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THE JACOBI FIELD OF A LÉVY PROCESS

ПОЛЕ ЯКОБІ ПРОЦЕСУ ЛЕВІ

We derive an explicit formula for the Jacobi field that is acting in an extended Fock space and corresponds to an ($\mathbb{R}$-valued) Lévy process on a Riemannian manifold. The support of the measure of jumps in the Lévy – Khintchine representation for the Lévy process is supposed to have an infinite number of points. We characterize the gamma, Pascal, and Meixner processes as the only Lévy processe whose Jacobi field leaves the set of finite continuous elements of the extended Fock space invariant.

Виведено явну формулу для поля Якобі, що діє в розширеному фоківському просторі і відповідає деякому ($\mathbb{R}$-значному) процесу Леві на рімановому многовиді. Припускається, що міра стрибків у зображенні Леві – Хінчина для процесу Леві має носій з нескінченного числа точок. Гамма-, Паскаль- і Мейкснер-процеси характеризуються як такі, для яких відповідне поле Якобі залишає інваріантною множину фінітних неперервних елементів розширеного фоківського простору.

The aim of this notice is to derive an explicit formula for the Jacobi field [1 – 5] that is acting in an extended Fock space [6 – 10] and corresponds to an ($\mathbb{R}$-valued) Lévy process on a Riemannian manifold. The support of the measure of jumps in the Lévy – Khintchine representation for the process is supposed to have an infinite number of points. The proof of this formula will be based on a result of [10], see also [11, 12]. We will characterize the gamma, Pascal, and Meixner processes as the only Lévy processes whose Jacobi field leaves the set of finite continous elements of the extended Fock space invariant.

So, let X be a complete, connected, oriented C^∞ (noncompact) Reimannian manifold and let $\mathcal{B}(X)$ be the Borel σ-algebra on X. Let σ be a Radon measure on $(X, \mathcal{B}(X))$ that is nonatomic and nondegenerate (i. e., $\sigma(O) > 0$ for any open set $O \subset \subset X$). We denote by $\mathcal{D}$ the space $C_0^\infty(X)$ of all infinitely differentiable, real-valued functions on X with compact support. It is known that $\mathcal{D}$ can be endowed with a topology such that the natural embedding of $\mathcal{D}$ into the real L^2-space $L^2(X;\sigma)$ is dense, continuous, and nuclear. Thus, we can consider the standard nuclear triple $\mathcal{D}' \supset L^2(X;\sigma) \supset \mathcal{D}$, where $\mathcal{D}'$ is the dual space of $\mathcal{D}$ with respect to the zero space $L^2(X;\sigma)$. The dual pairing between $\omega \in \mathcal{D}'$ and $f \in \mathcal{D}$ will be denoted by $\langle \omega, f\rangle$. By $C(\mathcal{D}')$ we will denote the cylinder σ-algebra on $\mathcal{D}'$.

Let $\mathcal{R} := \mathbb{R} \setminus \{0\}$. We endow $\mathcal{R}$ with the relative topology of $\mathbb{R}$ and let $\mathcal{B}(\mathcal{R})$ be the Borel σ-algebra on $\mathcal{R}$. Let ν be a Radon measure on $(\mathcal{R}, \mathcal{B}(\mathcal{R}))$ whose support contains an infinite number of points. Let $\tilde{\nu}(ds) := s^2\nu(ds)$. We suppose that $\tilde{\nu}$ is a finite measure on $(\mathcal{R}, \mathcal{B}(\mathcal{R}))$, and furthermore, there exists $\varepsilon > 0$ such that

$$\int_{\mathcal{R}} \exp(\varepsilon|s|)\tilde{\nu}(ds) < \infty. \tag{1}$$

Therefore, the measure $\tilde{\nu}$ has all moments finite, and the set of all polynomials is dence in $L^2(\mathcal{R};\tilde{\nu})$.

We now define a centered Lévy process on X as a generalized process on $\mathcal{D}'$

whose law is the probability measure $\rho_{\nu,\sigma}$ on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$ given by its Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\rho_{\nu,\sigma}(d\omega) = \exp\left[\int_{\mathcal{R}\times X}\left(e^{is\varphi(x)} - 1 - is\varphi(x)\right)\nu(ds)\sigma(dx)\right], \quad \varphi\in\mathcal{D} \quad (2)$$

(compare with [13, 14]). The existence of $\rho_{\nu,\sigma}$ follows from the Bochner – Minlos theorem. Formula (2) is the Lévy – Khintchine representation for a Lévy process.

We will now construct a decomposition of the L^2-space $L^2(\mathcal{D}';\rho_{\nu,\sigma})$ following the idea of orthogonalization of continuous polynomials with respect to a probability measure that is defined on a co-nuclear space, cf. [15] (Sect. 11).

We denote by $\mathcal{P}(\mathcal{D}')$ the set of continuous polynomials on $\mathcal{D}'$, i. e., functions on $\mathcal{D}'$ of the form $F(\omega) = \sum_{i=0}^n\langle\omega^{\otimes i}, f_i\rangle$, $\omega^{\otimes 0} := 1$, $f_i\in\mathcal{D}^{\hat{\otimes} i}$, $i = 0,\ldots,n$, $n\in\mathbb{Z}_+$. Here, $\hat{\otimes}$ stands for symmetric tensor product. The greatest number i for which $f^{(i)}\neq 0$ is called the power of a polynomial. We denote by $\mathcal{P}_n(\mathcal{D}')$ the set of continuous polynomials of power $\le n$.

By (1), (2), and [15] (Sect. 10, Theorem 1) or [16] and [17],. $\mathcal{P}(\mathcal{D}')$ is a dense subset of $L^2(\mathcal{D}';\rho_{\nu,\sigma})$. Let $\mathcal{P}_n^\sim(\mathcal{D}')$ denote the closure of $\mathcal{P}_n(\mathcal{D}')$ in $L^2(\mathcal{D}';\rho_{\nu,\sigma})$, let $\mathbf{P}_n(\mathcal{D}')$, $n\in\mathbb{N}$, denote the orthogonal difference $\mathcal{P}_n^\sim(\mathcal{D}')\ominus\mathcal{P}_{n-1}^\sim(\mathcal{D}')$, and let $\mathbf{P}_0(\mathcal{D}') := \mathcal{P}_0^\sim(\mathcal{D}')$. Then, we evidently have:

$$L^2(\mathcal{D}';\rho_{\nu,\sigma}) = \bigoplus_{n=0}^\infty\mathbf{P}_n(\mathcal{D}'). \quad (3)$$

The set of all projections $:\langle\cdot^{\otimes n}, f_n\rangle:$ of continuous monomials $\langle\cdot^{\otimes n}, f_n\rangle$, $f_n\in\mathcal{D}^{\hat{\otimes} n}$, onto $\mathbf{P}_n(\mathcal{D}')$ is dense in $\mathbf{P}_n(\mathcal{D}')$. For each $n\in\mathbb{N}$, we define a Hilbert space $\mathfrak{F}_n$ as the closure of the set $\mathcal{D}^{\hat{\otimes} n}$ in the norm generated by the scalar product

$$(f_n, g_n)_{\mathfrak{F}_n} := \frac{1}{n!}\int_{\mathcal{D}'}:\langle\omega^{\otimes n}, f_n\rangle::\langle\omega^{\otimes n}, g_n\rangle:\rho_{\nu,\sigma}(d\omega), \quad f_n, g_n\in\mathcal{D}^{\hat{\otimes} n}. \quad (4)$$

Denote

$$\mathfrak{F} := \bigoplus_{n=0}^\infty\mathfrak{F}_n n!, \quad (5)$$

where $\mathfrak{F}_0 := \mathbb{R}$. By (3) – (5), we get the unitary operator $\mathcal{U}\colon\mathfrak{F}\to L^2(\mathcal{D}';\rho_{\nu,\sigma})$ that is defined through $\mathcal{U}f_n := :\langle\cdot^{\otimes n}, f_n\rangle:$, $f_n\in\mathcal{D}^{\hat{\otimes} n}$, $n\in\mathbb{Z}_+$, and then extended by linearity and continuity to the whole space $\mathfrak{F}$.

We will now write down an explicit formula for the scalar product $(\cdot,\cdot)_{\mathfrak{F}_n}$. In the case of the gamma process, this formula is due to [7] and [6], in the case of the Pascal and Meixner process due to [9] and [18], and in the case of a general Lévy process due to [10].

We denote by $\mathbb{Z}_{+,0}^\infty$ the set of all sequences α of the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$, $\alpha_i\in\mathbb{Z}_+$, $n\in\mathbb{N}$. Let $|\alpha| := \sum_{i=1}^\infty\alpha_i$. Let for each $\alpha\in\mathbb{Z}_{+,0}^\infty$, $1\alpha_1 +$

$+2\alpha_2+\ldots=n$, $n\in\mathbb{N}$, and for any function $f_n\colon X^n\to\mathbb{R}$ we define a function $D_\alpha f_n\colon X^{|\alpha|}\to\mathbb{R}$ by setting

$$(D_\alpha f_n)(x_1,\ldots,x_{|\alpha|}) := f(x_1,\ldots,x_{\alpha_1},\underbrace{x_{\alpha_1+1},x_{\alpha_1+1}}_{2\text{ times}},$$

$$\underbrace{x_{\alpha_1+2},x_{\alpha_1+2}}_{2\text{ times}},\ldots,\underbrace{x_{\alpha_1+\alpha_2},x_{\alpha_1+\alpha_2}}_{2\text{ times}},\underbrace{x_{\alpha_1+\alpha_2+1},x_{\alpha_1+\alpha_2+1},x_{\alpha_1+\alpha_2+1}}_{3\text{ times}},\ldots). \tag{6}$$

Let $(P_n(\cdot))_{n=0}^\infty$ be the system of polynomials with leading coefficient 1 that are orthogonal with respect to the measure $\tilde{\nu}(ds)$ on $\mathcal{R}$. We have, for any $f_n,\ g_n\in\mathcal{D}^{\hat{\otimes}n}$, $n\in\mathbb{N}$,

$$(f_n,g_n)_{\mathfrak{F}_n} = \sum_{\alpha\in\mathbb{Z}_{+,0}^\infty:\ 1\alpha_1+2\alpha_2+\ldots=n} K_\alpha \int_{X^{|\alpha|}} (D_\alpha f_n)(x_1,\ldots,x_{|\alpha|})\times$$

$$\times\ (D_\alpha g_n)(x_1,\ldots,x_{|\alpha|})\,\sigma^{\otimes|\alpha|}(dx_1,\ldots,dx_{|\alpha|}), \tag{7}$$

where

$$K_\alpha = \frac{n!}{\alpha_1!\alpha_2!\cdots}\prod_{k\geq1}\left(\frac{\|P_{k-1}\|_{L^2(\mathcal{R};\tilde{\nu})}}{k!}\right)^{2\alpha_k}. \tag{8}$$

By using (7), (8), one derives the following representation of $\mathfrak{F}_n$ (see [10], formula (5.19)):

$$\mathfrak{F}_n := \bigoplus_{\alpha\in\mathbb{Z}_{+,0}^\infty:\ 1\alpha_1+2\alpha_2+\ldots=n}\mathfrak{F}_{n,\alpha},\qquad \mathfrak{F}_{n,\alpha} := L_\alpha^2\big(X^{|\alpha|};\sigma^{\otimes|\alpha|}\big)K_\alpha. \tag{9}$$

Here,

$$L_\alpha^2\big(X^{|\alpha|};\sigma^{\otimes|\alpha|}\big) = L^2(X;\sigma)^{\hat{\otimes}\alpha_1}\otimes L^2(X;\sigma)^{\hat{\otimes}\alpha_2}\otimes\cdots,$$

and for each $f_n\in\mathcal{D}^{\hat{\otimes}n}\subset\mathfrak{F}_n$, the $\mathfrak{F}_{n,\alpha}$-coordinate of f_n is equal to $D_\alpha f_n$. Thus, we can extend D_α by continuity to the orthogonal projection of $\mathfrak{F}_n$ onto $\mathfrak{F}_{n,\alpha}$. In what follows, we will also denote by S_α the orthogonal projection of $L^2(X^{|\alpha|};\sigma^{\otimes|\alpha|})$ onto $L_\alpha^2(X^{|\alpha|};\sigma^{\otimes|\alpha|})$. Taking (9) into account, we will call $\mathfrak{F}$ an extended Fock space (compare with [7, 8]).

For an arbitrary $\varphi\in\mathcal{D}$, we consider in the space $L^2(\mathcal{D}';\rho_{\nu,\sigma})$ the operator $M(\varphi)$ of multiplication by the function $\langle\cdot,\varphi\rangle$, and let $J(\varphi) := \mathcal{U}M(\varphi)\mathcal{U}^{-1}$. We denote by $\mathfrak{F}_{\text{fin}}(\mathcal{D})$ the set of all vectors of the form $(f_0,f_1,\ldots,f_n,0,0,\ldots)$, $f_i\in\mathcal{D}^{\hat{\otimes}i}$, $i=0,\ldots,n$, $n\in\mathbb{Z}_+$. Evidently, $\mathfrak{F}_{\text{fin}}(\mathcal{D})$ is a dense subset of $\mathfrak{F}$.

Theorem 1. *For any $\varphi\in\mathcal{D}$, we have:*

$$\mathfrak{F}_{\text{fin}}(\mathcal{D})\subset\operatorname{Dom}(J(\varphi)),\qquad J(\varphi)\restriction\mathfrak{F}_{\text{fin}}(\mathcal{D}) = J^+(\varphi)+J^0(\varphi)+J^-(\varphi), \tag{10}$$

the linear operators $J^+(\varphi)$, $J^0(\varphi)$, $J^-(\varphi)$ being defined as follows: for any $f_n\in\mathcal{D}^{\hat{\otimes}n}$, $n\in\mathbb{Z}_+$,

$$J^+(\varphi)f_n = \varphi\,\hat{\otimes}\,f_n, \tag{11}$$

$J^0(\varphi)f_n \in \mathfrak{F}_n$ *and each* $\mathfrak{F}_{n,\alpha}$*-coordinate of* $J^0(\varphi)f_n$ *is equal to*

$$\left(J^0(\varphi)f_n\right)_\alpha(x_1,\ldots,x_{|\alpha|}) =$$

$$= \sum_{k=1}^{\infty} \alpha_k a_{k-1} S_\alpha\Big(\varphi(x_{\alpha_1+\ldots+\alpha_k})\big(D_\alpha f_n\big)(x_1,\ldots,x_{|\alpha|})\Big) \quad \sigma^{\otimes|\alpha|}\text{-a. e.,} \tag{12}$$

$J^-(\varphi)f_n = 0$ *if* $n = 0$, $J^-(\varphi)f_n \in \mathfrak{F}_{n-1}$ *if* $n \in \mathbb{N}$, *and each* $\mathfrak{F}_{n-1,\alpha}$*-coordinate of* $J^-(\varphi)f_n$ *is equal to*

$$\left(J^-(\varphi)f_n\right)_\alpha(x_1,\ldots,x_{|\alpha|}) = n\tilde{\nu}(\mathcal{R})S_\alpha\left(\int_X \varphi(x)\big(D_{\alpha+1_1}f_n\big)(x,x_1,\ldots,x_{|\alpha|})\sigma(dx)\right) +$$

$$+ \sum_{k\ge 2} \frac{n}{k}\alpha_{k-1}b_{k-1}S_\alpha\Big(\varphi(x_{\alpha_1+\ldots+\alpha_k})\big(D_{\alpha-1_{k-1}+1_k}f_n\big)(x_1,\ldots,x_{|\alpha|})\Big) \quad \sigma^{\otimes|\alpha|}\text{-a. e.} \tag{13}$$

In formulas (12) *and* (13), *we denoted*

$$\alpha \pm 1_n := (\alpha_1,\ldots,\alpha_{n-1},\alpha_n \pm 1,\alpha_{n+1},\ldots)$$

for $\alpha \in \mathbb{Z}_{+,0}^{\infty}$ *and* $n \in \mathbb{N}$; *the real numbers* a_n *and positive numbers* b_n *are given through the recurrence relation*

$$sP_n(s) = P_{n+1}(s) + a_nP_n(s) + b_nP_{n-1}(s), \quad n \in \mathbb{Z}_+, \quad P_{-1}(s) := 0. \tag{14}$$

Finally, $J(\varphi)$ *is essentially self-adjoint on* $\mathfrak{F}_{\mathrm{fin}}(\mathcal{D})$.

By (10), the operator $J(\varphi) \restriction \mathfrak{F}_{\mathrm{fin}}(\mathcal{D})$ is a sum of creation, neutral, and annihilation operators, and hence $J(\varphi) \restriction \mathfrak{F}_{\mathrm{fin}}(\mathcal{D})$ has a Jacobi operator structure. The family of operators $(J(\varphi))_{\varphi\in\mathcal{D}}$ is called the Jacobi field corresponding to the Lévy process with law $\rho_{\nu,\sigma}$.

The proof of the representation (10) follows from [10] (Theorem 5.1, Corollaries 4.2 and 5.1). The essential self-adjointness of $J(\varphi)$ on $\mathfrak{F}_{\mathrm{fin}}(\mathcal{D})$ follows from (10), (11) and [4] (Theorem 4.1) whose proof admits a direct generalization to the case of the extended Fock space $\mathfrak{F}$.

We notice that the operator $J^+(\varphi)$ leaves the set $\mathfrak{F}_{\mathrm{fin}}(\mathcal{D})$ invariant, while the operators $J^0(\varphi)$ and $J^-(\varphi)$, in general do not.

Corollary 1. *Suppose that, for each* $\varphi \in \mathcal{D}$, *we have* $J^0(\varphi)\mathfrak{F}_{\mathrm{fin}}(\mathcal{D}) \subset \mathfrak{F}_{\mathrm{fin}}(\mathcal{D})$ *and* $J^-(\varphi)\mathfrak{F}_{\mathrm{fin}}(\mathcal{D}) \subset \mathfrak{F}_{\mathrm{fin}}(\mathcal{D})$, *so that* $J(\varphi)\mathfrak{F}_{\mathrm{fin}}(\mathcal{D}) \subset \mathfrak{F}_{\mathrm{fin}}(\mathcal{D})$. *Then,* $\tilde{\nu}$ *is a finite measure on* $\mathcal{R}$ *such that* $a_n = \lambda(n+1)$ *and* $b_n = \kappa n(n-1)$, $n \in \mathbb{Z}_+$. *Here,* a_n *and* b_n *are the coefficients from* (14), *and* $\lambda \in \mathbb{R}$ *and* $\kappa > 0$ *are arbitrarily chosen parameters. Furthermore, we have in this case, for each* $f_n \in \mathcal{D}^{\hat{\otimes} n}$, $n \in \mathbb{Z}_+$:

$$(J^0(\varphi)f_n)(x_1,\ldots,x_n) = \lambda n(\varphi(x_1)f_n(x_1,\ldots,x_n))^\wedge \quad \sigma^{\otimes n}\text{-a. e.,}$$

$$(J^-(\varphi)f_n)(x_1,\ldots,x_{n-1}) = n\tilde{\nu}(\mathcal{R})\int_{\mathcal{R}} \varphi(x)f_n(x,x_1,\ldots,x_{n-1})\sigma(dx) +$$

$$+ \kappa n(n-1)(\varphi(x_1)f_n(x_1,x_1,x_2,x_3,\ldots,x_{n-1}))^\wedge \quad \sigma^{\otimes(n-1)}\text{-a. e.}$$

Here, $(\cdot)^{\wedge}$ *denotes symmetrization of a function. The choice* $|\lambda| = 2$ *corresponds to a gamma process,* $|\lambda| > 2$ *corresponds to a Pascal process, and* $|\lambda| < 2$ *corresponds to a Meixner process.*

Corollary 1 is derived from Theorem 1 and [10] (Corollary 5.1), by using the idea that the off-diagonal values of a continuous function of several variables uniquely determine the on-diagonal values of this function.

The Jacobi fields of the gamma, Pascal, and Meixner processes (in the case $\tilde{\nu}(\mathcal{R}) = 1$ and $\kappa = 1$) were studied in [9, 10], see also [18, 7, 11, 12].

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