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W. I. Skrypnik (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

ON POLYMER EXPANSION FOR GIBBSIAN STATES OF NONEQUILIBRIUM SYSTEMS OF INTERACTING BROWNIAN OSCILLATORS

ПРО ПОЛІМЕРНИЙ РОЗКЛАД ДЛЯ ГІББСІВСЬКИХ СТАНІВ НЕРІВНОВАЖНИХ СИСТЕМ ВЗАЄМОДІЮЧИХ БРОУНІВСЬКИХ ОСЦИЛЯТОРІВ

Convergence of polymer cluster expansions for correlation functions of general Gibbs oscillator-type systems and related nonequilibrium systems of Brownian oscillators is established. The initial states for the latter are Gibbsian. It is proven that the sequence of the constructed correlation functions of the nonequilibrium system is a generalized solution of the diffusion BBGKY-type hierarchy.

Встановлено збіжність кластерних розкладів для кореляційних функцій загальних граткових гіббсівських систем осциляторного типу та споріднених нерівноважних систем броунівських осциляторів у термодинамічній границі. Початкові стани для останніх є гіббсівськими. Доведено, що послідовність побудованих функцій нерівноважної системи є узагальненим розв'язком дифузійної ієрархії типу ББГКІ.

1. Introduction. In this paper we generalize the results of our previous paper [1] devoted to the thermodynamic limit transition in Gibbs systems of oscillators interacting via ternary interaction potential. In [1] we showed that reduced density matrices (quantum correlation functions) of the quantum systems are expressed in terms of the correlation functions of Gibbs path oscillator systems with interaction determined by a pair complex potential, satisfying the Kunz condition [2] and worked out the conditions implying convergence of a high-temperature polymer expansion. Here the pair potential is more general.

In this paper we also perform the thermodynamic limit at high temperatures and a small time interval in lattice nonequilibrium systems of Brownian oscillators interacting via a pair (general) superstable potential whose states are described by the sequence of correlation functions. These systems can be reduced to Gibbs oscillator path lattice systems with a complex pair interaction potentials (the reduction is fulfilled in terms of correlation functions) whose real part satisfy the superstability and regularity condition [3] which is more general than the Kunz condition. This requires a generalization of the conditions leading to convergence of the polymer expansion.

We consider, as in [1], a general Gibbs system on a lattice $\mathbb{Z}^d$, whose sites label variables from the measure space (Ω, P_0), where Ω is a complete metric space, P_0 is a positive σ-finite measure, which is finite on compact sets, with the potential energy U, being the continuous function, expressed through the one-particle potential $u(\omega)$ and the two-particle complex-valued potential $u_{x-y}(\omega_x, \omega_y)$

$$U(\omega_\Lambda) = \sum_{x\in\Lambda} u(\omega_x) + \sum_{x,y\in\Lambda} u_{x-y}(\omega_x, \omega_y), \tag{1.1}$$

where Λ is a finite set with the cardinality $|\Lambda|$. We will require that

$$|\mathrm{Re}\, u_{x-y}(\omega_x, \omega_y)| \le \frac{1}{2} J(|x-y|)(v(\omega_x) + v(\omega_y)), \quad J_* = ||J||_1 < \infty, \quad J, v \ge 0, \tag{1.2}$$

$$e^{\kappa v} \in L^1(\Omega, P), \quad e^{-\beta u} \in L^1(\Omega, P_0) \quad P = e^{-\beta u} P_0, \quad \beta, \kappa > 0, \tag{1.3}$$

$$\begin{gathered} \mathrm{Im}\, u_{x-y}(\omega_x, \omega_y) = J_0(|x-y|)\phi(\omega_x, \omega_y), \\ ||\phi||_2 = \int |\phi(\omega, \omega')|^2 P(d\omega) P(d\omega') < \infty, \ \ ||J_0||_1 < \infty, \end{gathered} \tag{1.4}$$

where $|x|$ is the Euclidean norm of $x \in \mathbb{Z}^d$ and by $||F||_q$ we denote the norm of the Banach space $L^q(\mathbb{Z}^d)$.

The first condition is more general then the Kunz condition (on the right-hand side of (1.2) there is product of two square roots of the positive function v, depending on q_x, q_y, respectively, instead of their sum). Conditions (1.2), (1.3) imply the superstability and regularity conditions [3].

Gibbs correlation functions are given by

$$\begin{gathered} \rho^\Lambda(\omega_X) = Z_\Lambda^{-1} \int e^{-\beta U(\omega_\Lambda)} P_0(d\omega_{\Lambda \setminus X}), \\ Z_\Lambda = \int e^{-\beta U(\omega_\Lambda)} P_0(d\omega_\Lambda) > 0, \quad \beta \in \mathbb{R}^+. \end{gathered} \tag{1.5}$$

Here the integration is performed over $\Omega^{|\Lambda \setminus X|}$ and $\Omega^{|\Lambda|}$, respectively, $P_0(d\omega_X) = \prod_{x \in X} P_0(d\omega_x)$. Both potentials and P_0 may depend on the inverse temperature β.

The polymer high temperature expansion is given by

$$\rho_{\Lambda, X}(\omega_X) = \left(\int e^{-\beta u(\omega)} P_0(d\omega) \right)^{-|X|} e^{\beta \sum_{x \in \Lambda} u(\omega_x)} \rho^\Lambda(\omega_X) =$$

$$= \sum_{Y \in \Lambda \setminus X} \frac{Z_{\Lambda \setminus (X \cup Y)}}{Z_\Lambda} \int P(d\omega_Y) F_{\omega_X}(\omega_Y), \tag{1.6}$$

where $F_{\omega_X}(\omega_Y)$ are the truncated Boltzmann functions satisfying the Kirkwood–Saltsburg recursion relation (5.1) [1, 2]

$$P(d\omega) = \left(\int e^{-\beta u(\omega)} P_0(d\omega) \right)^{-1} e^{-\beta u(\omega)} P_0(d\omega).$$

The polymer expansion is derived with the help of the Ruelle [3] algebraic technique [2].

Polymer correlation functions

$$\bar{\rho}_\Lambda(X \cup Y) = \frac{Z_{\Lambda \setminus (X \cup Y)}}{Z_\Lambda}$$

satisfy the Kirkwood–Saltsburg (KS) polymer equation [1, 2].

Standard arguments [2, 3] (see Appendix and Proposition 2.1 in [3]) which demands only condition (1.2) yield the bound

$$\operatorname{ess}\sup_{X\in A}\sum_{Y\in A':|Y|=m}\int P(d\omega_Y)|F_{\omega_X}(\omega_Y)|\le e^{|X|}eB_{A,A'}(eB)^{m-1}e^{\beta\sum_{x\in X}\bar v(\omega_x)},\qquad \bar v\ge 0,\quad \beta>0, \tag{1.7}$$

where

$$B_{A,A'}=\operatorname{ess}\sup_{\omega\in\Omega,x\in A}\sum_{y\in A'}b_{x-y}(\omega),$$

$$b_x(\omega)=e^{-\beta(\bar v(\omega)-J_*v(\omega))}\int e^{\beta\bar v(\omega')}|e^{-\beta u_x(\omega,\omega')}-1|P(d\omega'),$$

$$B=B_{A,A'},\quad A=A'=\mathbb{Z}^d.$$

In order to establish convergence of the polymer expansion in the thermodynamic limit, i.e. $\Lambda\to\mathbb{Z}^d$, it is necessary to prove that [2, 3]

$$\lim_{\beta\to 0}B=0,\qquad\qquad \lim_{\beta\to 0}BD=0, \tag{1.8}$$

where

$$D=\int e^{\beta\bar v(\omega)}P(d\omega),$$

(1.7) implies that there exists $\bar\rho(X)$ such that (see [1, 2])

$$\bar\rho(X),\bar\rho_\Lambda(X)\le M\xi^{|X|},\quad |\bar\rho_\Lambda(X)-\bar\rho(X)|\le\xi^{|X|}\varepsilon_0(\lambda),\quad \xi=e^{-1}(B+\sqrt{BD})^{-1}, \tag{1.9}$$

where $A\subset\Lambda$ and λ is the distance of A from $\Lambda^c=\mathbb{Z}^d\backslash\Lambda$ and ε is a decreasing at infinity function.

From (1.8), (1.9) it follows that $eB\xi<1, e\xi>1$ for sufficiently small β. In simplest cases D is non-zero and finite, but in a general case this parameter may tend to infinity at the zero temperature.

In [3] we derived (1.7) with

$$\bar v(\omega)=J_*\gamma v(\omega)+\beta^{-1}\ln(1+b(\omega)),\qquad b^2(\omega)=\int|\phi(\omega,\omega')|^2P(d\omega'),\qquad \gamma>3,$$

and proved (1.8) if the real part of the complex pair potential satisfy the Kunz condition with the function v. This was done with the help of the Kunz-type estimate of B. But the Kunz condition together with (1.3) are quite restricting and do not allow to consider the case, important for our paper, of the following real-valued potential energy for classical and quantum systems

$$U(q_\Lambda)=\sum_{x\in\Lambda}(\partial_x u^0(q_x))^2+\sum_{x,y\in\Lambda}J_0(|x-y|)\varphi(q_x,q_y),\qquad \partial_x=\frac{\partial}{\partial q_x},$$

$$2\varphi(q_x,q_y)=(\partial_x u^0(q_x))\partial_x u^0(q_x,q_y)+(\partial_y u^0(q_y))\partial_y u^0(q_y,q_x),\quad q_x,q_y\in\mathbb{R}.$$

The obvious inequality $2ab \le a^2+b^2$, $a = \partial_x u^0(q_x)$, $b = \partial_x u^0(q_x, q_y)$, cannot yield the condition (1.3) even if the derivatives of the pair potential $u^0(q, q')$ satisfy (1.2). But we will show that a more refined estimate yields (1.2), (1.3) (see Proposition 4.1). Using (1.2), instead of the Kunz condition, one is obliged to generalize the first term in the function $\bar{v}$ in order to have (1.8). For such the generalization we find conditions, leading to (1.8), in the following proposition.

Proposition 1.1. *Let* (1.2) – (1.4) *hold,*

$$\bar{v}(\omega) = \gamma(\omega) + \beta^{-1}\ln(1 + b(\omega)), \qquad \gamma(\omega) > 0,$$

and $||e^{\beta\gamma}||_2$, $||e^{\beta J_* v}||_2$, $||e^{-\beta(\gamma - J_* v)}||_\infty$ *be bounded at* $\beta = 0$. *Then there exists a positive constant* $\bar{B}$ *such that* $B_{A,A'} \le J(A, A')\bar{B}$, *where* $J(A, A') = \operatorname{ess\,sup}_{x\in A} \sum_{t \in A'} J(|x - -y|)$, $B \le ||J||_1 \bar{B}$, *and* (1.8) *holds if*

$$\lim_{\beta \to 0} \beta(1 + ||\phi||_2)^2 |||ve^{-\beta[\gamma - 2J_* v]}||_\infty = 0, \quad \lim_{\beta\to 0} \beta(1 + ||\phi||_2)^2 ||ve^{\beta J_* v}||_2 = 0. \tag{1.10}$$

It is not difficult to check that in the case of $\phi = 0$ and classical system ($\Omega = \mathbb{R}$, P_0 is the Lebesque measure) the conditions of the proposition and inequalities in (1.10) are satisfied if $n > m$, where $2n, 2m$ are the degrees of the positive polynomials $u(q), v(q)$, respectively (it is necessary to rescale the variable $q \to \beta^{-\frac{1}{2n}} q$ in the above norms). The similar condition is required when the Kunz condition is used.

In formulating our main theorem we will use the Banach space $\mathbb{B}_{\xi;-\beta\bar{v}}$ of sequences $F = \{F_X, X \in \mathbb{Z}^d\}$ of function $F_X(\omega_X)$ with the norm

$$||F||_{\xi;-\beta\bar{v}} = \operatorname{ess}\sup_{|X|,\omega_X} \xi^{-|X|} \exp\left\{-\beta \sum_{x\in X} \bar{v}(\omega_x)\right\} |F_X(\omega_X)|.$$

We will say that $f_\Lambda \in \mathbb{B}_{\xi;-\beta\bar{v}}$, $\Lambda \subset \mathbb{Z}^d$, locally converges to f in $\mathbb{B}_{\xi;-\beta\bar{v}}$ if for every bounded A there exists a decreasing at infinity function ε such that

$$||\chi_A(f - f_\Lambda)||_{\xi;-\beta\bar{v}} \le \varepsilon(\lambda),$$

where χ_A is the operator of multiplication by the characteristic function of A and $\lambda = \operatorname{dist}\{A, \Lambda^c\}$, i. e. λ is the distance of A from $\Lambda^c = \mathbb{Z}^d \backslash \Lambda$.

A choice of $\bar{v}$ in (1.7) should yield the analog of the Ruelle superstability bound [3] for the correlation functions in (1.5) at high temperatures. This choice is quite obvious and is written down in the following theorem.

Theorem 1.1. *Let the conditions of Proposition* 1.1 *be satisfied*

$$\gamma(\omega) = \varepsilon u(\omega), \quad \varepsilon < \frac{1}{2},$$

and for sufficiently small β *the inequality* $||e^{-\beta(u - J_* v)}||_\infty \le e$ *holds.*

Let, also, the conditions (1.2) – (1.4), (1.10) *be satisfied. Then for sufficiently small* β $\bar{\rho}_\Lambda$ *converges uniformly on compact sets to* $\bar{\rho}(X)$ *and relations* (1.8), (1.9) *hold.*

Moreover, the sequence $\rho_\Lambda = \{\rho_{\Lambda,X}, X \subset \mathbb{Z}^d\}$ *in* (1.6) *locally converges in* $\mathbb{B}_{e\xi;-\beta\bar{v}}$ *to the sequence* $\rho = \{\rho_X, X \subset \mathbb{Z}^d\} \in \mathbb{B}_{e\xi;-\beta\bar{v}}$, *given by*

$$\rho_X(\omega_X) = \sum_{Y \subset X^c} \bar{\rho}(X \cup Y) \int P(d\omega_Y) F_{\omega_X}(\omega_Y), \tag{1.11}$$

$F_{\omega_X}(\omega_Y)$ *satisfy the KS-recursion relation* (5.1) *and* $||\rho||_{e\xi;-\beta\bar{v}} \le M(1-\xi e B)^{-1}$.

This theorem generalizes Theorem 1.1 from [3]. The only difference in the proof is that one has to demand that $||e^{-\beta(u-J_*v)}||_\infty \le e$ for sufficiently small inverse temperatures. This condition is necessary for the proof of (1.7) (see Appendix) and is obvious for classical systems.

Let $A \in S_r$, where S_r is the sphere of the radius r centered at the origin (we imbed $\mathbb{Z}^d$ in $\mathbb{R}^d$). When one estimates the difference of the expressions in (1.11) and (1.6) it is necessary to estimate the sums over the sets Λ^c, $S_{\frac{1}{2}\lambda+r}$, $\Lambda \backslash S_{\frac{1}{2}\lambda+r}$. The sums over the first and third sets are estimated with the help of (1.7), the first inequality in (1.9) and the inequality

$$J(A, \Lambda^c) \le \sum_{|y| \ge \lambda} J(|y|) = \varepsilon^0(\lambda).$$

This inequality becomes obvious if one makes a translation of the set Λ in the sum in such a way that the set A after this translation touches the origin in the point at the distance dist $\{A, \Lambda^c\}$ from Λ^c. Here one uses the translation invariance $J(A + x, A' + +x) = J(A, A')$. Making estimates of the sum over Y in the expression for the polymer expansion one has to bound it by the sum over m and the sum over subsets Y such that $|Y| = m$ and then apply (1.7). The second sum in this way is estimated with the help of (1.7) and the second inequality in (1.9). As a result the function ε will also contain the three terms:

$$\varepsilon(\lambda) = 2(1 - eB\xi)^{-1} \left[\varepsilon^0(\lambda) + \varepsilon^0\left(\frac{\lambda}{2}\right) + \varepsilon_0\left(\frac{\lambda}{2}\right) \right], \qquad eB\xi < 1,$$

where we used the inequality dist $\{A, S^c_{r+r'}\} \le r'$.

Our main application of Theorem 1.1 concerns systems with $\Omega = \mathbb{R} \times \Omega_0 \times \Omega_0 = \mathbb{R} \times \times \Omega_0^2$, where Ω_0 is the probability space of the Wiener measure $P_q(dw)$, concentrated on paths starting from q, and the conditional loop Wiener measure $P^\beta_{q,q}(dw)$, concentrated on paths starting from q and arriving at q at the time β. The measures P_0 in these cases are given by

$$P_0(d\omega) = dq P_q(dw) P_0(dw^*), \tag{1.12}$$

$$P_0(d\omega) = dq P^\beta_{q,q}(dw) P_0(dw^*), \tag{1.13}$$

respectively.

In both cases the potential ϕ is given in terms of the stochastic integrals

$$\phi(\omega, \omega') = \frac{1}{4} \left[\int_0^{t'} dw'^*(\tau) \partial u^0(w(\tau), w'(\tau)) + \int_0^{t'} dw'^*(\tau) \partial' u^0(w(\tau), w'(\tau)) \right], \tag{1.14}$$

where $\omega = (q, w, w^*)$, $\partial(\partial')$ is the derivative in $w(w')$. The real part of the interaction potential will not depend on w^*.

Using the formula

$$\int \left(\int_0^{t'} f(\tau) dw^*(\tau) \right)^2 P_0(dw^*) = \int_0^{t'} f^2(\tau) d\tau$$

and the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$ we obtain

$$\int P_0(dw^*) P_0(dw'^*) \phi^2(\omega, \omega') \leq$$

$$\leq 8^{-1} \int_0^{t'} [(\partial u^0(w(\tau), w'(\tau)))^2 + (\partial' u^0(w(\tau), w'(\tau)))^2] d\tau. \tag{1.15}$$

This formula will be used for deriving the estimate of $||\phi||_2$.

In this paper we will not consider measures (1.13) which correspond to quantum systems of oscillators interactiong through a factorized ternary potential as in [3]. But all our estimates from the fourth section can be used for a proof of convergence of the polymer expansion for reduced density matrices of quantum systems with the potential energy U_2 (see the beginning of the third section) in the thermodynamic limit. This is an interesting system since it has the Gibbs ground state with the pair potential from the Smoluchowski equation.

Our paper is organized as follows. In the second section we consider systems of interacting Brownian oscillators and formulate Theorem 2.1 which establishes an existence of the correlation functions in the thermodynamic limit. In the third section we show how a reduction of the nonequilibrium systems to Gibbs diffusion path systems with a complex pair potential (1.14) and a pair potential φ is performed and formulate Theorem 3.1 which is a version of Theorem 1.1. Theorem 2.1 follows from Theorem 3.1. The fourth section is devoted to a proof of Theorem 3.1. In Appendix we prove Proposition 1.1.

Earlier (see [4, 5] and references there) we proved existence of the correlation functions of nonequilibrium systems of interacting Brownian particles in the thermodynamic limit for Gibbs initial correlation of functions utilizing the same reduction of the system to Gibbs path systems with a factorized three-particle potential. Here we demonstrate that oscillator systems are easier to treat since there is no necessity of applying complicated, as in the case of particle systems, L^p bounds.

2. Brownian oscillators. Dynamics in the system of finite number of Brownian linear oscillators, whose one-dimensional coordinates q_x are indexed by the site x of the d-dimensional lattice $\mathbb{Z}^d$, interacting via a pair potential $J(|x-y|)u^0(q_x, q_y)$, is governed by the Smoluchowski equation for the density $\rho^0(q_X; t)$ of a probability distribution, where $q_X = (q_x, x \in X) \in \mathbb{R}^{|X|}, X \subset \mathbb{Z}^d$, $|X| < \infty$ ($|X|$ is the cardinality of X):

$$\frac{\partial}{\partial t} \rho^0(q_X; t) = \sum_{x \in X} \partial_x \{\beta^{-1} \partial_x + \partial_x U^0(q_X)\} \rho^0(q_X; t), \tag{2.1}$$

$$U^0(q_X) = \sum_{x\in X} u^0(q_x) + \sum_{x,y\in X} u^0_{x-y}(q_x, q_y), \quad u^0_{x-y}(q_x, q_y) = J_0(|x-y|)u^0(q_x, q_y), \tag{2.2}$$

where β is the inverse temperature, $u^0(q)$ is an even bounded from below polynomial of the $2n$-th degree, $u^0(q, q') = u^0(q', q)$,

$$|u^0(q, q')| \le \frac{1}{2}[v^0(q) + v^0(q')], \tag{2.3}$$

$$|\partial u^0(q, q')| \le \frac{1}{2}[v'(q) + v'(q')], \quad |\partial^2 u^0(q, q')| \le \frac{1}{2}[v''(q) + v''(q')], \tag{2.4}$$

where v^0, v', v'' are positive polynomials in $|q|$ of degrees $2m, 2m-1, 2m-2$, respectively, and $m < n$ and ∂ is the partial derivative in q.

Equations (2.1) is the forward Kolmogorov equation for the stochastic oscillator equations

$$\dot{q}_x(t) = -\partial_x U^0(q_\Lambda(t)) + \beta^{-\frac{1}{2}}\dot{w}_x(t), \qquad x \in \Lambda,$$

where $\dot{w}_x(t)$ are independent processes of the white noise.

Solutions of the infinite system ($x \in \mathbb{Z}^d$) were proven to exist in [6, 7]. A convergent (high temperature) cluster expansion for associated measures in the case of Gibbsian initial measures is proposed in [8] in the simplest case. In [9, 10] the nonequilibrium systems, described by the infinite systems of stochastic equations, are treated as Gibbs lattice oscillator path systems if the equations have a solution.

Let us consider the nonequilibrium correlation functions for the set Λ of finite cardinality $|\Lambda|$, assuming that initial correlation functions are Gibbsian and generated by the potential energy $\tilde{U}$

$$\rho^\Lambda(q_X, t) = Z_\Lambda^{-1} \int \rho^0(q_\Lambda; t)dq_{\Lambda\setminus X}, \quad Z_\Lambda = \int \rho^0(q_\Lambda; t)dq_\Lambda,$$

where the integrations are performed over $\mathbb{R}^{|\Lambda\setminus X|}$ and $\mathbb{R}^{|\Lambda|}$, respectively,

$$\tilde{U}(q_\Lambda) = \eta_0 U^0(q_\Lambda) + U^1(q_\Lambda), \quad \eta_0 > \frac{1}{2}, \tag{2.5}$$

$$U^1(q_\Lambda) = \sum_{x\in\Lambda} u^1(q_x) + \sum_{x,y\in\Lambda} u^1_{x-y}(q_x, q_y),$$

$$|u^1_x(q, q')| \le J^1(|x|)[v^1(q) + v^1(q')], \quad ||J^1||_1 < \infty, \tag{2.6}$$

where v^1 is a positive polynomial of the $2m^1$-th degree, $m^1 < n^1$ and $u^1(q)$ is an even bounded from below polynomial of the $2n^1$-th degree, $n^1 < n$.

From the required conditions and the Feynman – Kac (FK) formula it will follow that the correlation functions exponentially decrease at infinity in every variable. This makes it possible to prove the law of conservation of probability following from the gradient character of the Smoluchowski equation.

Due to the law of conservation of probability the partition function Z_Λ does not depend on the time:

$$Z_\Lambda = \int e^{-\beta \bar{U}(q_\Lambda)} dq_\Lambda.$$

Taking into account this law we derive the following hierarchy:

$$\frac{\partial}{\partial t}\rho^\Lambda(q_X;t) = \sum_{x\in X}\partial_x\Bigg\{\beta^{-1}\partial_x\rho^\Lambda(q_X;t) + \rho^\Lambda(q_X;t)\partial_x U^0(q_X) +$$

$$+ \sum_{y\in\Lambda\setminus X}\int(\partial_x u^0_{x-y})(q_x,q_y)\rho^\Lambda(q_{X\cup y};t)dq_y\Bigg\}.$$

In the thermodynamic limit the following hierarchy is written as

$$\frac{\partial}{\partial t}\rho(q_X;t) = \sum_{x\in X}\partial_x\Bigg\{\beta^{-1}\partial_x\rho(q_X;t) + \rho(q_X;t)\partial_x U^0(q_X) +$$

$$+ \sum_{y\in X^c}\int(\partial_x u^0_{x-y})(q_x,q_y)\rho(q_{X\cup y};t)dq_y\Bigg\}, \tag{2.7}$$

where $X^c = \mathbb{Z}^d\setminus X$.

We will say that a sequence of correlation functions is a generelized solution of the diffusion hierarchy if it satisfies the last hierarchy which is averaged in the time and the oscillator variables with an infinitely-differentiable test function with a compact support and all the derivatives in which are acting not on the correlation functions but on the test functions (the first derivatives change signs).

Theorem 2.1. *Let β be sufficiently small, conditions (2.3), (2.4), (2.7) be satisfied, $\bar{\rho}(X)$ be the limit of the polymer correlation functions $\bar{\rho}_\Lambda(X)$ of the initial Gibbs system, $t' = t\beta^{-1} \leq \infty$ and $n - \bar{m} - 2m + 1 > 0$, $\bar{m} = \max(m, m^1, n^1)$, $m^1 < m$. Then there exist bounded functions $\rho_Y(q_X;t)$ and positive numbers $M', \xi'(\beta,t'), \epsilon \leq \frac{1}{4}$ such that*

1) *the finite volume correlation functions $\rho^\Lambda(q_X;t)$ are given by*

$$\rho^\Lambda(q_X;t) = \sum_{Y\in\Lambda\setminus X}\bar{\rho}_\Lambda(X\cup Y)\rho_Y(q_X;t),$$

their sequence $\rho^\Lambda(t)$ belongs to $\mathbb{B}_{\xi';\frac{\beta}{2}(1-\epsilon)u^0}$ and $||\rho^\Lambda(t)||_{\xi';\frac{\beta}{2}(1-\epsilon)u^0} \leq M'$;

2) *$\rho^\Lambda(t)$ converges locally in $\mathbb{B}_{\xi';\frac{\beta}{2}(1-\epsilon)u^0}$ to the sequence $\rho(t) = \{\rho(q_X;t), X \subset \subset \mathbb{Z}^d\} \in \mathbb{B}_{\xi';\frac{\beta}{2}(1-\epsilon)u^0}$ with the same norm whose elements are given by*

$$\rho(q_X;t) = \sum_{Y\subset X^c}\bar{\rho}(X\cup Y)\rho_Y(q_X;t); \tag{2.8}$$

3) *the sequence $\rho(t)$ is a generalized solution of the diffusion hierarchy (2.7).*

This theorem is a consequence of Theorem 1.1 and a reduction of the considered nonequilibrium systems to diffusion Gibbs path systems in which interaction is determined by a pair and a ternary interaction potentials. It is proven with the help of Theorem 3.1(a version of Theorem 1.1), formulated in the third section. We believe that the condition on n, m, n^1 can be removed for the systems with the quadratic interaction potential $u^0(q, q') = (q - q')^2$. In this case the ternary potential degenerates into a pair potential depending on Λ.

3. Diffusion Gibbs path system. After the rescaling time we obtain the following Smoluchowski equation for $t' = \beta^{-1}t$

$$\frac{\partial}{\partial t'}\rho^0(q_X;\beta t') = \sum_{x\in X}\partial_x\{\partial_x\rho^0(q_X;\beta t') + \rho^0(q_X;\beta t')\partial_x\beta U^0(q_X)\}. \tag{3.1}$$

The reduction of the above nonequilibrium system to a Gibbs oscillator path system is fulfilled in two steps.

The first one is the transformation of the Smoluchowski equation into the heat equation. In the second one we solve the latter with the help of the FK-formula. Indeed, after the substitution

$$\rho^0(q_X,;\beta t') = e^{-\frac{\beta}{2}U^0(q_X)}\psi(q_X;t'),$$

the following heat equation for ψ is obtained:

$$\frac{\partial}{\partial t}\psi(q_X;t) = \sum_{x\in X}\partial_x^2\psi(q_X;t) + \beta U_2(q_X)\psi(q_X;t), \tag{3.2}$$

$$U_2(q_X) = \frac{1}{2}\sum_{x\in X}\left[-\partial_x^2 U^0(q_\Lambda) + \frac{\beta}{2}(\partial_x U^0(q_X))^2\right].$$

We solve the Cauchy problem for the heat equation with the help of the well-known [11,12] FK (Feynman–Kac) formula. The obtained solutions are the (generalized) L^2-solutions. From the Kato–Relich theorem it follows that the the operator on the right-hand side of the heat equation is essential self-adjoint on the Schwartz space [12]. This together with the Trotter formula implies that the FK-formula produces the generalized solution of the heat equation (2.1). This formula also produces the generalized solution of the Smoluchowski equation (3.1) via the above substitution for the initial data chosen in the previous section.

Application of the FK-formula gives

$$\rho^\Lambda(q_X;t) = \int \rho^\Lambda((q,w)_X)P_{q_X}(dw_X),$$

$$\rho^\Lambda((q,w)_X) = Z_\Lambda^{-1}\int e^{-\beta U((q,w)_\Lambda)}P_0((dqdw)_{\Lambda\setminus X}), \tag{3.3}$$

where $P_q(dw)$ is the Wiener measure on paths starting from q, $P_0(dqdw) = dqP_q(dw)$,

$$Z_\Lambda = \int e^{-\beta U((q,w)_\Lambda)}P_0((dqdw)_\Lambda), \quad P_0((dqdw)_X) = \prod_{x\in X}P_0(dq_x dw_x),$$

$$U((q,w)_\Lambda) = \frac{1}{2}U^0(q_\Lambda) - \frac{1}{2}U^0(w_\Lambda(t')) + \bar{U}(w_\Lambda(t')) + \int_0^{t'} U_2(w_\Lambda(\tau))d\tau =$$

$$= U_0(q_\Lambda) + U_1(w_\Lambda(t')) + \int_0^{t'} U_2(w_\Lambda(\tau))d\tau, \tag{3.4}$$

where integration in two integrals is performed over the spaces $\Omega_0^{|X|}$, $\Omega_0^{|\Lambda\setminus X|}$, respectively, and Ω_0 is the probability path space.

It is obvious that we arrived at the path Gibbs system with a three-particle potential hidden in the second term in the expression for U_2.

Let us decompose the U_2-term into three terms

$$\sum_{x\in X}(\partial_x U^0(q_X))^2 = \sum_{x\in X}(\partial u^0(q_x))^2 + \sum_{x,y\in X} J_0(|x-y|)\varphi(q_x,q_y) + U_2'(q_X),$$

where the pair potential φ is given by

$$\varphi(q_x,q_y) = (\partial_x u^0(q_x))\partial_x u^0(q_x,q_y) + (\partial_y u^0(q_y))\partial_y u^0(q_y,q_x), \quad q_x,q_y\in\mathbb{R}, \tag{3.5}$$

and

$$U_2'(q_X) = \sum_{x\in X}\left(\sum_{y\in X} J_0(|x-y|)\partial_x u^0(q_x,q_y)\right)^2.$$

Hence

$$U((q,w)_X) - \int_0^{t'} U_2'(w_X(\tau))d\tau = \sum_{x\in X} u((q,w)_x) + \sum_{x,y\in X} u_{x-y}((q,w)_x,(q,w)_y), \tag{3.6}$$

where

$$u(q,w) = u_0(q) + u_1(w(t')) + \int_0^{t'} u_2(w(\tau))d\tau,$$

$$u_{x-y}((q,w)_x,(q,w)_y) =$$

$$= J_0(|x-y|)u_0(q_x,q_y) + u_{1,x-y}(w_x(t'),w_y(t')) + J_0(|x-y|)\int_0^{t'} u_2(w_x(\tau),w_y(\tau))d\tau,$$

$$u_0(q) = 2^{-1}u^0(q), \quad u_1(q) = \left(\eta_0 - \frac{1}{2}\right)u^0(q) + u^1(q),$$

$$u_2(q) = \frac{1}{2}\left[-\partial^2 u^0(q) + \frac{1}{2}\beta(\partial u^0(q))^2\right], \quad u_0(q_x,q_y) = \frac{1}{2}u^0(q_x,q_y),$$

$$u_{1,x-y}(q_x,q_y) = u^1_{x-y}(q_x,q_y) - \frac{1}{2}J_0(|x-y|)u^0(q_x,q_y),$$

$$u_2(q_x,q_y) = -\frac{1}{4}\left[\partial_x^2 u^0(q_x,q_y) + \partial_y^2 u^0(q_y,q_x)\right] + \beta\varphi(q_x,q_y).$$

In order to transform the U_2'-term we must use the formula

$$\exp\left\{-\frac{\beta^2}{4}\int_0^{t'} U_2'(w_X(\tau))d\tau\right\}=$$

$$=\int \exp\left\{i\frac{\beta}{2}\sum_{x,y\in X} J_0(|x-y|)\int_0^{t'} dw_x^*(\tau)\partial_x u^0(w_x(\tau),w_y(\tau))\right\}P_0(dw_X^*),$$

where on the right-hand side of the equality the stochastic integral is written and P_0 is the Wiener measure concentrated on the probability space Ω_0 consisting of paths which start from the origin.

Hence we are dealing with the systems with the potential energy (1.1) and the complex pair potential whose imaginary part is given by (1.14) and

$$\operatorname{Im} u_{x-y}(\omega_x,\omega_y)=J_0(|x-y|)\phi(\omega_x,\omega_y), \tag{3.7}$$

and the real part

$$\operatorname{Re} u_{x-y}(\omega_x,\omega_y)=u_{x-y}((q,w)_x,(q,w)_y), \qquad u(\omega)=u(q,w). \tag{3.8}$$

As a result

$$\rho^\Lambda((q,w)_X)=\int \rho^\Lambda(\omega_X)P_0(dw_X^*), \tag{3.9}$$

and the function $\rho^\Lambda(\omega_X)$ is given by (1.2). We see that only the imaginary part of the pair potential depends on w^*. This fact will be exploited by us in our estimates.

Theorem 3.1. *Let the conditions of Theorem* 2.1 *be satisfied. Then the condition* (1.10) *and the conclusions of Theorem* 1.1 *hold for the system with the potential energy* (I:1), *measure* $P_0(d\omega)$ *and the complex pair potential determined by* (1.12) *and* (1.4), (3.7), (3.8), *respectively. Moreover the functions* ρ_Y *in Theorem* 2.1 *are expressed as*

$$\rho_Y(q_X,t)=\int P_0(dw_X^*)P_{q_X}(dw_X)\int P(d\omega_Y)F_{\omega_X}(\omega_Y),$$

$$P(d\omega_Y)=\left(\int \exp\Big\{-\beta\sum_{y\in Y}u(\omega_y)\Big\}P_0(\omega_Y)\right)^{-1}\exp\Big\{-\beta\sum_{y\in Y}u(\omega_y)\Big\}P_0(d\omega_Y).$$

Corollary **3.1.** *For the functions* ρ_Y *and the correlation functions from Theorem* 2.1 *the following bounds are valid*

$$|\rho_Y(q_X;t)|\le M(eB\xi)^{|Y|}\bar\xi^{|X|}\exp\Big\{-\beta\sum_{x\in X}\bar u(q_x)\Big\}, \tag{3.10}$$

$$|\rho(q_X;t)|\le M(1-eB\xi)^{-1}\bar\xi^{|X|}\exp\Big\{-\beta\sum_{x\in X}\bar u(q_x)\Big\}, \tag{3.11}$$

where

$$\beta\bar{u}(q) = -\ln\int P_q(dw)P_0(dw^*)e^{-\beta[u(\omega)-\bar{v}(\omega)]}, \quad \bar{\xi} = e\xi\int P_0(d\omega)e^{-\beta u(\omega)}.$$

We will use the following inequalities in the next section in the proof of the theorem:

$$u^0(q) \geq (\eta-\varepsilon)q^{2n} - c, \quad u_1(q) \geq (\eta_0 - 2^{-1} - \varepsilon)q^{2n} - c,$$
$$(\partial u^0)^2 \geq (\eta^2(2n)^2 - \varepsilon)q^{2(2n-1)} - c. \tag{3.12}$$

We will also need the following inequality:

$$|p(q)| \leq (\varepsilon|q|^{2l} + c), \tag{3.13}$$

where c is a sufficiently large positive constant, ε is a sufficiently small positive constant and p is a polynomial with a degree strictly less than $2l$. Inequalities (3.12) are a concequence of (3.13).

4. Proof of Theorem 3.1. In order to prove Theorem 3.1 one has to check that the potentials of introduced Gibbs diffusion path system satisfy the conditions of Theorem 1.1.

Proposition 4.1. *The pair potential* u_{x-y} *satisfies* (1.2) *with* $J = |J_0| + J^1$,

$$v(\omega) = v_0(q) + v_1(w(t')) + \int_0^{t'} v_2(w(\tau))d\tau,$$

$$v_0 = \left(\eta_0 - \frac{1}{2}\right)v^0, \quad v_1 = v^1, \quad v_2 = v_2' + \beta v_2'',$$

where $v_2' = v''$, $v_2''(q) = c'(q^{2(n+m-1)} + 1)$ *and* c' *is a constant.*

Proof. The proof follows from (2.3), (2.4). Functions v_2', v_2'' are the contribution from $\partial^2 u^0(q,q'), \varphi$ respectively. The nontrivial bound of the pair potential φ from (3.4) follows from the fact that $\partial u^0(q)$ is a polynomial of the $(2n-1)$-th degree, the bound

$$|\varphi(q,q')| = [|\partial_x u^0(q_x)| + |\partial_y u^0(q_y)|][|\partial_x u^0(q_x,q_y)| + |\partial_y u^0(q_y,q_x)|] \leq$$
$$\leq c'(|q_x|^{2n-1} + |q_y|^{2n-1} + 1)(|q_x|^{2m-1} + |q_y|^{2m-1} + 1) =$$
$$= c'(|q_x|^{2(n+m-1)} + |q_y|^{2(n+m-1)} + |q_x|^{2n-1}|q_y|^{2m-1} + |q_x|^{2m-1}|q_y|^{2n-1}),$$

where c is a positive constant (here we use the inequality (3.13) for all the terms in the expressions for $\partial^2 u^0(q), v''$ with $l = 2(n-1)$ and $l = 2(m-1)$, respectively) and the inequality

$$|q_x|^{n_1}|q_y|^{n_2} + |q_x|^{n_2}|q_y|^{n_1} \leq 2^{n_1+n_2}(|q_x|^{n_1+n_2} + |q_y|^{n_1+n_2}).$$

The last inequality is proven in the following way ($n_2 > n_1$):

$$|q_x^{n_1}||q_y^{n_2}| + |q_y^{n_1}||q_x^{n_2}| \leq |q_x|^{n_1}|q_y|^{n_1}[|q_x|^{n_2-n_1} + |q_y|^{n_2-n_1}] \leq$$

$$\leq 2^{-1}(|q_x|^{2n_1}+|q_y|^{2n_1})(|q_x|^{n_2-n_1}+|q_y|^{n_2-n_1}) \leq$$

$$\leq 2^{-1}(|q_x|+|q_y|)^{2n_1}(|q_x|+|q_y|)^{n_2-n_1} = 2^{-1}(|q_x|+|q_y|)^{n_1+n_2} \leq$$

$$\leq 2^{n_1+n_2-1}(|q_x|^{n_1+n_2}+|q_y|^{n_1+n_2}).$$

Proposition is proved.

Proposition 4.2. *If $t' < \infty$ then the following inequalities hold*

$$\lim_{\beta\to 0}||e^{\beta\varepsilon u}||_2 < \infty, \quad \lim_{\beta\to 0}||e^{\beta J v}||_2 < \infty.$$

Proof. We will rescale the variables in the integrals by $g = \beta^{-\frac{1}{2n}}$ and have to use the following formula:

$$\int P_{gq}(dw)f(w(t_1),\ldots,w(t_k)) = \int P_q(dw)f(gw(g^{-2}t_1),\ldots,w(g^{-2}t_k)) \quad (4.1)$$

which follows from the well-known definition of the Wiener measure by rescaling the oscillator variables by g $(q_j \to gq_j)$

$$\int P_{gq}(dw)f(w(t_1),\ldots,w(t_k)) =$$

$$= \int f(q_1,\ldots,q_k)P_0^{t_1}(q_1,gq)\prod_{j=1}^{k-1}P_0^{t_{j+1}-t_j}(q_{j+1},q_j')dq_1\ldots dq_k$$

and the formula

$$P_0^t(gq,gq') = \exp\{t\partial^2\}(gq;gq') = (4\pi t)^{-\frac{1}{2}}\exp\left\{-\frac{g^2|q-q'|^2}{4t}\right\} = g^{-1}P_0^{tg^{-2}}(q,q').$$

After the rescaling the integral is multiplied by g^k in the previous formula and this multiplier is cancelled by the g^{-k} which results from the right-hand side of last formula and the fact that there is a product of k similar terms on the right-hand side of the previous formula.

Let us define the operator S_g of scaling by

$$S_gF(q,w) = F(gq,gw_g), \quad w_g(t) = w(g^{-2}t).$$

For our measure P we have (u, v do not depend on w^*)

$$||e^{\beta\varepsilon u}||_2^2 = \left(\int e^{-\beta S_g u(q,w)}dqP_q(dw)\right)^{-1}\int e^{-\beta(1-2\varepsilon)S_g u(q,w)}dqP_q(dw), \quad (4.2)$$

$$||e^{\beta J_* v}||_2^2 = \left(\int e^{-\beta S_g u(q,w)}dqP_q(dw)\right)^{-1}\int e^{-\beta[S_g u(q,w)-2J_* S_g v(q,w)]}dqP_q(dw).$$
(4.3)

Relations (4.2), (4.3) follow from (4.1) and the Trotter formula.

It is evident that

$$S_g u(q,w) = u_0(gq) + u_1(gw(g^{-2}t')) + \int_0^{t'} u_2(gw(g^{-2}\tau))d\tau. \quad (4.4)$$

Now, we have to find the limits of the functions under the sighs of integrals in (4.2), (4.3) when g tends to zero, make uniform estimates in g.

It easy to check that

$$\lim_{g\to\infty} g^{-2n} \dot{S}_g u(q,w) = \lim_{g\to\infty} g^{-2n} u(gq, gw_g) = \eta\eta_0 q^{2n}, t' < \infty \quad g^{-2n} = \beta, \quad (4.5)$$

where η is the positive coefficient before q^{2n} in the expression for u^0(by definition). Here one has to take into account

$$\lim_{g\to\infty} w(g^{-2}\tau) = w(0) = q,$$

and the fact that only u^0 and u^1 make contribution to the right-hand side of (4.5). Indeed, the coefficient before the term q^{2k} in $S_g\partial^2 u^0(q)$, $\beta S_g(\partial u^0)^2$ is less than $g^{2(n-1)} = g^{-2}g^{2n}$, $g^{-2n+2(2n-1)} = g^{-2}g^{2n}$, respectively, since $k \le 2(n-1)$, $k \le 2(2n-1)$ for the first and second polynomials, respectively. The coefficient before the term q^{2k} in the expressions for $S_g v_2'$, $\beta S_g v_2''$, also, is less than $g^{-2}g^{2n}$.

As a result

$$\lim_{g\to\infty} g^{-2n} S_g v(q,w) = 0, \quad g^{-2n} = \beta, \quad (4.6)$$

since the degrees of the polynomials v_0, v_1 do not exceed $2(n-1)$.

Applying (3.13) for the functions v'', $\partial^2 u^0$ with $l = 2(2n-1)$ we see that

$$\beta(v_2(\beta^{-\frac{1}{2n}}q) + |\partial^2 u^0(\beta^{-\frac{1}{2n}}q)|) \le \beta^{\frac{1}{n}}(\varepsilon q^{2(2n-1)} + c(\varepsilon)), \quad \beta < 1.$$

This inequality and the third inequality in (3.12) imply that

$$\beta(u_2(\beta^{-\frac{1}{2n}}q) - v_2(\beta^{-\frac{1}{2n}}q)) \ge \beta^{\frac{1}{n}}[(2\eta n)^2 - \varepsilon')q^{2(2n-1)} + c], \qquad \varepsilon' << (2\eta n)^2$$

and c depends on ε'.

That is

$$\exp\left\{-\beta\int_0^{t'} [S_g u_2(w(\tau)) - s_g v_2((w(\tau))]d\tau\right\} \le e^{t'\beta^{\frac{1}{n}}c}.$$

Two first inequalities in (3.12) show that the exponent of the part of u, containing u_0, u_1 are also bounded by a sufficiently large constant.

Hence the Lebesque dominated convergence theorem and (4.5) prove the equalities

$$\lim_{\beta\to 0} ||e^{\beta\varepsilon u}||_2^2 = \left(\int e^{-\eta\eta_0 q^{2n}} dq\right)^{-1} \int e^{-\eta\eta_0(1-2\varepsilon)q^{2n}} dq,$$

$$\lim_{\beta\to 0} ||e^{\beta J_* v}||_2 = 1, \quad t' < \infty. \quad (4.7)$$

This concludes the proof of the proposition.

Let us consider now the conditions in the Proposition 1.1.

From the Helder inequality for the norm induced by the Lebesque measure on $[0,t']$ $||fh||_1 \leq ||f||_p||h||_q, \quad p^{-1}+q^{-1}=1$ with $h=1$ we derive

$$\int_0^{t'} w^{2(m-1)}(\tau)d\tau \leq (t')^{1-\frac{m-1}{2n-1}} \left(\int_0^{t'} w^{2(2n-1)}(\tau)d\tau\right)^{\frac{m-1}{2n-1}} =$$

$$= (t')^{\frac{2n-m}{2n-1}} R^{\frac{m-1}{2n-1}}, \quad p = \frac{2(2n-1)}{2(m-1)},$$

$$\int_0^{t'} w^{2(n+m-1)}(\tau)d\tau \leq (t')^{1-\frac{n+m-1}{2n-1}} \left(\int_0^{t'} w^{2(2n-1)}(\tau)d\tau\right)^{\frac{n+m-1}{2n-1}} =$$

$$= (t')^{\frac{n-m}{2n-1}} R^{\frac{n+m-1}{2n-1}}, \quad p = \frac{2(2n-1)}{2(n+m-1)}, \quad m > 1,$$

where

$$R = \int_0^{t'} w^{2(2n-1)}(\tau)d\tau.$$

From Proposition 4.1 it follows that for some constant c the following inequalities hold:

$$\int_0^{t'} v_1'(w(\tau))d\tau \leq c\left[(t')^{\frac{2n-m}{2n-1}} R^{\frac{m-1}{2n-1}} + 1\right],$$

$$\int_0^{t'} v_2'(w(\tau))d\tau \leq c\left[(t')^{\frac{n-m}{2n-1}} R^{\frac{n+m-1}{2n-1}} + 1\right].$$

With the help of (3.13) ($\partial^2 u^0$ is a polynomial of the $2(n-1)$-th degree) and the Holder inequality for $p = \dfrac{2(2n-1)}{2(n-1}$ we derive $\left(1 - p^{-1} = \dfrac{n}{2n-1}\right)$

$$\int_0^{t'} |\partial^2 u^0(w(\tau))|d\tau \leq c\left[(t')^{\frac{n}{2n-1}} R^{\frac{n-1}{2n-1}} + 1\right].$$

These inequalities and the last inequality from (3.12) imply that

$$\exp\{-\beta[\varepsilon(u-u_0-u_1)+2J_*(v-v_0-v_1)]\} \leq \exp\{-\varepsilon((2\eta n)^2-\varepsilon)\beta^2 R + \theta(R)\},$$

$$\theta(R) = 2c\beta\left[(t')^{\frac{2n-m}{2n-1}} R^{\frac{m-1}{2n-1}} + 1\right] J_* + 2c\beta^2\left[(t')^{\frac{n-m}{2n-1}} R^{\frac{n+m-1}{2n-1}} + 1\right] J_* +$$

$$+ c\varepsilon\beta\left[(t')^{\frac{n}{2n-1}} R^{\frac{n-1}{2n-1}} + 1\right] + \varepsilon\beta^2 t' c.$$

After rescaling R by β^2 we obtain

$$||(v - v_0 - v_1)\exp\{-\beta[\varepsilon(u - u_0 - u_1) + 2J_*(v - v_0 - v_1)]\}||_\infty \leq$$

$$\leq \alpha\left(1 + \beta^{-\frac{2(m-1)}{2n-1}} + \beta^{-\frac{2m-1}{2n-1}}\right), \tag{4.8}$$

where

$$\alpha = c\sup_{R\geq 0}\Big[1 + \Big[(t')^{\frac{2n-m}{2n-1}}R^{\frac{m-1}{2n-1}} + 1\Big] +$$

$$+2\Big[(t')^{\frac{n-m}{2n-1}}R^{\frac{n+m-1}{2n-1}} + 1\Big]\Big]\exp\{-\varepsilon((2\eta m)^2 - \varepsilon)R + \theta_0(R)\}$$

and $\theta_0(R) = \max\limits_{\beta<1}\theta(\beta^{-2}R)$. The function $\theta(\beta^{-2}R)$ equals zero at zero temperature. Indeed, the temperature has the following powers in the coefficients before the square brackets in its expression

$$1 - \frac{2(m-1)}{2n-1} = \frac{2(n-m)+1}{2n-1} > 0,\ 2 - \frac{2(n+m-1)}{2n-1} = \frac{2(n-m)}{2n-1} > 0,\ \frac{1}{2n-1} > 0.$$

α is finite since the powers of R in $\theta(\beta^{-2}R)$ are strictly less than unity. The power of β in the last term on the right-hand side of (4.8) we also derived from the equality

$$1 - \frac{2(n+m-1)}{2n-1} = -\frac{2m-1}{2n-1}$$

taking into account that β stands before v_2'' in the expression for v_2 (see Proposition 4.1).

From (3.12) for u_0, u_1 and (3.13) for v_0, v_1 after the rescaling of the variables by $\beta^{\frac{1}{2n}}$ we deduce

$$||(v_0 + v_1)\exp\{-\beta[\varepsilon(u_0 + u_1) + 2J_*(v_0 + v_1)]\}||_\infty \leq$$

$$\leq ||v_0\exp\{-\beta[\varepsilon u_0 + 2J_*v_0]\}||_\infty||\exp\{-\beta[\varepsilon u_1 + 2J_*v_1]\}||_\infty +$$

$$+||v_1\exp\{-\beta[\varepsilon u_1 + 2J_*v_1]\}||_\infty||\exp\{-\beta[\varepsilon u_0 + 2J_*v_0]\}||_\infty \leq \alpha'\left(1 + \beta^{-\frac{m}{n}} + \beta^{-\frac{m^1}{n}}\right),$$

where

$$\alpha' = e^{4\beta c}c\sup_{R_1,R_2\geq 0}[2 + R_1^{2m} + R_2^{2m^1}]\times$$

$$\times\exp\Big\{-2^{-1}\varepsilon(\eta - \varepsilon)R_1^{2n} - \varepsilon\left((\eta_0 - 2^{-1})\eta - \varepsilon\right)R_2^{2n} +$$

$$+cJ_*\left(\beta^{\frac{n-m}{n}}R_1^{2m} + \beta^{\frac{n-m^1}{n}}R_2^{2m^1}\right)\Big\},\quad \beta = 1.$$

Inequality (4.8) and the last inequality yield the following inequality:

$$||ve^{-\beta(\varepsilon u - 2J_*v)}||_\infty \leq (\alpha + \alpha')\left(1 + \beta^{-\frac{2(m-1)}{2n-1}} + \beta^{-\frac{2m-1}{2n-1}} + \beta^{-\frac{m}{n}} + \beta^{-\frac{m^1}{n}}\right) \leq$$

$$\leq 4(\alpha + \alpha')\left(1 + 4\beta^{-\frac{2\bar{m}-1}{2n-1}}\right), \tag{4.9}$$

where $\bar{m} = \max(m, m^1, n^1)$ and the inequality $\dfrac{m}{n} \geq \dfrac{m-1}{n-1}$ was applied.

Applying the bound

$$||ve^{\beta J_* v}||_2 \le ||v^2 e^{-\beta(\epsilon u - 2J_* v)}||_\infty^{\frac{1}{2}} ||e^{\beta\frac{\epsilon}{2}u}||_2,$$

the bound

$$(a_1 + \ldots + a_n)^2 \le n(a_1^2 + \ldots + a_n^2), \quad (a_1 + \ldots + a_n)^{\frac{1}{2}} \le \sqrt{a_1} + \ldots + \sqrt{a_n}, a_s \ge 0,$$

for $n = 4$ (there are four terms in the expression for v) and repeating the previous arguments we derive

$$||ve^{\beta J_* v}||_2 \le ||e^{\beta\frac{\epsilon}{2}u}||_2 \bar{\alpha} \left(1 + \beta^{-\frac{2(m-1)}{2n-1}} + \beta^{-\frac{2m-1}{2n-1}} + \beta^{-\frac{m}{n}} + \beta^{-\frac{m^1}{n}}\right) \le$$

$$\le ||e^{\beta\frac{\epsilon}{2}u}||_2 \bar{\alpha} \left(1 + 4\beta^{-\frac{2\tilde{m}-1}{2n-1}}\right), \tag{4.10}$$

where $\bar{\alpha} = 2(\sqrt{\alpha_1} + \sqrt{\alpha_1'})$, α_1 and α_1' are obtained, respectively, from α, α' by squaring the powers of $R. t'$ only in the square brackets.

Let us consider the expression for $||\phi||_2^2$. Inequality (1.15) and independence of u on w^* yield

$$||\phi||_2^2 \le 8^{-1} \left(\int dq P_q(dw) e^{-\beta u(q,w)}\right)^{-2} \times$$

$$\times \int dq dq' P_q(dw) P_{q'}(dw') e^{-\beta[u(q',w') + u(q',w')]} \int\limits_0^{t'} [(\partial u^0(w(\tau), w'(\tau)))^2 +$$

$$+ (\partial' u^0(w'(\tau), w(\tau)))^2] d\tau \le 8^{-1} \left(\int dq P_q(dw) e^{-\beta u(q,w)}\right)^{-2} \times$$

$$\times \int dq dq' P_q(dw) P_{q'}(dw') e^{-\beta[u(q,w) + u(q',w')]} [v_\partial(w) + v_\partial(w')] = 4^{-1} ||v_\partial||_1,$$

$$v_\partial(w) = \int\limits_0^{t'} v'^2(w(\tau)) d\tau.$$

Therefore

$$||\phi||_2^2 \le 4^{-1} ||v_\partial||_1 \le 4^{-1} ||v_\partial e^{-\beta\epsilon u}||_\infty ||e^{\beta\epsilon u}||_1. \tag{4.11}$$

Applying once more (3.13) and the Holder inequality for $p = \dfrac{2(2n-1)}{2(2m-1)}$ and the previous arguments we derive $\left(1 - p^{-1} = \dfrac{2(n-m)}{2n-1}\right)$

$$||\phi||_2^2 \le \alpha'' ||e^{\beta\epsilon u}||_1 \left(1 + \beta^{-\frac{2(2m-1)}{(2n-1)}}\right), \tag{4.12}$$

where

$$\alpha'' = c \sup_{R \ge 0} \left[1 + \left((t')^{\frac{2(n-m)}{2n-1}} R^{\frac{2m-1}{2n-1}} + 1\right)\right] \exp\left\{-\epsilon((2\eta n)^2 - \epsilon)R + \theta_0(R)\right\}.$$

From the expression for $\theta_0(R) = \beta^\kappa\theta'(R)$, where $\kappa > 0$, θ' is finite at zero temperature and depends on powers of R less than unity, it follows that $||e^{\beta(\epsilon u - J_* v)}||_\infty$ tends to unity at $\beta = 0$ and that the conditions of Proposition 1.1 and Theorem 2.1 are satisfied.

Proposition (4.2), (4.9), (4.10), (4.12) show that the condition (1.10) is satisfied if

$$1 - \frac{2\bar{m} - 1}{2n - 1} + \frac{2(2m - 1)}{2n - 1} > 0, \qquad n - \bar{m} - 2m + 1 > 0. \tag{4.13}$$

This fact proves Theorem 3.1 and part of Theorem 2.1 concerning expressions for the elements of the sequences ρ and ρ^Λ. The norms of the sequences are derived from the Corollary 3.1.

From (3.11) it follows that

$$M' = M(1 - eB\xi)^{-1}. \tag{4.14}$$

$$\xi' = e\xi \left(\int P_0(d\omega) e^{-\beta u(\omega)} \right) ||e^{-\beta(1-\epsilon)(u - \frac{1}{2}u_0)}||_\infty (1 + ||\phi||_2). \tag{4.15}$$

Here we applied the Schwartz inequality and as a result obtained the inequality $\int P(d\omega) b(\omega) \le ||\phi||_2$. Items 1 and 2 of Theorem 2.1 are proved.

Now, we have to prove that the sequence of the correlation functions is a generalized solution of the diffusion hierarchy. In order to do this it is necessary to prove that the both sides of the averaged with a test function finite volume diffusion hierarchy converges to the averaged infinite volume diffusion hierarchy.

Proof of the third item is a standard one. If follows along the lines of the argument, given after Theorem 1.1, taking into account the second item of the Theorem. We have to prove that the averaged left- and right-hand sides of the finite volume hierarchy converges to the corresponding averaged sides of (2.7). We have to rely on the local convergence of the sequence $\rho^\Lambda(t)$ to the sequence $\rho(t)$ in the Banach space $\mathbb{B}_{\xi', \frac{\beta}{2}(1-\epsilon)u^0}$. This convergence leads immediately to the convergence of the first two non-integral terms on the right-hand side and the left-hand side of the averaged finite volume hierarchy. In order to prove the convergence of the third right-hand side term one has to estimate the integrals for $X \in A$

$$\sum_{y \in \Lambda \setminus X} \int f(q_X; t) dq_X dt \int (\partial_x u^0_{x-y})(q_x, q_y) |\rho(q_{X \cup y}; t) - \rho^\Lambda(q_{X \cup y}; t)| dq_y.$$

$$\sum_{y \in \Lambda'} \int f(q_X; t) dq_X dt \int (\partial_x u^0_{x-y})(q_x, q_y) \rho(q_{X \cup y}; t) dq_y,$$

where f is a function with a compact support. The second integral converges to zero since (2.2), (2.4), $\rho(t) \in \mathbb{B}_{\xi', \frac{\beta}{2}(1-\epsilon)u^0}$ and $||J_0||_1 < \infty$ hold. Indeed, from these inequalities we deduce that the second integral is bounded by

$$\xi'^{|X|+1} ||\rho||_{\xi', \frac{\beta}{2}(1-\epsilon)} ||f||_1 J(A, \Lambda^c) \int dq_y \frac{1}{2} (v'(q_y) +$$

$$+ v'(q_x)) e^{-\frac{\beta}{2}(1-\epsilon)[u^0(q_x) + u^0(q_y)]} \left(||e^{-\frac{\beta}{2}(1-\epsilon)u^0}||_\infty \right)^{|X|-1} \le$$

$$\leq \frac{1}{2}\xi'^{|X|+1}||\rho||_{\xi',\frac{\beta}{2}(1-\varepsilon)}||f||_1\varepsilon^0(\lambda)\Big[||e^{-\frac{\beta}{2}(1-\varepsilon)u^0}||_\infty||e^{-\frac{\beta}{2}(1-\varepsilon)u^0}v'||_1+$$

$$+||e^{-\frac{\beta}{2}(1-\varepsilon)u^0}v'||_\infty||e^{-\frac{\beta}{2}(1-\varepsilon)u^0}||_1\Big]\left(||e^{-\frac{\beta}{2}(1-\varepsilon)u^0}||_\infty\right)^{|X|-1}. \qquad (4.16)$$

This expression tends to zero in the thermodynamic limit since ε^0 tends to zero at infinity.

After decomposing the sum over $\Lambda\backslash X$ into the sum over two sets $S_{\frac{1}{2}\lambda+r}$, $\Lambda\backslash S_{\frac{1}{2}\lambda+r}$ from the first section (2.2) and (2.4) we see that the first integral also converges to zero. Indeed, in the first set we have to take into consideration the fact of the local convergence of ρ^Λ to ρ. So, this sum is bounded by the last expression in which $\varepsilon\left(\frac{1}{2}\lambda\right)||J||_1$ is inserted instead of ε^0. Applying the bounds, used for obtaining (4.16), we see that the second sum is bounded by the expression on the right-hand side of (4.16) in which $2\varepsilon^0\left(\frac{1}{2}\lambda\right)$ is inserted instead of $\varepsilon^0(\lambda)$.

This concludes the proof of Theorem 2.1.

5. Appendix. The KS-relation is given by

$$F_{\omega_X}(\omega_Y) = e^{-\beta W(\omega_x|\omega_X)}\Bigg[F_{\omega_{X\backslash x}}(\omega_Y)+$$

$$+\sum_{Z\in Y,|Z|>0} K(\omega_x|\omega_Z)F_{\omega_{X\backslash x},\omega_Z}(\omega_{Y\backslash Z})\Bigg], \quad x\in X, \qquad (5.1)$$

where

$$W(\omega_x|\omega_X) = U(\omega_X) - U(\omega_{X\backslash x}), \quad K(\omega_x|\omega_Z) = \prod_{z\in Z}\left(e^{-\beta u_{x-y}(\omega_x,\omega_y)} - 1\right),$$

$$F_{\omega_X}(\emptyset) = \exp\left\{-\beta\sum_{x,y\in X} u_{x-y}(\omega_x,\omega_y)\right\},$$

$$F_\emptyset(\omega_Y) = 0, \qquad F_{\omega_x}(\omega_y) = e^{-\beta u_{x-y}(\omega_x,\omega_y)} - 1.$$

***Proof of inequality* 1.7.** From (1.2) it follows that

$$I(0,n) \leq ||e^{-\beta(u-J_*v)}||_\infty^n.$$

Now we have to estimate the following expression for $m>0$:

$$I_{A,A'}(m,n) = \operatorname{ess}\sup_{X'\subset A,|X'|=n-1,\omega_{x\cup X'}}\ \sum_{Y\subset A',|Y|=m} F^Y(\omega_{x\cup X'}),$$

$$I(m,n) = I_{A,A'}(m,n), \quad A = A' = \mathbb{Z}^d,$$

where

$$F^Y(\omega_X)=\prod_{x\in X}e^{-\beta\bar v(\omega_x)}\int P(d\omega_Y)|F_{\omega_X}(\omega_Y)|,\quad F^{\emptyset}(\omega_X)=\prod_{x\in X}e^{-\beta\bar v(\omega_x)}F_{\omega_X}(\emptyset).$$

Our estimate will be derived with the help of the symmetrized (5.1) and induction. Symmetrization of (1.5) is based on the inequality (1.2) and is analogical to the symmetrization of the KS-recursion relation for particle systems [3]. It is performed by multiplication of both sides of (5.1) by the normalized characteristic function χ_x of the set where $W(\omega_x|\omega_X)\le J_*v(\omega_x)$ and summation over $x\in X$. The normalized characteristic functions mean that $\sum_{x\in X}\chi_x=1$. So from the symmetrized (5.1) by integration and summation of both its sides we obtain

$$I(m,n)\le I_{A,A'}(m,n-1)+\sup_{x,\omega_x}\sum_{k=0}^{m-1}\sum_{Z,|Z|=m-k}\prod_{z\in Z}b_{x-z}(\omega_x)I(k,n-1+m-k)\le$$

$$\le I(m,n-1)+\sum_{l=1}^{m}\frac{B^l}{l!}I(m-l,n-1+l),$$

where sum over Z is performed over $\mathbb{Z}^d$. Here we used the following relation:

$$\sum_{Y,\ |Y|=s}=\frac{1}{s!}\sum_{y_1,\dots,y_s}\ I_{A,A'}(m,n)\le I_{A,A'}(m,n-1)+$$

$$+\sup_{x\in A,\omega_x\in\Omega}\sum_{k=0}^{m-1}\sum_{Z\in A',|Z|=m-k}\prod_{z\in Z}b_{x-z}(\omega_x)I(k,n-1+m-k)\le$$

$$\le I_{A,A'}(m,n-1)+\sum_{l=1}^{m}\frac{B^l_{A,A'}}{l!}I(m-l,n-1+l)\le$$

$$\le I_{A,A'}(m,n-1)+B_{A,A'}\sum_{l=1}^{m}\frac{B^{l-1}}{l!}I(m-l,n-1+l).$$

Here we changed the order of the summation in the following way:

$$\sum_{Y\subset A'}\sum_{Z\subseteq Y}=\sum_{Z\subset A'}\sum_{Y\setminus Z\subset A'},$$

and used the inequalities $B_{A,A'}\le B,\quad I_{A,A'}(m,n-1)\le I(m,n-1)$.

Since $I(1,0)=0,\ I(0,1)=1,\ I(1,1)\le B$, by induction we easily conclude from the obtained (recursion) inequality that, for $I(m,n)$ and arbitrary positive a,

$$I(m,n)\le a^n(e^{aB}a^{-1})^{m+n}.$$

As a result, for $a=B^{-1}$

$$I(m,n)\le(eB)^me^n.$$

Hence, (1.7) is true for $A=A'=\mathbb{Z}^d$ only for small temperatures since $I(0,n)\le e^{2n}$ only for them.

From this inequality and the recursion inequality for $I_{A,A'}(m,n)$, by induction we derive (1.7). Here one has to take into consideration that

$$I_{A,A'}(1,1) \le ||e^{-\beta\varepsilon u}||_\infty B_{A,A'} \le ||e^{-\beta(u-J_*v)}||_\infty B_{A,A'} \le eB_{A,A'}.$$

***Proof of Proposition* 1.1.** From

$$e^a - 1 = e^{i\mathrm{Im}\, a}(e^{\mathrm{Re}\, a} - 1) + (e^{i\mathrm{Im}\, a} - 1), \quad |e^{i\mathrm{Im}\, a} - 1| \le |\mathrm{Im}\, a|,$$

it follows that

$$b_x(\omega) \le e^{-\beta[\bar v(\omega) - J_* v(\omega)]} \int e^{\beta \bar v(\omega)} \left[|e^{-\beta \mathrm{Re}\, u_x(\omega,\omega')} - 1| + \beta |J(|x|)||\phi(\omega,\omega')|\right] P(d\omega'). \tag{5.2}$$

From the definition of b and the Schwartz inequality, applied to both terms in the square bracket, we derive

$$b_x(\omega) \le ||e^{\beta\bar v}||_2 (b_x^0(\omega) + \beta ||b e^{-\beta(\bar v - v)}||_\infty |J(|x|)|), \tag{5.3}$$

$$b_x^0(\omega) = e^{-\beta[\bar v - J_* v(\omega)]} \left(\int |e^{-\beta \mathrm{Re}\, u_x(\omega,\omega')} - 1|^2 P(d\omega')\right)^{\frac12},$$

where by $||.||_p$ the norm of the space $L^p(\Omega, P)$ is denoted.

Now, we have to estimate the term with b_x^0. We have to start from the standard inequality (it differs from the bound used in [1, 2]) taking into account (1.2)

$$|e^{-\beta \mathrm{Re}\, u_x(\omega,\omega')} - 1| \le \beta |\mathrm{Re}\, u_x(\omega,\omega')| e^{\beta|\mathrm{Re}\, u_x(\omega,\omega')|} \le$$

$$\le \frac{\beta J(|x|)}{2}(v(\omega) + v(\omega')) e^{\frac{J(|x|)\beta}{2}(v(\omega)+v(\omega'))}.$$

It is clear that $J(A, \mathbb{Z}^d) = ||J||_1$. From the last inequality and the inequalities $(a+b)^2 \le 2(a^2+b^2)$, $\sqrt{a+b} \le \sqrt a + \sqrt b$, $J(|x|) \le J_*$ we derive ($v \ge 0$)

$$\operatorname{ess\,sup}_{x\in A} \sum_{t\in A'} b_{x-y}^0(\omega) \le$$

$$\le \frac{\beta J(A,A')}{2} e^{-\beta[\bar v(\omega) - 2J_* v(\omega)]} \left(\int (v(\omega) + v(\omega'))^2 e^{J_*\beta v(\omega')} P(d\omega')\right)^{\frac12} \le$$

$$\le \beta J(A,A') \left[e^{-\beta[\bar v(\omega) - 2J_* v(\omega)]} v(\omega) ||e^{\beta J_* v}||_2 + ||v e^{\beta J_* v}||_2\right] =$$

$$\le \beta J(A,A') \left[||v e^{-\beta[\bar v - 2J_* v]}||_\infty ||e^{\beta J_* v}||_2 + ||v e^{\beta J_* v}||_2\right].$$

From (5.3) and the last inequality we obtain

$$B_{A,A'} \le J(A,A')\bar B, \quad B \le ||J||_1 \bar B,$$

$$\bar B = \beta ||e^{\beta \bar v}||_2 \left(||v e^{-\beta[\bar v - 2J_* v]}||_\infty ||e^{\beta J_* v}||_2 + ||v e^{\beta J_* v}||_2 + ||b e^{-\beta(\bar v - v)}||_\infty\right).$$

Applying the Schwartz inequality we derive (P is a probability measure)

$$D \le ||e^{\beta\bar v}||_2 \le ||e^{\beta\gamma}||_2 ||1 + b||_2 \le 2||e^{\beta\gamma}||_2 (1 + ||\phi||_2).$$

These inequalities yield the Proposition 1.1 since

$$||be^{-\beta(\bar{v}-v)}||_\infty = ||b(1+b)^{-1}e^{-\beta(\gamma-v)}||_\infty \leq ||e^{-\beta(\gamma-v)}||_\infty.$$

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