_s) S_s(t_1, (x)_s) \times$$
$$\times \sum_{i=1}^{s} \int dx_{s+1} \delta(q_i - q_{s+1}) \int_{S_2^+} d\eta_{is+1} Q(\eta_{is+1} \cdot (p_i - p_{s+1})) \times$$

$$\times\Big[S_{s+1}(-t_1,(x)^*_{s+1})-S_{s+1}(-t_1,(x)_{s+1})\Big]\dots$$

$$\dots S_{s+n-1}(t_{n-1},(x)_{s+n-1})\sum_{i=1}^{s+n-1}\int dx_{s+n}\delta(q_i-q_{s+n})\times$$

$$\times\int_{S_2^+} d\eta_{is+n}Q(\eta_{is+n}\cdot(p_i-p_{s+n}))\times$$

$$\times\Big[S_{s+n}(-t_n,(x)^*_{s+n})-S_{s+n}(-t_n,(x)_{s+n})\Big]F_{s+n}(0,(x)_{s+n}) \tag{3.8}$$

where $(x)^*_{s+n}=(x_1,\dots,x_i^*,\dots,x_s,x^*_{s+n})$ in i-th term and $S_s(\pm t,(x)_s)$ is operator of shift along the trajectory $X(\pm t,(x)_s)$ of s particles with initial data $(x)_s$ at $t=0$.

3.2. Chaos property. It has been shown in our papers [3, 4] (see also book [8]) that if phase points $(x)_s$ are outside the hypersurfaces V_{ij} where s particles interact (on hypersurfaces V_{ij} vectors q_i-q_j is parallel to vectors $p_i-p_j,(ij)\subset(1,\dots,s)$) then all the operators $S_{s+i}(\pm t,(x)_{s+i})$ in (3.7) should be replaced by the operators of free evolution $S^0_{s+i}(\pm t,(x)_{s+i}),\ 0\le i\le n$. If the initial correlation functions satisfy the chaos property

$$F_s(0,x_1,\dots,x_s)=F_1(0,x_1)\dots F_1(0,x_s) \tag{3.9}$$

then all correlation functions $F_s(t,x_1,\dots,x_s)$ outside of all V_{ij} satisfy the chaos property

$$F_s(t,x_1,\dots,x_s)=F_1(t,x_1)\dots F_1(t,x_s) \tag{3.10}$$

and one-particle correlation function satisfies nonlinear Boltzmann equation

$$\frac{\partial F_1(t,x_1)}{\partial t}=-p_1\frac{\partial}{\partial q_1}F_1(t,x_1)+\int dx_2\delta(q_1-q_2)\int_{S_2^+}d\eta_{12}Q(\eta_{12}\cdot(p_1-p_2))\times$$

$$\times\Big[F_1(t,x_1^*)F_1(t,x_2^*)-F_1(t,x_1)F_1(t,x_2)\Big] \tag{3.11}$$

with initial condition $F_1(t,x_1)|_{t=0}=F_1(0,x_1)$.

Note that the corresponding proof of the above formulated assertion has been performed for the differential scattering crosssection $\eta_{ij}\cdot(p_i-p_j)$ of hard spheres but it can be repeated word by word for the stochastic dynamics with arbitrary crosssection $Q(\eta_{ij}\cdot(p_i-p_j))$, because the hypersurfaces V_{ij} where stochastic particles interact do not depend on form of differential scattering crosssection.

In order do prove existence of solutions of hierarchy (3.4) represented by series (3.7), (3.8) one needs to impose some restriction on differential scattering crosssection $Q(\eta_{ij}\cdot(p_i-p_j))$. If

$$Q(\eta_{ij}\cdot(p_i-p_j))\le|p_i-p_j|,\quad \eta_{ij}\in S_2^+, \tag{3.12}$$

then all results obtained for hard spheres also hold for hierarchies with the crosssection $Q(\eta_{ij}\cdot(p_i-p_j))$.

Namely, series (3.8) *is uniformly convergent with respect to* $(x)_s$ *on compacts and with respect to time on finite interval* $[-t_0,t_0]$ *if sequence of initial functions* $F(0)$ *belong to the space* E_ξ *with norm*

$$\|F(0)\| = \sup_{s\geq 1} \frac{1}{\xi^s} \sup_{(x)_s} e^{\beta \sum_{i=1}^{s} p_i^2} |F_s(0,(x)_s)|, \qquad \xi > 0, \quad \beta > 0, \tag{3.13}$$

the number t_0 depends on $Q,\ \xi,\ \beta,\ \xi > 0,\ \beta > 0$ [1, 3, 4, 8, 9].

Series (3.8) *is uniformly convergent with respect to* $(x)_s$ *on compacts and with respect to time on arbitrary interval, i.e., globally in time, if sequence of initial functions are exponentially decreasing with respect to squared momenta and coordinates (see for detail* [8, 9]).

The correlation functions $F_s(t, x_1, \dots, x_s)$, $s \geq 1$, are defined in the entire phase space of s particles as usual (not generalized) functions represented by series (3.8) and are discontinuous on hyperplanes V_{ij} where s particles interact. (Outside V_{ij} all operators $S_{s+i}(\pm t, (x)_{s+i})$ should be replaced by $S^0_{s+i}(\pm t, (x)_{s+i})$ and series (3.8) is uniformly convergent for $t \subset [-t_0, t_0]$ and for $(x)_s$ from compacts for $F(0) \in E_\xi$, or globally in time for $F(0)$ exponentially decreasing with respect to squared momenta and coordinates).

In series (3.8) one can perform the integration with respect to $(q_{s+1}, \dots, q_{s+n})$ using δ-functions. In the integration with respect to $(p_{s+1}, \dots, p_{s+n})$ one can neglect the hypersurfaces V_{ij} of lower dimension where particles with numbers $(s+1, \dots, s+n)$ interact with themselves and with particles with numbers $(1, \dots, s)$, because it is the usual Lebesque integrals. One obtains the following representation for $F_s(t, (x)_s)$:

$$F_s(t,(x)_s) = \sum_{n=0}^{\infty} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n S_s\Big(-t,(x)_s) S_s(t_1,(x)_s\Big) \times$$

$$\times \sum_{i=1}^{s} \int dp_{s+1} \int_{S_2^+} d\eta_{is+1} Q(\eta_{is+1} \cdot (p_i - p_{s+1})) \times$$

$$\times \Big[S_s(-t_1) S_1^0(-t_1)((x)^*_{s+1}) - S_s(-t_1) S_1^0(-t_1)((x)_{s+1})\Big]\Big|_{q_{s+1}=q_i} \dots$$

$$\dots S_s(t_{n-1}) S^0_{n-1}(t_{n-1})((x)_{s+n-1}) \times$$

$$\times \sum_{i=1}^{s+n-1} \int dp_{s+n} \int_{S_2^+} d\eta_{is+n} Q(\eta_{is+n} \cdot (p_i - p_{s+n})) \times$$

$$\times \Big[S_s(-t_n) S_n^0(-t_n)((x)^*_{s+n}) - S_s(-t_n) S_n^0(-t_n)((x)_{s+n})\Big] F_{s+n}(0,(x)_{s+n})\Big|_{q_i = q_{s+n}} \tag{3.8$'$}$$

where the operators $S_s(-t_i)$ depend on phase points with numbers $1, \dots, s$ and operators $S^0_{s+i}(-t_i)$ depend on phase points with numbers $s+1, \dots, s+i$.

To define the functional average with observable $\varphi_s(x_1, \dots, x_s)$ according to the principle of duality we need the correlation functions $\bar{F}_s(t, x_1, \dots, x_s)$ (see (2.9)). The functions $\bar{F}_s(t, x_1, \dots, x_s)$ can be obtained from series (3.8$'$) by replacement of operators $S_{s+i}(\pm t_i) S^0(\pm t_i)((x)_{s+i})$ by operators $\bar{S}_s(\pm t_i) S^0(\pm t_i)((x)_{s+i})$.

3.3. Spatially homogeneous initial data. Consider initial data $F(0)$ with initial correlation functions independent on positions, i.e.,

$$F_s(0, x_1, \dots, x_s) = F_s(0, p_1, \dots, p_s), \quad s \geq 1. \tag{3.14}$$

Outside the hypersurfaces V_{ij}, with $S^0_{s+i}(\pm t, (x)_{s+i})$ instead of $S_{s+i}(\pm t, (x)_{s+i})$ (see (3.8$'$)), the correlation functions $F_s(t, (x)_s)$ also do not depend on position, i.e.,

$$F(t, x_1, \dots, x_s) = F(t, p_1, \dots, p_s) \tag{3.15}$$

because

$$S^0_{s+i}(\pm t, (x)_{s+i}) F_{s+i}(0, p_1, \dots, p_{s+i}) = F_{s+i}(0, p_1, \dots, p_{s+i}).$$

Further, if initial correlation functions have the chaos property

$$F_s(0, p_1, \dots, p_s) = F_1(0, p_1) \dots F_1(0, p_s), \quad s \geq 1, \tag{3.16}$$

then $F_s(t, x_1, \dots, x_s)$ have the chaos property (3.10) outside the hypersupfaces V_{ij} and $F_1(t, x_1)$ satisfies the Boltzmann equation (3.11) with initial condition $F_1(t, p_1)|_{t=0} = F_1(0, p_1)$.

Therefore solution of the Boltzmann equation (3.11) does not depend on position $F_1(t, x_1) = F_1(t, p_1)$ and satisfy the spatially homogeneous equation

$$\frac{\partial F_1(t, p_1)}{\partial t} = \int dp_2 \int\limits_{S_2^+} d\eta_{12} Q(\eta_{12} \cdot (p_1 - p_2)) \Big[F_1(t, p_1^*) F_1(t, p_2^*) - F_1(t, p_1) F_1(t, p_2) \Big]. \tag{3.17}$$

Thus using the stochastic dynamics and the stochastic Boltzmann hierarchy we derived the spatially inhomogeneous (3.11) *and spatially homogeneous* (3.17) *Boltzmann equations without mean-field approximation.*

Recall that M.Kac [11, 12] proposed certain Markov process, in the momentum space of N particles, defined through the corresponding Kolmogorov equation in mean field approximation. He derived the hierarchy for the sequence of correlation functions and proved that in the thermodynamic limit as $N \to \infty$ solutions of hierarchy have chaos property if initial data have this property. This means that correlation function at arbitrary time, from interval where solutions of hierarchy exist, are product of one-particle correlation functions which satisfies the spatially homogeneous Boltzmann equation [12, 13].

In the next section we will show that the Kac's Markov process in momentum space and the corresponding Kolmogorov equation can be obtained from our stochastic process in the entire phase space by mean of some specific averaging procedure in the space of position.

4. Stochastic process in momentum space. ***4.1. Averaging procedure in spatially homogeneous case.*** Consider the functional average (1.4) in spatially homogeneous case when the functions f_N and φ_N do not depend on positions

$$\begin{gathered} f_N(q_1, p_1, \dots, q_N, p_N) = f_N(p_1, \dots, p_N), \\ \varphi_N(q_1, p_1, \dots, q_N, p_N) = \varphi_N(p_1, \dots, p_N), \\ f_N(q_1 - p_1 \Delta t, p_1, \dots, q_N - p_N t, p_N) = f_N(p_1, \dots, p_N), \\ f_N(q_1 - p_1 \Delta t, p_1, \dots, q_i - p_i \tau - p_i^* (\Delta t - \tau), \\ p_i^*, \dots, q_j - p_j \tau - p_j^* (\Delta t - \tau), p_j^*, \dots, q_N - p_N \Delta t, p_N) = \\ = f_N(p_1, \dots, p_i^*, \dots, p_j^*, \dots, p_N). \end{gathered} \tag{4.1}$$

For spatially homogeneous f_N and φ_N the functional average $(S_N(-\Delta t) f_N, \varphi_N)$ (1.4) is divergent, the first term and the second term are proportional to V^N and V^{N-1} respectively. Instead of functional (1.4) we introduce the following functional:

$$\lim_{V\to\infty}\frac{1}{V^N}\int\limits_{\Lambda} dq_1\int dp_1\ldots\int\limits_{\Lambda} dq_N\int dp_N f_N(p_1,\ldots,p_N)\varphi_N(p_1,\ldots,p_N)+$$
$$+\lim_{V\to\infty}\frac{1}{V^{N-1}}\sum_{i<j=1}^{N}\int\limits_{\Lambda} dq_1\int dp_1\ldots\int\limits_{\Lambda_i} dq_i\int dp_i\ldots$$
$$\ldots\int\limits_{\Lambda_j} dq_j\int dp_j\ldots\int\limits_{\Lambda} dq_N\int dp_N\int\limits_0^{\Delta t} d\tau\int\limits_{S_2^+} d\eta_{ij}\times$$
$$\times Q(\eta_{ij}\cdot(p_i-p_j))\delta(q_i-p_i\tau-q_j+p_j\tau)\times$$
$$\times\Big[f_N(p_1,\ldots,p_i^*,\ldots,p_j^*,\ldots,p_N)-$$
$$-f_N(p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N)\Big]\varphi_N(p_1,\ldots,p_N)=$$
$$=\int dp_1\ldots dp_N f_N(p_1,\ldots,p_N)\varphi_N(p_1,\ldots,p_N)+$$
$$+\Delta t\sum_{i<j=1}^{N}\int dp_1\ldots dp_N\int\limits_{S_2^+} d\eta_{ij}Q(\eta_{ij}\cdot(p_i-p_j))\times$$
$$\times\Big[f_N(p_1,\ldots,p_i^*,\ldots,p_j^*,\ldots,p_N)-$$
$$-f_N(p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N)\Big]\varphi_N(p_1,\ldots,p_N)=$$
$$=\int dp_1\ldots dp_N S_N(-\Delta t)f_N(p_1,\ldots,p_N)\varphi_N(p_1,\ldots,p_N)=$$
$$=(S_N(-\Delta t)f_N,\varphi_N). \tag{4.2}$$

By Λ, Λ_i, and Λ_j are denoted the spheres with centers in the origin, in the points $-p_i\tau$ and $-p_j\tau$ respectively, and with the volume $V=V(\Lambda)=V(\Lambda_i)=V(\Lambda_j)$.

The functional $(S_N(-\Delta t)f_N,\varphi_N)$ was obtained from the functional (1.4) by averaging over configurational space (space of positions) and is average of the state $S_N(-\Delta t)\times\times f_N(p_1,\ldots,p_N)$ over the observable $\varphi_N(p_1,\ldots,p_N)$.

From (4.2) one obtains

$$S_N(-\Delta t)f_N(p_1,\ldots,p_N)=$$
$$=f_N(\Delta t,p_1,\ldots,p_N)=f_N(p_1,\ldots,p_N)+\Delta t\sum_{i<j=1}^{N}\int\limits_{S_2^+} d\eta_{ij}\Theta(\eta_{ij}\cdot(p_i-p_j))\times$$
$$\times\Big[f_N(p_1,\ldots,p_i^*,\ldots,p_j^*,\ldots,p_N)-f_N(p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N)\Big]. \tag{4.3}$$

Note that the operator $S_N(-\Delta t)$ (4.3) is defined on functions that depend only on momenta.

Formula (4.3) holds for infinitesimal Δt and it defines the operator of evolution $S_N(-\Delta t)$. For $N=2$ it is true for arbitrary $\Delta t>0$. For arbitrary $t>0$ we define formally the group of operators of evolution $S_N(-t)$ as follows:

$$S_N(-t)=\lim_{n\to\infty}\prod_{i=1}^{n}S_N(-\Delta t_i),\qquad \sum_{i=1}^{n}\Delta t_i=t \tag{4.4}$$

where $S_N(-\Delta t_i)$ are already defined according to (4.3).

The state $f_N(t+\Delta t, p_1, \ldots, p_N)$ is defined as follows through $f_N(t, p_1, \ldots, p_N)$:

$$f_N(t+\Delta t, p_1, \ldots, p_N) = S_N(-\Delta t) f_N(t, p_1, \ldots, p_N) =$$

$$= f_N(t, p_1, \ldots, p_N) + \Delta t \sum_{i<j=1}^{N} \int_{S_2^+} d\eta_{ij} Q(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times \Big[f_N(t, p_1, \ldots, p_i^*, \ldots, p_j^*, \ldots, p_N) - f_N(t, p_1, \ldots, p_i, \ldots, p_j, \ldots, p_N) \Big]. \quad (4.5)$$

The functional average $(f_N(t+\Delta t), \varphi_N)$ is defined by (4.2) if instead of $f_N(p_1, \cdots, p_N)$ one puts $f_N(t, p_1, \ldots, p_N)$.

4.2. Differential equation for $f_N(t, p_1, \ldots, p_N)$. If follows from (4.5) the following differential equation:

$$\frac{\partial f_N(t, p_1, \ldots, p_N)}{\partial t} = \sum_{i<j=1}^{N} \int_{S_2^+} d\eta_{ij} Q(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times \Big[f_N(t, p_1, \ldots, p_i^*, \ldots, p_j^*, \ldots, p_N) - f_N(t, p_1, \ldots, p_i, \ldots, p_j, \ldots, p_N) \Big] \quad (4.6)$$

with initial data

$$f_N(t, p_1, \ldots, p_N)|_{t=0} = f_N(0, p_1, \ldots, p_N) = f_N(p_1, \ldots, p_N).$$

It is the Kolmogorov equation for certain Markov process in momentum space.

We will also consider the modification of equation (4.6) in mean-field approximation, namely

$$\frac{\partial f_N(t, p_1, \ldots, p_N)}{\partial t} = \frac{1}{N} \sum_{i<j=1}^{N} \int_{S_2^+} d\eta_{ij} Q(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times \Big[f_N(t, p_1, \ldots, p_i^*, \ldots, p_j^*, \ldots, p_N) - f_N(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_N) \Big], \quad (4.7)$$

$$f_N(t, p_1, \ldots, p_N)|_{t=0} = f_N(p_1, \ldots, p_N).$$

Equation (4.7) can be obtained if one considers functional (4.2) in mean-field approximation with additional factor $\frac{1}{N}$.

4.3. Hierarchy for correlation functions in mean-field approximation. Define the sequence of correlation functions

$$F_s^{(N)}(t, p_1, \ldots, p_s) = \int dp_{s+1} \ldots dp_N f_N(t, p_1, \ldots, p_N), \quad s \geq 1, \quad (4.8)$$

where $f_N(t, p_1, \ldots, p_N)$ is solution of (4.7). It is easy to obtain the following hierarchy of integro-differential equations:

$$\frac{\partial F_s^{(N)}(t, p_1, \ldots, p_N)}{\partial t} = \frac{1}{N} \sum_{i<j=1}^{s} \int_{S_2^+} d\eta_{ij} Q_i(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times \Big[F_s^{(N)}(t, p_1, \ldots, p_i^*, \ldots, p_j^*, \ldots, p_s) - F_s^{(N)}(t, p_1, \ldots, p_i, \ldots, p_j, \ldots, p_s) \Big] +$$

$$+ \frac{N-s}{N} \sum_{i=1}^{s} \int dp_{s+1} \int d\eta_{is+1} Q_i(\eta_{is+1} \cdot (p_i - p_{s+1})) \times$$

$$\times\Big[F_{s+1}^{(N)}(t,p_1,\dots,p_i^*,\dots,p_s,p_{s+1}^*) -$$
$$-F_{s+1}^{(N)}(t,p_1,\dots,p_i,\dots,p_s,p_{s+1})\Big], \quad 1\le s\le N. \tag{4.9}$$

Performing formally thermodynamic limit $N\to\infty$ and supposing that limit correlation functions exist in some sense

$$\lim_{N\to\infty} F_s^{(N)}(t,p_1,\dots,p_N) = F_s(t,p_1,\dots,p_s), \quad s\ge 1,$$

one obtains the limit hierarchy

$$\frac{\partial F_s(t,p_1,\dots,p_s)}{\partial t} = \sum_{i=1}^{s}\int dp_{s+1}\int d\eta_{is+1} Q_i(\eta_{is+1}\cdot(p_i-p_{s+1}))\times$$
$$\times\Big[F_{s+1}(t,p_1,\dots,p_i^*,\dots,p_s,p_{s+1}^*) - F_{s+1}(t,p_1,\dots,p_i,\dots,p_s,p_{s+1})\Big], \quad s\ge 1, \tag{4.10}$$

with initial data

$$F_s(t,p_1,\dots,p_s)|_{t=0} = F_s(p_1,\dots,p_s).$$

For justification of existence of the thermodynamic limit see [12, 13]. Consider the initial correlation functions that have the chaos property

$$F_s(p_1,\dots,p_s) = F_1(p_1)\dots F_1(p_s).$$

Hierarchy (4.10) *admit the method of separation of variables because in the right-hand side of* (4.10) *we have sum of* s *operators acting on each* s *variables. As result we obtain that* $F_s(t,p_1,\dots,p_s)$ *also have the chaos property*

$$F_s(t,p_1,\dots,p_s) = F_1(t,p_1)\dots F_1(t,p_s)$$

and one-particle correlation function $F_1(t,p_s)$ *satisfy the spatially homogeneous Boltzmann equation*

$$\frac{\partial F_s(t,p_1)}{\partial t} =$$
$$= \int dp_2\int_{S_2^+} d\eta_{12} Q(\eta_{12}\cdot(p_1-p_2))\Big[F_1(t,p_1^*)F_1(t,p_2^*) - F_1(t,p_1)F_1(t,p_2)\Big]. \tag{4.11}$$

Thus we obtained the spatially homogeneous Boltzmann equation by using the stochastic dynamics in momentum space by averaging the stochastic dynamics in phase space and in mean-field approximation. In Section 3 we obtained the spatially inhomogeneous and homogeneous Boltzmann equation by using solutions of the stochastic Boltzmann hierarchy outside the hypersurfaces V_{ij} of lower dimension where point-wise particles interact *and without the mean-field approximation.*

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Received 23.12.2003