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PROPERTIES OF A SUBCLASS OF AVAKUMOVIĆ FUNCTIONS AND THEIR GENERALIZED INVERSES*

ВЛАСТИВОСТІ ОДНОГО ПІДКЛАСУ ФУНКЦІЙ АВАКУМОВІЧА ТА ЇХНІХ УЗАГАЛЬНЕНИХ ОБЕРНЕНИХ ФУНКЦІЙ

We study properties of a subclass of ORV functions introduced by Avakumović and provide their applications for the strong law of large numbers for renewal processes.

Вивчаються властивості одного підкласу ORV функцій, означених Авакумовічем, та наводяться деякі застосування до посиленого закону великих чисел для процесів відновлення.

1. Introduction. In the paper [1] the relationship between the strong law of large numbers for sequences of random variables and its counterpart for renewal processes constructed by these sequences is studied. Namely, given a sequence of random variables $\{Z_n,\ n \geq 0\}$, the generalized renewal processes are defined as follows

$$L(t) = \sup\{n \geq 0 : Z_n \leq t\},$$

$$M(t) = \sup\{n \geq 0 : \max(Z_0, Z_1 \ldots, Z_n) \leq t\},$$

$$N(t) = \sum_{n=1}^{\infty} I(Z_n \leq t),$$

i. e. $L(t)+1$ is the last-exit time of $\{Z_n,\ n \geq 0\}$ from $(-\infty, t]$, $M(t)+1$ is the first-passage time of $\{Z_n,\ n \geq 0\}$ from $(-\infty, t]$, and $N(t)$ is the total time spent by $\{Z_n,\ n \geq 0\}$ in $(-\infty, t]$, respectively. If the sequence $\{Z_n,\ n \geq 0\}$ increases, then all the three functions coincide. Otherwise they are different and further "natural" definitions of renewal processes can be given.

Under some mild conditions, it is proved in [1] that, if $\lim_{n\to\infty} Z_n / a_n = 1$ almost surely (a. s.), then

$$\lim_{t\to\infty} \frac{L(t)}{a^{-1}(t)} = 1, \qquad \lim_{t\to\infty} \frac{M(t)}{a^{-1}(t)} = 1, \qquad \lim_{t\to\infty} \frac{N(t)}{a^{-1}(t)} = 1 \quad \text{a. s.},$$

where $a_n = a(n)$, $a^{-1}(\cdot)$ is the inverse to $a(\cdot)$, and $a(\cdot)$ is a continuous increasing unbounded function. The main assumption posed on the function $a(\cdot)$ in [1] is that either $g(\cdot) = a(\cdot)$ or $g(\cdot) = a^{-1}(\cdot)$ or both of them (the choice depends on a result desired) satisfy the following condition:

$$\lim_{\varepsilon \downarrow 0} \lim_{t\to\infty} \left| \frac{g((1 \pm \varepsilon)t)}{g(t)} - 1 \right| = 0. \tag{1.1}$$

Note that every regularly varying function $g(\cdot)$ satisfies this condition.

A natural further step is to prove the converse, namely if the strong law of large numbers is satisfied for renewal processes, then so does the strong law of large numbers for the original sequence of random variables. Such results are obtained in [2] for the partial case of sequences of random variables formed by sums of nonnegative independent identically distributed random variables.

Another interesting problem is to obtain similar relationships for the scheme, where

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the starting point is a stochastic process $\{Z(t),\ t \geq 0\}$ instead of a sequence $\{Z_n,\ n \geq 0\}$.

Both of these problems require further properties of normalizing functions $a(\cdot)$ and their inverses $a^{-1}(\cdot)$. As it will be seen later most of those properties are consequences of the main assumption (1.1).

Prof. E. Seneta kindly informed the authors that property (1.1) is related to the notion of the so called ORV functions and provided us with the references [3 – 5] which helped us to discover the origin of and to prove some basic results for functions possessing property (1.1). It turns out that the functions $g(\cdot)$ satisfying (1.1) form a subclass of functions introduced by V. Avakumović [3] and which are natural to call *Avakumović functions*. More detail concerning the Avakumović functions is given by Karamata [6] and Aljančić and Arandelović [4] where those functions are called O-regularly varying (ORV) functions. Bari and Stechkin [7] independently studied Avakumović functions and discussed their applications in the theory of function approximation.

The main aim of this paper is to obtain some key properties of functions satisfying (1.1). Applications of these results to limit theorems of the probability theory are planned to appear elsewhere, however in this paper we indicate some of possible applications to the relationship between strong laws for sequences of random variables and corresponding renewal processes.

2. Definitions and preliminaries. We assume throughout the paper that real-valued functions $g(\cdot) = (g(t),\ t \geq 0)$ are measurable and positive for sufficiently large arguments.

Definition 2.1. *A function* $g(\cdot)$ *is called regularly varying (RV) if the limit*

$$\kappa(c) = \lim_{t \to \infty} \frac{g(ct)}{g(t)}$$

exists for all $c > 0$.

For any RV function $g(\cdot)$, $\kappa(c) = c^{\alpha}$ for some number $\alpha \in \mathbb{R} = (-\infty, \infty)$ which is called the index of the function $g(\cdot)$. Definition 2.1 is due to Karamata [8] who gave in [9] an account of properties of regularly varying functions. Note that RV functions of zero index are called *slowly varying* (SV) functions.

A generalization of RV functions is due to Avakumović [3] and it was Karamata [6] who obtained characteristic properties of such functions. For given function $g(\cdot)$, introduce

$$r(c) = \limsup_{t \to \infty} \frac{g(ct)}{g(t)}, \qquad c > 0.$$

Definition 2.2. *A function* $g(\cdot)$ *is called* O*-regularly varying (ORV) if*

$$r(c) < \infty \quad \textit{for all} \quad c > 0.$$

It is obvious that any RV function is an ORV function. Some subclasses of ORV functions are known in the literature. For example, Drasin and Seneta [5] studied the so-called OSV functions.

Definition 2.3. *A function* $g(\cdot)$ *is called* O*-slowly varying (OSV) if it is an ORV function such that*

$$\sup_{c > 0} r(c) < \infty.$$

It is clear that any SV function is an OSV function.

Another property of $r(\cdot)$ has been used in [1]. Used in [1] the PRV notion, is introduced here in a somewhat different way which makes it unified with the preceding definitions. For any RV function $g(\cdot)$, we have $\kappa(c) = r(c) \to 1$ as $c \to 1$. In order to generalize this property to a wider class of functions we introduce the following definition.

Definition 2.4. *A function* $g(\cdot)$ *is called pseudo regularly varying (PRV) if*

$$\limsup_{c\to 1} r(c) = 1. \tag{2.1}$$

***Remark* 2.1.** We do not assume in Definition 2.4 that the function $g(\cdot)$ is ORV. But this property is easy to show by (2.1), since otherwise there exists $c>0$ for which $r(c)=\infty$. Using the property $r(c)\le \left(r(\sqrt{c})\right)^2$ we get $r(\sqrt{c})=\infty$. Repeating this procedure we obtain a sequence $c_n = c^{1/2^n}$ tending to 1 as $n\to\infty$ and such that $r(c_n)=\infty$, which contradicts (2.1). Therefore any PRV function is an ORV function.

For a given function $g(\cdot)$, we also introduce

$$l(c) = \liminf_{t\to\infty}\frac{g(ct)}{g(t)}, \qquad c>0.$$

It is easy to see that condition (2.1) coincides with the following one:

$$\liminf_{c\to 1} l(c) = 1, \tag{2.2}$$

since $l(c)=1/r(1/c)$ if we agree that $(1/\infty)=0$ and $(1/0)=\infty$.

It turns out that PRV property (2.1) coincides with condition (1.1) which can be rewriten in an equivalent form:

$$\lim_{c\to 1}\limsup_{t\to\infty}\left|\frac{g(ct)}{g(t)}-1\right| = 0. \tag{2.3}$$

Further, the PRV property (2.1) can be expressed only in terms of lim instead of lim sup, that is

$$\lim_{c\to 1} r(c) = 1. \tag{2.4}$$

Finally, the PRV property (2.1) can be expressed in terms of one-sided limits, namely either

$$\lim_{c\downarrow 1} r(c) = \lim_{c\downarrow 1} l(c) = 1 \tag{2.5}$$

or

$$\lim_{c\uparrow 1} r(c) = \lim_{c\uparrow 1} l(c) = 1. \tag{2.6}$$

We summarize all of these equivalences in the following result.

Proposition 2.1. *Conditions* (2.1), (2.2), (2.3), (2.5), (2.6), *and* (2.4) *are equivalent.*

***Proof of Proposition* 2.1.** By (2.2) we have

$$\limsup_{c\to 1}\limsup_{t\to\infty}\left|\frac{g(ct)}{g(t)}-1\right| = \limsup_{c\to 1}\limsup_{t\to\infty}\max\left\{\frac{g(ct)}{g(t)}-1; 1-\frac{g(ct)}{g(t)}\right\} =$$

$$= \limsup_{c\to 1}\max\{r(c)-1; 1-l(c)\} = \max\left\{\limsup_{c\to 1} r(c)-1;\ 1-\liminf_{c\to 1} l(c)\right\} =$$

$$= \max\left\{\limsup_{c\to 1} r(c)-1;\ 1-\left[\limsup_{c\to 1} r(c)\right]^{-1}\right\}.$$

So, conditions (2.1), (2.2), and (2.3) are equivalent. Also, by (2.2), we have (2.5) $\Leftrightarrow$ (2.6) $\Rightarrow$ (2.4) $\Rightarrow$ (2.3).

Now let (2.3) hold. Then (2.1) and (2.2) hold and

$$1 = \liminf_{c \to 1} l(c) \le \liminf_{c \to 1} r(c) \le \limsup_{c \to 1} r(c) = 1.$$

Therefore $\lim_{c \to 1} r(c) = 1$, hat is (2.4) holds. Thus (2.3) $\Rightarrow$ (2.4) and Proposition 2.1 is proved.

***Remark* 2.2.** We often make use of the following result: a function $g(\cdot)$ is not PRV if and only if there are two sequences $\{c_n\}$ and $\{t_n\}$ such that $c_n \to 1$ and $t_n \to \infty$ as $n \to \infty$, but either

$$\limsup_{n \to \infty} \frac{g(c_n t_n)}{g(t_n)} > 1$$

or

$$\liminf_{n \to \infty} \frac{g(c_n t_n)}{g(t_n)} < 1.$$

It is also worth mentioning that the preceding two conditions can be weaken in some extend, namely a function $g(\cdot)$ is not PRV if and only if there are two sequences $\{c_n\}$ and $\{t_n\}$ such that $c_n \to 1$ and $t_n \to \infty$ as $n \to \infty$, but either

$$\limsup_{n \to \infty} \frac{g(c_n t_n)}{g(t_n)} \ne 1$$

or

$$\liminf_{n \to \infty} \frac{g(c_n t_n)}{g(t_n)} \ne 1.$$

***Remark* 2.3.** Proposition 2.1 also given another characterization of the PRV property, namely a function $g(\cdot)$ is PRV if and only if its function $r(\cdot)$ is continuous at the point $c = 1$.

An important case of PRV functions is presented by nondecreasing functions, where the PRV property (2.1) can be expressed only in terms of one-sided limits of either $r(\cdot)$ or $l(\cdot)$.

***Corollary* 2.1.** *A nondecreasing function $g(\cdot)$ is PRV if and only if*

$$\lim_{c \downarrow 1} r(c) = 1. \tag{2.7}$$

The same statement holds for $c \uparrow 1$, namely: A nondecreasing function $g(\cdot)$ is PRV if and only if

$$\lim_{c \uparrow 1} r(c) = 1. \tag{2.8}$$

***Example* 2.1.** The function $g(t) = 2 + (-1)^{[t]}$ is ORV, but is not PRV.

***Example* 2.2.** Let α be a real number. The function

$$g(t) = \begin{cases} 0 & \text{for } t = 0; \\ t^{\alpha} \exp\{\sin(\ln t)\} & \text{for } t > 0, \end{cases}$$

is PRV and is not RV.

***Example* 2.3.** The function

$$g(t) = \begin{cases} 1 & \text{for } t \in [0, 1); \\ 2^k & \text{for } t \in \left[2^{2k}, 2^{2k+1}\right),\ k = 0, 1, 2, \ldots; \\ t/2^{k+1} & \text{for } t \in \left[2^{2k+1}, 2^{2(k+1)}\right),\ k = 0, 1, 2, \ldots. \end{cases}$$

is PRV, but is not RV.

3. Functions preserving equivalences. In this section, functions $u(\cdot)$ and $v(\cdot)$ are nonnegative and positive for sufficiently large arguments.

Definition 3.1. *Two functions $u(\cdot)$ and $v(\cdot)$ are called equivalent if*

$$\lim_{t \to \infty} \frac{u(t)}{v(t)} = 1. \tag{3.1}$$

The equivalence of functions is denoted by $u \sim v$.

It is natural to study the functions $g(\cdot)$ preserving this relation, in other word, to investigate which functions $g(\cdot)$ satisfy the following condition

$$\lim_{t \to \infty} \frac{g(u(t))}{g(v(t))} = 1 \tag{3.2}$$

for all equivalent functions $u(\cdot)$ and $v(\cdot)$? If $u(t) \to u_0$ as $t \to \infty$, then so does $v(\cdot)$, namely $v(t) \to u_0$ as $t \to \infty$. It is then clear that any function $g(\cdot)$, continuous at u_0, preserves equivalence of those $u(\cdot)$ and $v(\cdot)$. Varying u_0 and considering corresponding $u(\cdot)$ and $v(\cdot)$ we prove that $g(\cdot)$ should be continuous on the semi axis in order to preserve (3.1). A different case arises if

$$\lim_{t \to \infty} u(t) = \infty \quad \text{and} \quad \lim_{t \to \infty} v(t) = \infty. \tag{3.3}$$

There are other cases, where $u(\cdot)$ and $v(\cdot)$ do not have limits but vary in an agreed way such that (3.1) holds. However in what follows we restrict ourselves to the case (3.3).

Definition 3.2. *A function $g(\cdot)$ preserves the equivalence of functions if* (3.2) *holds for all equivalent functions $u(\cdot)$ and $v(\cdot)$ satisfying* (3.3). *In other words, a function $g(\cdot)$ preserves equivalence of functions if $g \circ u \sim g \circ v$ for all $u(\cdot)$ and $v(\cdot)$ such that $u \sim v$ and* (3.3) *holds.*

In a similar way, one can introduce the notion of functions $g(\cdot)$ preserving equivalence of sequences. All the sequences $\{u_n,\ n \geq 0\}$ and $\{v_n,\ n \geq 0\}$ below are assumed to be nonnegative and positive for sufficiently large indices.

Definition 3.3. *Two sequences $\{u_n,\ n \geq 0\}$ and $\{v_n,\ n \geq 0\}$ are called equivalent if*

$$\lim_{n \to \infty} \frac{u_n}{v_n} = 1. \tag{3.4}$$

Equivalent sequences $\{u_n,\ n \geq 0\}$ and $\{v_n,\ n \geq 0\}$ are denoted by $u \sim v$.

Definition 3.4. *A function $g(\cdot)$ preserves equivalence of sequences if*

$$\lim_{n \to \infty} \frac{g(u_n)}{g(v_n)} = 1 \tag{3.5}$$

for all equivalent sequences $\{u_n,\ n \geq 0\}$ and $\{v_n,\ n \geq 0\}$ such that

$$\lim_{n \to \infty} u_n = \infty \quad \text{and} \quad \lim_{n \to \infty} v_n = \infty. \tag{3.6}$$

One of the most important properties of PRV functions is that they and only they preserve equivalence of both functions and sequences.

Theorem 3.1. *The following three conditions are equivalent:*

(a) *a function $g(\cdot)$ preserves equivalence of functions;*

(b) *a function $g(\cdot)$ preserves equivalence of sequences;*

(c) *a function $g(\cdot)$ is PRV.*

***Remark* 3.1.** All the conditions of Theorem 3.1 are equivalent to the following one (which seems to be the most useful in applications):

(d) *a function* $g(\cdot)$ *preserves equivalence of continuous increasing to infinity functions.*

***Proof of Theorem* 3.1.** The equivalence (a) $\Leftrightarrow$ (b) is trivial.

If (b) does not hold, then there exist two sequences $\{s_n, n \ge 0\}$ and $\{t_n, n \ge 0\}$ such that $s_n \to \infty$, $t_n \to \infty$, and $c_n = (s_n / t_n) \to 1$ as $n \to \infty$, but either

$$\limsup_{n\to\infty} \frac{g(s_n)}{g(t_n)} = \limsup_{n\to\infty} \frac{g(c_n t_n)}{g(t_n)} \ne 1$$

or

$$\liminf_{n\to\infty} \frac{g(s_n)}{g(t_n)} = \liminf_{n\to\infty} \frac{g(c_n t_n)}{g(t_n)} \ne 1.$$

By Remark 2.2 this means that condition (c) does not hold, whence the implication (c) $\Rightarrow$ (b) follows.

Now if (c) is not satisfied and (b) holds, then by Remark 2.2 there exist two sequences $\{c_n, n \ge 0\}$ and $\{t_n, n \ge 0\}$ such that $c_n \to 1$ and $t_n \to \infty$ as $n \to \infty$ but either

$$\limsup_{n\to\infty} \frac{g(c_n t_n)}{g(t_n)} \ne 1 \quad \text{or} \quad \liminf_{n\to\infty} \frac{g(c_n t_n)}{g(t_n)} \ne 1.$$

On the other hand $\lim_{n\to\infty} (c_n t_n)/t_n = 1$ and by (b) $\lim_{n\to\infty} g(c_n t_n)/g(t_n) = 1$. This contradiction shows that (b) $\Rightarrow$ (c). Theorem 3.1 is completely proved.

4. Representation theorems. There are two basic result on RV functions, namely the representation theorem and the uniform convergence theorem. As it is pointed [10] these results are in fact equivalent. Several proofs for RV functions are known in the literature (see, for example, [11]).

For ORV functions the representation theorem is proved in [6] (also see [4]) and the uniform convergence theorem has been obtained in [4]. Our current goal is to obtain a representation for PRV functions in the manner of Karamata's theory of RV functions. We recall the representation theorem for RV functions: *a function* $g(\cdot)$ *is RV if and only if*

$$g(t) = \exp\left\{ \alpha(t) + \int_{t_0}^{t} \beta(s) \frac{ds}{s} \right\} \tag{4.1}$$

for some $t_0 > 0$ *and all* $t \ge t_0$, *where* $\alpha(\cdot)$ *and* $\beta(\cdot)$ *are bounded functions such that the limits*

$$\lim_{t\to\infty} \alpha(t) \quad \text{and} \quad \lim_{t\to\infty} \beta(t)$$

exist.

For SV functions, one additionally has $\lim_{t\to\infty} \beta(t) = 0$.

In the case of OSV functions it is proved in [5] that they also have the characterization representation (4.1), where $\alpha(\cdot)$ and $\beta(\cdot)$ are bounded functions such that $\lim_{t\to\infty} \beta(t) = 0$.

ORV functions also have the same characterization representation (4.1), where $\alpha(\cdot)$ and $\beta(\cdot)$ are bounded functions (see [6] or [4]).

Note that all of these representations are not unique. For example, one can start from a discontinuous function $\beta(\cdot)$ and obtain the same representation with other functions $\tilde{\alpha}(\cdot)$ and $\tilde{\beta}(\cdot)$, where $\tilde{\beta}(\cdot)$ is continuous or even differentiable as many times as one wants.

The proof of the representation of PRV functions is based on that for ORV functions (see [4]).

Theorem 4.1. *A function* $g(\cdot)$ *is PRV if and only if it has representation* (4.1), *where* $\alpha(\cdot)$ *and* $\beta(\cdot)$ *are bounded functions such that*

$$\lim_{c \to 1} \limsup_{t \to \infty} |\alpha(ct) - \alpha(t)| = 0. \tag{4.2}$$

***Remark* 4.1.** Condition (4.2) characterizes the so-called *slowly oscillating* functions (see [11]). Using this notion, Theorem 4.1 can be stated as follows: *a function* $g(\cdot)$ *is PRV if and only if it has representation* (4.1), *where* $\alpha(\cdot)$ *is bounded and slowly oscillating and* $\beta(\cdot)$ *is bounded.*

***Proof of Theorem* 4.1.** Let $g(\cdot)$ be a PRV function. Since it is an ORV function, representation (4.1) holds for it, where $\alpha(\cdot)$ and $\beta(\cdot)$ are bounded. It only remains to show that condition (4.2) is satisfied. Note that

$$\left|\int_t^{ct} \beta(s)\frac{ds}{s}\right| \le \ln(c) \operatorname*{ess\,sup}_{t \ge t_0} |\beta(t)|,$$

whence in view of the boundness of $\beta(\cdot)$

$$\lim_{c \to 1} \limsup_{t \to \infty} \left|\int_t^{ct} \beta(s)\frac{ds}{s}\right| = 0.$$

If (4.2) is not satisfied, then there are a sequence $\{c_k\}$ such that $c_k \downarrow 1$ as $k \to \infty$ and $\delta > 0$ for which

$$\limsup_{t \to \infty} |\alpha(c_k t) - \alpha(t)| \ge \delta \quad \text{for all } k.$$

This means that for any k, there exists a sequence $\{t_{ki}\}$ such that $t_{ki} \uparrow \infty$ as $i \to \infty$ and either

$$\alpha(c_k t_{ki}) - \alpha(t_k) \ge \frac{\delta}{2} \quad \text{for all } i, \tag{4.3}$$

or

$$\alpha(c_k t_{ki}) - \alpha(t_k) \le -\frac{\delta}{2} \quad \text{for all } i. \tag{4.4}$$

Condition (4.3) and representation (4.1) imply that

$$\frac{g(c_k t_{ki})}{g(t_{ki})} \ge \exp\left\{\frac{\delta}{2} + \int_{t_{ki}}^{c_k t_{ki}} \beta(s)\frac{ds}{s}\right\} \to \exp\{\delta/2\} \quad \text{as } i, k \to \infty,$$

which contradicts the PRV property (2.1). Similarly, (4.4) also contradicts (2.1), which completes the proof of (4.2).

On the other hand, any function $g(\cdot)$ possessing representation (4.1), where $\alpha(\cdot)$ and $\beta(\cdot)$ are bounded and (4.2) holds, is a PRV function. Indeed

$$\frac{g(ct)}{g(t)} = \exp\{\alpha(c_k t) - \alpha(t)\} \exp\left\{\int_t^{ct} \beta(s)\frac{ds}{s}\right\}$$

and the second factor tends to 1 as $t \to \infty$ and then $c \to 1$, in view of the boundedness of $\beta(\cdot)$. The first factor approaches 1 as $t \to \infty$ and then $c \to 1$ by condition (4.2). Theorem 4.1 is proved.

Another representation for PRV functions is based on that for RV functions.

Theorem 4.2. *A function* $g(\cdot)$ *is PRV if and only if*

$$g(t) = \exp\left\{a(t) + \int_{t_0}^{t} b(u)\,du\right\} \tag{4.5}$$

for some $t_0 > 0$ *and all* $t \ge t_0$ *where the functions* $a(\cdot)$ *and* $b(\cdot)$ *are such that the limit* $\lim_{t\to\infty} a(t)$ *exists,* $\lim_{t\to\infty} b(t) = 0$, *and*

$$\lim_{c\to 1}\limsup_{t\to\infty}\int_{t}^{ct} b(u)\,du = \lim_{c\to 1}\liminf_{t\to\infty}\int_{t}^{ct} b(u)\,du = 0.$$

Proof. Let a function $g(\cdot)$ be PRV. Then $g(\ln(\cdot))$ is a slowly varying function, that is, a RV function of zero index. We use representation (4.1) for $g(\ln(\cdot))$:

$$g(\ln(t)) = \exp\left\{\alpha(t) + \int_{t_0}^{t} \beta(s)\frac{ds}{s}\right\},$$

where the limit $\lim_{t\to\infty}\alpha(t)$ exists and $\lim_{t\to\infty}\beta(t) = 0$. Therefore representation (4.5) holds with $a(t) = \alpha(e^t)$ and $b(t) = \beta(e^t)$. It is easy to see that the functions $a(\cdot)$ and $b(\cdot)$ satisfy the conditions of Theorem 4.2.

The converse statement is also easy to prove.

***Remark* 4.2.** The key point in the proof of Theorem 4.2 is the property that $g \circ \ln$ is SV for any PRV function $g(\cdot)$. The example $g(t) = \exp\{\sqrt{t}\}$ shows that the converse to this property is not true. Also $g \circ \ln$ is not necessarily SV if $g(\cdot)$ is PRV (see $g(\cdot)$ of Example 2.1). However one can prove that $g \circ \ln$ is OSV in this case. Again the converse is not true (confer the function $g(t) = \exp\{\sqrt{t}\}$).

5. Uniform convergence theorem. We use the method introduced in [10] for RV functions and applied in [4] for ORV functions.

Theorem 5.1. *Let* $g(\cdot)$ *be a PRV function. Then*

$$\lim_{a\downarrow 1}\limsup_{t\to\infty}\sup_{1\le c\le a}\frac{g(ct)}{g(t)} = 1. \tag{5.1}$$

Proof. Condition (5.1) can be given in an equivalent form in terms of the function $\alpha(\cdot)$ involved in the representation (4.1), namely

$$\lim_{a\downarrow 1}\limsup_{t\to\infty}\sup_{1\le c\le a}|\alpha(ct)-\alpha(t)| = 0. \tag{5.2}$$

Here we prove (5.2). Set $a(t) = \alpha(e^t)$. Then $a(x+\mu) - a(x) = \alpha(e^{\mu}e^{x}) - \alpha(e^{x})$ and

$$\lim_{\mu\downarrow 0}\limsup_{x\to\infty}|a(x+\mu)-a(x)| = 0. \tag{5.3}$$

We show that

$$\lim_{\varepsilon\downarrow 0}\limsup_{x\to\infty}\sup_{0\le\mu\le\varepsilon}|a(x+\mu)-a(x)| = 0. \tag{5.4}$$

Otherwise, there exists $\delta > 0$ and sequences $\{x_n\}$ and $\{\mu_n\}$ such that $x_n \uparrow \infty$ and $\mu_n \downarrow 0$ as $n \to \infty$, and

$$|a(x_n+\mu_n)-a(x_n)| \ge \delta \quad \text{for all } n. \tag{5.5}$$

By condition (5.3) there is $\mu_0 > 0$ such that

$$\limsup_{x\to\infty} |a(x+\mu)-a(x)| \le \frac{\delta}{3} \quad \text{for all } 0 \le \mu \le \mu_0. \tag{5.6}$$

Define sets

$$U_n = \left\{ \mu \in [0, \mu_0] : |a(x_k+\mu)-a(x_k)| < \frac{\delta}{2} \quad \text{for all} \quad k \ge n \right\}, \tag{5.7}$$

$$V_n = \left\{ \lambda \in [0, \mu_0] : |a(x_k+\mu_k+\lambda)-a(x_k+\mu_k)| < \frac{\delta}{2} \quad \text{for all} \quad k \ge n \right\}. \tag{5.8}$$

Now $U_n \uparrow [0, \mu_0]$ and $V_n \uparrow [0, \mu_0]$ as $n \to \infty$. Moreover

$$\text{meas}(U_n) > \frac{3}{4}\mu_0, \qquad \text{meas}(V_n) > \frac{3}{4}\mu_0 \quad \text{for all} \quad n \ge n_0. \tag{5.9}$$

Set $V_n' = V_n + \mu_n$, so that $\text{meas}(V_n') = \text{meas}(V_n)$ and assume $\mu_n \in [0, \mu_0/2]$ for $n \ge n_0$. Note that $\text{meas}(V_n') = \text{meas}(V_n) > \frac{3}{4}\mu_0$, and

$$U_n \subset [0, \mu_0] \subset [0, 3\mu_0/2] \quad \text{for all} \quad n,$$

$$V_n' \subset [0, \mu_0] + \mu_n \subset [0, 3\mu_0/2] \quad \text{for all} \quad n \ge n_0.$$

So, in view of (5.9), $U_n \cap V_n' \ne \varnothing$. Thus there is $\mu_n' \in U_n$ such that $\mu_n' - \mu_n \in V_n$. Now by (5.7) and (5.8)

$$|a(x+\mu_n')-a(x_n)| < \frac{\delta}{2}, \qquad |a(x+\mu_n')-a(x_n+\mu_n)| < \frac{\delta}{2}.$$

By the triangle inequality, this implies $|a(x+\mu_n)-a(x_n)| < \delta$ which contradicts (5.5).

6. Quasiinverse functions satisfying the PRV property. We have mentioned in the Introduction that the PRV property is sometimes required for inverse functions to study relationships between limit theorems for sequences of random variables and those for their corresponding renewal processes. One can express the PRV property for an inverse function by putting $g^{-1}(\cdot)$ into condition (2.1). However, it is useful to reformulate this condition in terms of the function $g(\cdot)$ itself. In doing so we take nondecreasing functions into consideration, too. Since the inverse function is not necessarily exists for nondecreasing $g(\cdot)$, one has to consider a generalized inverse function in this case.

Let $\mathbb{F}^{(\infty)}(\mathbb{R}_0)$ be the space of real-valued functions $f(\cdot) = (f(t),\ t \ge 0)$ such that:

(i) $\sup_{0 \le t \le T} f(t) < \infty$ for all $T > 0$;

(ii) $\limsup_{t\to\infty} f(t) = \infty$;

(iii) there exists a number $s_0 = s_0(f) \ge 0$ such that $\mathcal{M}_s = \{t \ge 0 : f(t) = s\} \ne \varnothing$ for all $s \ge s_0$.

Also we introduce the following notations: $C^{(\infty)}(\mathbb{R}_0)$ for the space of real-valued continuous functions $(f(t),\ t \ge 0)$ such that $\limsup_{t\to\infty} f(t) = \infty$; $C^{\infty}(\mathbb{R}_0)$ for the space of real-valued continuous functions $(f(t),\ t \ge 0)$ such that $\lim_{t\to\infty} f(t) = \infty$; $C^{\infty}_{\text{ndec}}(\mathbb{R}_0)$ for the space of functions $f(\cdot) \in C^{(\infty)}(\mathbb{R}_0)$ such that functions $f(\cdot)$ is nondecreasing for large t; $C^{\infty}_{\text{inc}}(\mathbb{R}_0)$ for the space of functions $f(\cdot) \in C^{(\infty)}(\mathbb{R}_0)$ such that functions $f(\cdot)$ is strictly increasing for large t.

Now we introduce the notion of quasiinverse functions which is suitable for our goal.

Definition 6.1. *Let* $f(\cdot)\in\mathbb{F}^{(\infty)}(\mathbb{R}_0)$. *A function* $f^{(-1)}(\cdot)=\left(f^{(-1)}(s),\ s\geq 0\right)$ *is called quasiinverse function for* $f(\cdot)$ *if*

(i) $f^{(-1)}(\cdot)$ *is nondecreasing;*

(ii) $f^{(-1)}(s)\to\infty$, *as* $s\to\infty$;

(iii) *there exists a number* $s_0\geq 0$ *such that* $f\left(f^{(-1)}(s)\right)=s$ *for all* $s\geq s_0$.

For any $f(\cdot)\in C^{(\infty)}(\mathbb{R}_0)$, a quasiinverse function exists and, possibly, it is not unique. If $f(\cdot)\in C^{\infty}_{\mathrm{inc}}(\mathbb{R}_0)$ then there exists the inverse function $f^{-1}(\cdot)$ such that $f\left(f^{-1}(s)\right)=s$ and $f^{-1}(f(t))=t$ for sufficiently large s and t.

***Example* 6.1.** Let $x(\cdot)\in C^{(\infty)}(\mathbb{R}_0)$. Put $x_1^{(-1)}(s)=\min\{t\geq 0\colon x(t)=s\}$ for $s\geq$ $\geq s_0=x(0)$, and put $x_1^{(-1)}(s)=0$ for $0\leq s<s_0$ if $s_0>0$.

The function $x_1^{(-1)}(\cdot)=\left(x_1^{-1}(s),\ s\geq 0\right)$ is a quasiinverse for $x(\cdot)$.

***Example* 6.2.** Let $x(\cdot)\in C^{\infty}(\mathbb{R}_0)$. Put $x_2^{(-1)}(s)=\max\{t\geq 0\colon x(t)=s\}$ for $s\geq$ $\geq s_0=x(0)$, and put $x_2^{(-1)}(s)=0$ for $0\leq s<s_0$ if $s_0>0$.

The function $x_2^{(-1)}(\cdot)=\left(x_2^{(-1)}(s),\ s\geq 0\right)$ is a quasiinverse for $x(\cdot)$. Observe that $x_1^{(-1)}(s)\leq x_2^{(-1)}(s)$, $s_0>0$, and in general $x_1^{(-1)}(\cdot)\neq x_2^{(-1)}(\cdot)$.

Lemma 6.1. *Let* $g(\cdot)\in\mathbb{F}^{(\infty)}(\mathbb{R}_0)$. *Then its quasiinverse function* $g^{(-1)}(\cdot)$ *is PRV if and only if*

$$\lim_{c\downarrow 1}\limsup_{t\to\infty}\frac{g^{(-1)}(ct)}{g^{(-1)}(t)} = 1. \tag{6.1}$$

Proof. This lemma follows from Corollary 2.1.

Proposition 6.1. *Let* $g(\cdot)\in C^{\infty}_{\mathrm{ndec}}(\mathbb{R}_0)$ *and* $g^{(-1)}(\cdot)$ *be a quasiinverse function for* $g(\cdot)$. *If*

$$\liminf_{t\to\infty}\frac{g(ct)}{g(t)} > 1 \quad \text{for all} \quad c>1, \tag{6.2}$$

then the function $g^{(-1)}(\cdot)$ *is PRV.*

Proof. First we assume that condition (6.2) holds and (6.1) is not satisfied. If (6.1) is not satisfied, then there exist a number $\delta>0$ and sequences $\{c_n\}$ and $\{s_n\}$ such that $c_n\downarrow 1$ and $s_n\uparrow\infty$ as $n\to\infty$ and

$$g^{(-1)}(c_n s_n) > (1+\delta)g^{(-1)}(s_n) \quad \text{for} \quad n\geq 1.$$

In its turn condition (6.2) implies that for the above δ there exists a number $\gamma>1$ such that

$$g((1+\delta)t) > \gamma g(t) \quad \text{for sufficiently large} \quad t.$$

Therefore for sufficiently large n

$$c_n s_n = g\left(g^{(-1)}(c_n s_n)\right) \geq g\left((1+\delta)g^{(-1)}(s_n)\right) > \gamma g\left(g^{(-1)}(s_n)\right) = \gamma s_n,$$

whence $c_n\geq\gamma>1$. This contradiction proves the implication (6.2) $\Rightarrow$ (6.1). Therefore Proposition 6.1 is proved by Lemma 6.1.

Theorem 6.1. *Let* $g(\cdot)\in C^{\infty}_{\mathrm{inc}}(\mathbb{R}_0)$. *Then its inverse function* $g^{-1}(\cdot)$ *is PRV if and only if condition* (6.2) *holds.*

Proof. By Proposition 6.1, (6.2) implies that $g^{-1}(\cdot)$ is PRV.

Now we assume that $g^{-1}(\cdot)$ is PRV. If nevertheless (6.2) is not satisfied, then there exists a number $c_0 > 1$ such that $\liminf_{t\to\infty} g(c_0 t)/g(t) = 1$, whence we get that there exists a sequence $\{t_n\}$ such that $t_n \uparrow \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{g(c_0 t_n)}{g(t_n)} = 1.$$

This implies that the sequences $u_n = g(c_0 t_n)$ and $v_n = g(t_n)$ are equivalent and satisfy condition (3.6). By Theorem 3.1 $g^{-1}(\cdot)$ preserves equivalence of sequences and therefore

$$1 = \lim_{n\to\infty} \frac{g^{-1}(u_n)}{g^{-1}(v_n)} = \lim_{n\to\infty} \frac{c_0 t_n}{t_n} = c_0 > 1.$$

This contradiction proves (6.2). Theorem 6.1 is proved.

***Remark* 6.1.** The above theorem allows for a characterization of inverses of RV functions. Namely let $g(\cdot)$ be a RV function of an index α. Then $g^{-1}(\cdot)$ is PRV if and only if $\alpha \neq 0$.

***Corollary* 6.1.** *Let* $g(\cdot) \in C^{\infty}_{\text{inc}}(\mathbb{R}_0)$. *Then it is PRV if and only if*

$$\liminf_{t\to\infty} \frac{g^{-1}(ct)}{g^{-1}(t)} > 1 \quad \text{for all} \quad c > 1.$$

The following results shows that the limit behavior of the ratio of inverse functions is the same as that of the original functions.

Proposition 6.2. *Assume that* $x(\cdot) \in \mathbb{F}^{(\infty)}(\mathbb{R}_0)$ *and* $g(\cdot) \in C^{\infty}_{\text{inc}}(\mathbb{R}_0)$. *Moreover let condition* (6.2) *be satisfied. If*

$$\lim_{t\to\infty} \frac{x(t)}{g(t)} = a \quad \text{for some} \quad a \in (0, \infty), \tag{6.3}$$

then for any quasiinverse function $x^{(-1)}(\cdot)$ *we have*

$$\lim_{s\to\infty} \frac{x^{(-1)}(s)}{g^{-1}(s/a)} = 1. \tag{6.4}$$

Proof. Since $x^{(-1)}(\cdot)$ is nondecreasing and unbounded, by (6.3) we have

$$\lim_{s\to\infty} \frac{x\left(x^{(-1)}(s)\right)}{g\left(x^{(-1)}(s)\right)} = a$$

and thus

$$\lim_{s\to\infty} \frac{g\left(x^{(-1)}(s)\right)}{s/a} = 1.$$

By Proposition 6.1 and condition (6.2) we obtain

$$\lim_{s\to\infty} \frac{g^{-1}\left(g\left(x^{(-1)}(s)\right)\right)}{g^{-1}(s/a)} = 1.$$

Therefore

$$\lim_{s\to\infty} \frac{x^{(-1)}(s)}{g^{-1}(s/a)} = \lim_{s\to\infty} \frac{g^{-1}\left(g\left(x^{(-1)}(s)\right)\right)}{g^{-1}(s/a)} = 1.$$

Proposition 6.2 can also be proved for zero and infinite limits. However, an additional condition is required in this case.

Proposition 6.3. *Let* $g(\cdot) \in C_{\text{inc}}^{\infty}(\mathbb{R}_0)$ *and*

$$\liminf_{t \to \infty} \frac{g^{-1}(c_0 t)}{g^{-1}(t)} > 1 \tag{6.5}$$

for some $c_0 > 1$. *Assume that* $x(\cdot) \in \mathbb{F}^{(\infty)}(\mathbb{R}_0)$ *and* $x^{(-1)}(\cdot)$ *is its quasiinverse. If condition* (6.5) *holds, then the following relations are satisfied*:

$$\lim_{t \to \infty} \frac{x(t)}{g(t)} = \infty \Rightarrow \lim_{s \to \infty} \frac{x^{(-1)}(s)}{g^{-1}(s)} = 0, \tag{6.6}$$

$$\lim_{t \to \infty} \frac{x(t)}{g(t)} = 0 \Rightarrow \lim_{s \to \infty} \frac{x^{(-1)}(s)}{g^{-1}(s)} = \infty. \tag{6.7}$$

Proof. First we prove that

$$\lim_{c \to \infty} \hat{l}(c) = \infty \quad \text{and} \quad \lim_{c \to 0} \hat{r}(c) = 0, \tag{6.8}$$

where $\hat{l}(c) = \liminf_{t \to \infty} \left(g^{-1}(ct) / g^{-1}(t) \right)$ and $\hat{r}(c) = \limsup_{t \to \infty} \left(g^{-1}(ct) / g^{-1}(t) \right)$. Indeed

$$\hat{l}_{\infty} = \liminf_{c \to \infty} \hat{l}(c) = \liminf_{c \to \infty} \hat{l}(c^2) \geq \left(\liminf_{c \to \infty} \hat{l}(c) \right)^2 = \hat{l}_{\infty}^2$$

and the first relation in (6.8) follows by condition (6.5). The second one follows by $\hat{r}(c) = 1/\hat{l}(1/c)$.

Put $a(t) = x(t)/g(t)$ for $t > 0$. Then for sufficiently large $s > 0$

$$\frac{g^{-1}\left(a\left(x^{(-1)}(s)\right)g\left(x^{(-1)}(s)\right)\right)}{g^{-1}(s)} =$$

$$= \frac{g^{-1}\left(a\left(x^{(-1)}(s)\right)g\left(x^{(-1)}(s)\right)\right)}{x^{(-1)}(s)} \cdot \frac{x^{(-1)}(s)}{g^{-1}(s)} = 1. \tag{6.9}$$

It follows from (6.6) that

$$\liminf_{s \to \infty} \frac{g^{-1}\left(a\left(x^{(-1)}(s)\right)g\left(x^{(-1)}(s)\right)\right)}{x^{(-1)}(s)} \geq \liminf_{t \to \infty} \frac{g^{-1}(a(t)g(t))}{t} =$$

$$= \liminf_{t \to \infty} \frac{g^{-1}(a(t)t)}{g^{-1}(t)} \geq \lim_{c \to \infty} \liminf_{t \to \infty} \frac{g^{-1}(ct)}{g^{-1}(t)}.$$

By (6.8)

$$\lim_{c \to \infty} \liminf_{t \to \infty} \frac{g^{-1}(ct)}{g^{-1}(t)} = \infty.$$

Therefore

$$\liminf_{s \to \infty} \frac{g^{-1}\left(a\left(x^{(-1)}(s)\right)g\left(x^{(-1)}(s)\right)\right)}{x^{(-1)}(s)} = \infty$$

and in view of (6.9)

$$\limsup_{s\to\infty} \frac{x^{(-1)}(s)}{g^{-1}(s)} = 0.$$

Thus (6.6) is proved.

Similar reasonings prove (6.7).

For RV functions, the above results specialize as follows.

Corollary 6.2. *Let an increasing continuous RV function* $g(\cdot)$ *be of a positive index* α *and* $x(\cdot)\in \mathbb{F}^{(\infty)}(\mathbb{R}_0)$. *If*

$$\lim_{t\to\infty} \frac{x(t)}{g(t)} = a \in [0,\infty],$$

then

$$\lim_{s\to\infty} \frac{x^{(-1)}(s)}{g^{-1}(s)} = \left(\frac{1}{a}\right)^{1/\alpha}$$

for any quasiinverse function $x^{(-1)}(\cdot)$. *Here we assume that* $(1/\infty) = 0$ *and* $(1/0) = \infty$.

The preceding results allow us to describe the class of increasing unbounded functions $g(\cdot)$ for which the following relation holds:

$$x(\cdot) \sim g(\cdot) \Rightarrow x^{(-1)}(\cdot) \sim g^{-1}(\cdot) \qquad (6.10)$$

for $x(\cdot)\in \mathbb{F}^{(\infty)}(\mathbb{R}_0)$.

Theorem 6.2. *Let* $g(\cdot)\in C^{\infty}_{\text{inc}}(\mathbb{R}_0)$. *Then relation* (6.10) *holds if and only if condition* (6.2) *is satisfied.*

Proof. Let relation (6.10) hold. Consider the function $I(t) = t$, $t \ge 0$. Let $u(\cdot)$ and $v(\cdot)$ be equivalent continuous increasing unbounded functions.

Then

$$u \sim v \Leftrightarrow u\circ v^{-1} \sim I \Leftrightarrow u\circ v^{-1}\circ g \sim g.$$

By (6.10) this implies that

$$(u\circ v^{-1}\circ g)^{-1} \sim g^{-1} \Leftrightarrow g^{-1}\circ v\circ u^{-1} \sim g^{-1} \Leftrightarrow g^{-1}\circ v \sim g^{-1}\circ u.$$

This means that $g^{-1}(\cdot)$ preserves equivalence on the space of continuous increasing unbounded functions. By Remark 3.1 and Theorem 3.1 this implies that the function $g^{-1}(\cdot)$ is PRV and by Theorem 6.1 condition (6.2) holds.

The converse statement is proved in Proposition 6.2.

Theorem 6.2 can easily be converted for the following relation:

$$x(\cdot) \sim g^{-1}(\cdot) \Rightarrow x^{(-1)}(\cdot) \sim g(\cdot) \qquad (6.11)$$

for $x(\cdot)\in \mathbb{F}^{(\infty)}(\mathbb{R}_0)$.

Theorem 6.3. *Let* $g(\cdot)\in C^{\infty}_{\text{inc}}(\mathbb{R}_0)$. *Then relation* (6.11) *holds if and only if* $g(\cdot)$ *is PRV. Another criteria for* (6.11) *is*

$$\liminf_{t\to\infty} \frac{g^{-1}(ct)}{g^{-1}(t)} > 1 \quad \textit{for all} \quad c > 1.$$

As we have seen above condition (6.2) plays an important role in obtaining relations of the form (6.10). For a RV function, it means that the index of the function is positive. We use condition (6.2) to introduce the class of PRV functions similar to the class of RV functions of positive indices.

Definition 6.2. *A PRV function $g(\cdot)$ is said to have positive order of variation POV if it satisfies condition* (6.2).

Any slowly varying function $g(\cdot)$ is not POV, so does any fastly increasing function, say $g(t) = e^t$. On the other hand, any RV function of a positive index is POV. Example 2.3 above presents a PRV function which is not RV and is not POV. Example 2.2 gives a PRV function which is not RV but is POV.

Just for the sake of completeness we give the following characterization result.

Theorem 6.4. *Let $g = g(\cdot)$ be a continuous increasing unbounded function. The following five conditions are equivalent:*

(a) *g is POV;*

(b) *g^{-1} is POV;*

(c) *both g and g^{-1} are PRV;*

(d) *both g and g^{-1} preserves equivalence of functions;*

(e) *relations* (6.10) *and* (6.11) *hold.*

Theorem 6.4 follows from Theorem 3.1, Theorem 6.1, Theorem 6.2, Theorem 6.3, and Lemma 6.1.

7. Piecewise linear interpolations and their applications.

Definition 7.1. *The function*

$$\hat{x}(t) = ([t]+1-t)x_{[t]} + (t-[t])x_{[t]+1}, \qquad t \ge 0,$$

is called the piecewise linear interpolation of the sequence $\{x_n, n \ge 0\}$.

Definition 7.2. *The function*

$$\hat{x}(t) = ([t]+1-t)x([t]) + (t-[t])x([t]+1), \qquad t \ge 0,$$

is called the piecewise linear interpolation of the function $x(\cdot)$.

Lemma 7.1.

(a) *If a function $g(\cdot)$ is PRV, then $\hat{g}(\cdot)$ also is PRV and $g \sim \hat{g}$.*

(b) *If a function $g(\cdot)$ is POV, then $\hat{g}(\cdot)$ also is POV and $g \sim \hat{g}$.*

Proof. Let $g(\cdot)$ be PRV. Since it preserves equivalence of functions (see Theorem 3.1),

$$\lim_{t\to\infty} \frac{g([t])}{g(t)} = \lim_{t\to\infty} \frac{g([t]+1)}{g(t)} = 1.$$

Moreover

$$\left|\frac{\hat{g}(t)}{g(t)} - 1\right| \le \left|\frac{g([t])}{g(t)} - 1\right| + \left|\frac{g([t]+1)}{g(t)} - 1\right|.$$

Therefore

$$\lim_{t\to\infty}\left|\frac{\hat{g}(t)}{g(t)} - 1\right| = 0,$$

that is $\hat{g} \sim g$, and therefore

$$\limsup_{c\to 1}\limsup_{t\to\infty}\frac{\hat{g}(ct)}{\hat{g}(t)} = \limsup_{c\to 1}\limsup_{t\to\infty}\frac{g(ct)}{g(t)} = 1.$$

This means that $\hat{g}(\cdot)$ is PRV.

Statement (b) follows form (a).

Lemma 7.2. *Let $\{x_n, n \ge 0\}$ and $\{g_n, n \ge 0\}$ be two sequences with $g_n > 0$ for large n. The following equivalences hold:*

$$\lim_{n\to\infty}\frac{x_n}{g_n} = 1 \Leftrightarrow \lim_{t\to\infty}\frac{\hat{x}(t)}{\hat{g}(t)} = 1,$$

$$\lim_{n\to\infty}\frac{x_n}{g_n} = 0 \Leftrightarrow \lim_{t\to\infty}\frac{\hat{x}(t)}{\hat{g}(t)} = 0,$$

$$\lim_{n\to\infty}\frac{x_n}{g_n} = \infty \Leftrightarrow \lim_{t\to\infty}\frac{\hat{x}(t)}{\hat{g}(t)} = \infty.$$

Proof. Implications „$\Leftarrow$" are trivial.
Implications „$\Rightarrow$" follows from the estimates:

$$\left|\frac{\hat{x}(t)}{\hat{g}(t)} - 1\right| \le \left|\frac{x_{[t]}}{g_{[t]}} - 1\right| + \left|\frac{x_{[t]+1}}{g_{[t]+1}} - 1\right|,$$

$$\min\left\{\frac{x_{[t]}}{g_{[t]}}, \frac{x_{[t]+1}}{g_{[t]+1}}\right\} \le \frac{\hat{x}(t)}{\hat{g}(t)} \le \max\left\{\frac{x_{[t]}}{g_{[t]}}, \frac{x_{[t]+1}}{g_{[t]+1}}\right\}.$$

Theorem 7.1. *Let a continuous increasing unbounded function* $g(\cdot)$ *be POV and* $\{x_n, n \ge 0\}$ *be a sequence such that* $\limsup_{t\to\infty} x_n = \infty$. *Assume that* $\hat{x}^{(-1)}$ *is a quasiinverse function to the linear interpolation* $\hat{x}(\cdot)$ *of the sequence* $\{x_n, n \ge 0\}$. *Then the following statements hold:*

$$\lim_{n\to\infty}\frac{x_n}{g(n)} = 1 \Rightarrow \lim_{s\to\infty}\frac{\hat{x}^{(-1)}(s)}{g^{-1}(s)} = 1, \tag{7.1}$$

$$\lim_{n\to\infty}\frac{x_n}{g(n)} = a \in (0,\infty) \Rightarrow \lim_{s\to\infty}\frac{\hat{x}^{(-1)}(s)}{g^{-1}(s/a)} = 1, \tag{7.2}$$

$$\lim_{n\to\infty}\frac{x_n}{g(n)} = \infty \Rightarrow \lim_{s\to\infty}\frac{\hat{x}^{(-1)}(s)}{g^{-1}(s)} = 0, \tag{7.3}$$

$$\lim_{n\to\infty}\frac{x_n}{g(n)} = 0 \Rightarrow \lim_{s\to\infty}\frac{\hat{x}^{(-1)}(s)}{g^{-1}(s)} = \infty. \tag{7.4}$$

Proof. To prove (7.1) we note that its left hand side implies by Lemma 7.2 and Lemma 7.1

$$\lim_{t\to\infty}\frac{\hat{x}(t)}{g(t)} = 1.$$

Proposition 6.2 yields

$$\lim_{s\to\infty}\frac{\hat{x}^{(-1)}(s)}{g^{-1}(s)} = 1,$$

whence (7.1) follows. Similar reasonings prove (7.2), (7.3), and (7.4).

Corollary 7.1. *Let a continuous increasing unbounded function* $g(\cdot)$ *be POV and* $\{x_n, n \ge 0\}$ *be a sequence such that* $\limsup_{t\to\infty} x_n = \infty$. *Assume that* $\hat{x}_1^{(-1)}(\cdot)$ *and* $\hat{x}_2^{(-1)}(\cdot)$ *are two quasiinverse functions to the linear interpolation* $\hat{x}(\cdot)$ *of the sequence* $\{x_n, n \ge 0\}$ *and* $\psi(\cdot)$ *is a function such that*

$$\hat{x}_1^{(-1)}(s) - a_1(s) \le \psi(s) \le \hat{x}_2^{(-1)}(s) + a_2(s) \tag{7.5}$$

for sufficiently large s, *where* $a_1(\cdot)$ *and* $a_2(\cdot)$ *are two nonnegative functions for which*

$$\lim_{s\to\infty}\frac{a_1(s)}{g^{-1}(s)} = 0 \quad and \quad \lim_{s\to\infty}\frac{a_2(s)}{g^{-1}(s)} = 0.$$

Then the following statements hold:

$$\lim_{n\to\infty}\frac{x_n}{g(n)} = 1 \Rightarrow \lim_{s\to\infty}\frac{\psi(s)}{g^{-1}(s)} = 1, \tag{7.6}$$

$$\lim_{n\to\infty}\frac{x_n}{g(n)} = a \in (0,\infty) \Rightarrow \lim_{s\to\infty}\frac{\psi(s)}{g^{-1}(s/a)} = 1, \tag{7.7}$$

$$\lim_{n\to\infty}\frac{x_n}{g(n)} = \infty \Rightarrow \lim_{s\to\infty}\frac{\psi(s)}{g^{-1}(s)} = 0, \tag{7.8}$$

$$\lim_{n\to\infty}\frac{x_n}{g(n)} = 0 \Rightarrow \lim_{s\to\infty}\frac{\psi(s)}{g^{-1}(s)} = \infty. \tag{7.9}$$

8. Generalized renewal sample functions and processes for continuous functions and processes. Given a real-valued continuous function $x(\cdot)$ such that $\lim_{t\to\infty}x(t) = \infty$, the generalized renewal sample functions are defined as follows

$$f_{x(\cdot)}(s) = \min\left\{u \ge 0: \max_{0\le t\le u} x(t) \ge s\right\} = \min\{u \ge 0: x(t) = s\},$$

$$m_{x(\cdot)}(s) = \max\left\{u \ge 0: \max_{0\le t\le u} x(t) \le s\right\},$$

$$\tau_{x(\cdot)}(s) = \operatorname{meas}\{t \ge 0: x(t) \le s\} = \int_0^\infty I(x(t) \le s)\,dt,$$

$$l_{x(\cdot)}(s) = \max\{t \ge 0: x(t) = s\}$$

for $s \ge x(0)$, and we put $f_{x(\cdot)}(s) = m_{x(\cdot)}(s) = \tau_{x(\cdot)}(s) = l_{x(\cdot)}(s) = 0$ for $0 \le s < x(0)$ if $x(0) > 0$.

If the function $x(\cdot)$ is continuous increasing and unbounded, then all the four functions $f_{x(\cdot)}(\cdot)$, $m_{x(\cdot)}(\cdot)$, $\tau_{x(\cdot)}(\cdot)$, and $l_{x(\cdot)}(\cdot)$ coincide. Otherwise they are different and

$$f_{x(\cdot)}(s) \le m_{x(\cdot)}(s) \le \tau_{x(\cdot)}(s) \le l_{x(\cdot)}(s) \tag{8.1}$$

for $s \ge 0$. Observe also that the functions $f_{x(\cdot)}(\cdot)$ and $m_{x(\cdot)}(\cdot)$ are well defined for $x(\cdot) \in C^{(\infty)}(\mathbb{R}_0)$ and are quasiinverse functions for $x(\cdot)$ (see Example 6.1). Moreover, if $x(\cdot) \in C^{\infty}(\mathbb{R}_0)$, then the function $l_{x(\cdot)}(\cdot)$ is a quasiinverse functions for $x(\cdot)$ (see Example 6.2).

Propositions 6.2 and 6.3, Theorem 6.1, and formula (8.1) yield the following statements.

Proposition 8.1. *Assume that* $x(\cdot) \in C^{\infty}(\mathbb{R}_0)$, $g(\cdot) \in C^{\infty}_{\text{inc}}(\mathbb{R}_0)$, *and the function* $g^{-1}(\cdot)$ *is PRV. If*

$$\lim_{t\to\infty}\frac{x(t)}{g(t)} = a \quad \textit{for some} \quad a \in (0,\infty),$$

then

$$\lim_{s\to\infty}\frac{f_{x(\cdot)}(s)}{g^{(-1)}(s/a)} = \lim_{s\to\infty}\frac{m_{x(\cdot)}(\cdot)}{g^{-1}(s/a)} = \lim_{s\to\infty}\frac{\tau_{x(\cdot)}(\cdot)}{g^{-1}(s/a)} = \lim_{s\to\infty}\frac{l_{x(\cdot)}(\cdot)}{g^{-1}(s/a)} = 1.$$

Proposition 8.2. *Assume that* $x(\cdot) \in C^{\infty}(\mathbb{R}_0)$, $g(\cdot) \in C^{\infty}_{\text{inc}}(\mathbb{R}_0)$ *and relation (6.5) holds. Then the following statements hold:*

(a) *if* $\lim_{t\to\infty}\frac{x(t)}{g(t)} = \infty$, *then*

$$\lim_{s\to\infty}\frac{f_{x(\cdot)}(s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{m_{x(\cdot)}(s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{\tau_{x(\cdot)}(s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{l_{x(\cdot)}(s)}{g^{-1}(s)} = 0;$$

(b) *if* $\lim_{t\to\infty}\frac{x(t)}{g(t)} = 0$, *then*

$$\lim_{s\to\infty}\frac{f_{x(\cdot)}(s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{m_{x(\cdot)}(s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{\tau_{x(\cdot)}(s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{l_{x(\cdot)}(s)}{g^{-1}(s)} = \infty.$$

For RV functions these results become as follows.

Corollary 8.1. *Let an increasing continuous RV function* $g(\cdot)$ *be of a positive index* α *and* $x(\cdot)\in C^{\infty}(\mathbb{R}_0)$. *If*

$$\lim_{t\to\infty}\frac{x(t)}{g(t)} = a\in[0,\infty],$$

then

$$\lim_{s\to\infty}\frac{f_{x(\cdot)}(s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{m_{x(\cdot)}(s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{\tau_{x(\cdot)}(s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{l_{x(\cdot)}(s)}{g^{-1}(s)} = \left(\frac{1}{a}\right)^{1/\alpha}.$$

Here we assume that $(1/\infty)=0$ *and* $(1/0)=\infty$.

Let $\{\Omega, \mathfrak{F}, \mathrm{P}\}$ be a probability space, and $\mathbb{X} = (X(\omega,t),\ \omega\in\Omega,\ t\geq 0)$ be real-valued stochastic process. Put $X(\omega) = (X(\omega,t),\ t\geq 0)$, $\omega\in\Omega$.

Given a real-valued stochastic process $\mathbb{X}$ such that $X(\omega)\in C^{\infty}(\mathbb{R}_0)$ almost surely, the generalized renewal processes are defined as follows

$$F_{\mathbb{X}}(\omega,s) = f_{X(\omega)}(s), \qquad M_{\mathbb{X}}(\omega,s) = m_{X(\omega)}(s),$$
$$T_{\mathbb{X}}(\omega,s) = \tau_{X(\omega)}(s), \qquad L_{\mathbb{X}}(\omega,s) = l_{X(\omega)}(s),$$

for $\omega\in\Omega$ and $s\geq 0$.

Propositions 8.1, 8.2 yield the following theorems.

Theorem 8.1. *Let* $\mathbb{X}$ *be a stochastic process such that* $\mathbb{X}\in C^{\infty}(\mathbb{R}_0)$ *a. s. Assume that* $g(\cdot)\in C^{\infty}_{\mathrm{inc}}(\mathbb{R}_0)$ *and the function* $g^{-1}(\cdot)$ *is PRV. If* $\lim_{t\to\infty}\frac{X(\omega,t)}{g(t)} = a$ *a .s. for some* $a\in(0,\infty)$, *then*

$$\lim_{s\to\infty}\frac{F_{\mathbb{X}}(\omega,s)}{g^{-1}(s/a)} = \lim_{s\to\infty}\frac{M_{\mathbb{X}}(\omega,s)}{g^{-1}(s/a)} = \lim_{s\to\infty}\frac{T_{\mathbb{X}}(\omega,s)}{g^{-1}(s/a)} = \lim_{s\to\infty}\frac{L_{\mathbb{X}}(\omega,s)}{g^{-1}(s/a)} = 1 \quad a.\ s.$$

Theorem 8.2. *Let* $\mathbb{X}$ *be a stochastic process such that* $\mathbb{X}\in C^{\infty}(\mathbb{R}_0)$ *a. s.,* $g(\cdot)\in C^{\infty}_{\mathrm{inc}}(\mathbb{R}_0)$ *and relation* (6.5) *holds. Then the following statements hold*:

(a) *if* $\lim_{t\to\infty}\frac{X(\omega,t)}{g(t)} = \infty$, *then*

$$\lim_{s\to\infty}\frac{F_{\mathbb{X}}(\omega,s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{M_{\mathbb{X}}(\omega,s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{T_{\mathbb{X}}(\omega,s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{L_{\mathbb{X}}(\omega,s)}{g^{-1}(s)} = 0 \quad a.s.;$$

(b) *if* $\lim_{t\to\infty}\frac{X(\omega,t)}{g(t)} = 0$, *then*

$$\lim_{s\to\infty}\frac{F_{\mathbb{X}}(\omega,s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{M_{\mathbb{X}}(\omega,s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{T_{\mathbb{X}}(\omega,s)}{g^{-1}(s)} = \lim_{s\to\infty}\frac{L_{\mathbb{X}}(\omega,s)}{g^{-1}(s)} = \infty \quad a.\ s.$$

***Corollary* 8.2.** *Let* $\mathbb{X}$ *be a stochastic process such that* $\mathbb{X} \in C^{\infty}(\mathbb{R}_0)$ *a. s. and let an increasing continuous RV function* $g(\cdot)$ *be of a positive index* α. *If* $\lim_{t\to\infty} \frac{X(\omega,t)}{g(t)} = a$ *a. s. for some* $a \in [0,\infty]$, *then*

$$\lim_{s\to\infty}\frac{F_{\mathbb{X}}(\omega,s)}{g^{-1}(s/a)} = \lim_{s\to\infty}\frac{M_{\mathbb{X}}(\omega,s)}{g^{-1}(s/a)} = \lim_{s\to\infty}\frac{T_{\mathbb{X}}(\omega,s)}{g^{-1}(s/a)} = \lim_{s\to\infty}\frac{L_{\mathbb{X}}(\omega,s)}{g^{-1}(s/a)} = \left(\frac{1}{a}\right)^{1/\alpha} \quad a.\ s.$$

Here we assume that $(1/\infty) = 0$ *and* $(1/0) = \infty$.

***Example* 8.1.** Let $X(t)$, $t \geq 0$, be an almost surely continuous process with independent increments such that $\mathrm{E}\,X(1) = a > 0$. Then $X(t)/t \to a$ a. s. and by Corollary 8.2

$$\lim_{s\to\infty}\frac{F_{\mathbb{X}}(s)}{s} = \lim_{s\to\infty}\frac{M_{\mathbb{X}}(s)}{s} = \lim_{s\to\infty}\frac{T_{\mathbb{X}}(s)}{s} = \lim_{s\to\infty}\frac{L_{\mathbb{X}}(s)}{s} = \frac{1}{a} \quad \text{a. s.} \tag{8.2}$$

***Example* 8.2.** A number of examples can be given for additive functionals of stochastic processes. Let $Y(t) = (Y(\omega,t),\ \omega\in\Omega)$, $t \in \mathbb{R}$ be a strictly stationary measurable stochastic process such that $\mathrm{E}|Y(0)|^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$ (this guarantees that $X(\cdot)$ is a. s. continuous) and $\mathrm{E}[Y(0)/\mathcal{F}] = a > 0$ a. s., where $\mathcal{F}$ is the σ-algebra of shift invariant events. Put

$$X(t) = X(\omega,t) = \int_0^t Y(\omega,u)\,du, \qquad t \geq 0.$$

Then by the ergodic theorem

$$\lim_{t\to\infty}\frac{X(t)}{t} = a \quad \text{a. s.,} \tag{8.3}$$

whence (8.2) follows by Corollary 8.2.

Note that a is nonrandom if $Y(\cdot)$ is ergodic.

***Example* 8.3.** Assume that the process $Y(\cdot)$ is second order stationary, measurable (this guarantees that $X(\cdot)$ is a. s. continuous), and such that $\mathrm{E}Y(0) = a > 0$. Necessary and sufficient conditions for (8.3) are given in [12] in this case. We use a weaker form of those conditions which also is given in [12]. If $R(\cdot)$ is the correlation function of the process $Y(\cdot)$, then the ("best possible" sufficient) condition is as follows:

$$\text{the integral } \int_3^{\infty}\frac{R(t)}{t\log t}\log\log t\,dt \quad \text{converges.}$$

Thus the latter condition implies (8.3), whence (8.2) follows by Corollary 8.2.

9. Generalized renewal sample functions and processes for nonrandom and random sequences. Given a real-valued sequence $\{z_n, n \geq 0\}$, $z(0) = 0$, the generalized renewal sample functions are defined as follows

$$f_{\{z_n\}}(s) = \min\{n \geq 0 \colon \max(z_0, z_1, \ldots, z_n) \geq s\},$$

$$m_{\{z_n\}}(s) = \sup\{n \geq 0 \colon \max(z_0, z_1, \ldots, z_n) \leq s\},$$

$$\#_{\{z_n\}}(s) = \sum_{n=1}^{\infty} I(z_n \leq s),$$

$$l_{\{z_n\}}(s) = \sup\{n \geq 0 \colon z_n \leq s\}.$$

If $\lim_{n\to\infty} z_n = \infty$, then $f_{\{z_n\}}(s)$, $m_{\{z_n\}}(s)$, $\#_{\{z_n\}}(s)$, $l_{\{z_n\}}(s)$ are well defined for $s \ge 0$, and we can replace sup by max. If the sequence $\{z_n\}$ increases, then the three functions $m_{\{z_n\}}(\cdot)$, $\#_{\{z_n\}}(\cdot)$, $l_{\{z_n\}}(\cdot)$ coincide. Otherwise they are different and

$$f_{\{z_n\}}(s) \le m_{\{z_n\}}(s) \le \#_{\{z_n\}}(s) \le l_{\{z_n\}}(s) \tag{9.1}$$

for $s \ge 0$.

Along with a sequence $\{z_n\}$ we consider its piecewise linear interpolation $\hat{z}(\cdot)$ (see Definition 7.1) and two quasiinverse functions $f_{\hat{z}(\cdot)}(\cdot)$ and $l_{\hat{x}(\cdot)}(\cdot)$ for the function $\hat{z}(\cdot)$ (see Section 7).

It is clear that

$$f_{\hat{z}(\cdot)}(s) \le f_{\{z_n\}}(s), \qquad l_{\{z_n\}}(s) \le l_{\hat{z}(\cdot)}(s) \tag{9.2}$$

for all $s > 0$.

From Corollary 7.1 and inequalities (9.1) and (9.2) we obtain the following result.

Proposition 9.1. *Let a continuous increasing unbounded function $g(\cdot)$ be POV and $\{z_n, n \ge 0\}$ be a sequence such that $\lim_{n\to\infty} z_n = \infty$. Then the following statements hold:*

(a) *if* $\lim\limits_{n\to\infty} \dfrac{z_n}{g(n)} = a \in (0,\infty)$, *then*

$$\lim_{s\to\infty} \frac{f_{\{z_n\}}(s)}{g^{-1}(s/a)} = \lim_{s\to\infty} \frac{m_{\{z_n\}}(s)}{g^{-1}(s/a)} = \lim_{s\to\infty} \frac{\#_{\{z_n\}}(s)}{g^{-1}(s/a)} = \lim_{s\to\infty} \frac{l_{\{z_n\}}(s)}{g^{-1}(s/a)} = 1;$$

(b) *if* $\lim\limits_{n\to\infty} \dfrac{x_n}{g(n)} = \infty$, *then*

$$\lim_{s\to\infty} \frac{f_{\{z_n\}}(s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{m_{\{z_n\}}(s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{\#_{\{z_n\}}(s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{l_{\{z_n\}}(s)}{g^{-1}(s)} = 0;$$

(c) *if* $\lim\limits_{n\to\infty} \dfrac{x_n}{g(n)} = 0$, *then*

$$\lim_{s\to\infty} \frac{f_{\{z_n\}}(s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{m_{\{z_n\}}(s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{\#_{\{z_n\}}(s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{l_{\{z_n\}}(s)}{g^{-1}(s)} = \infty.$$

Let $\{\Omega, \mathfrak{F}, \mathbb{P}\}$ be a probability space, and $\mathbb{Z} = (Z_n(\omega),\ \omega\in\Omega,\ n \ge 0)$ be a real-valued random sequence. Put $\{Z_n(\omega)\} = \{Z_n(\omega), n \ge 0\}$, $\omega \in \Omega$.

Given a real-valued random sequence $\mathbb{Z}$ such that $Z_0 = 0$ and $\lim_{n\to\infty} Z_n(\omega) = \infty$ almost surely, the generalized renewal processes are defined as follows

$$F_{\mathbb{Z}}(\omega, s) = f_{\{Z_n(\omega)\}}(s), \qquad M_{\mathbb{Z}}(\omega, s) = m_{\{Z_n(\omega)\}}(s),$$

$$N_{\mathbb{Z}}(\omega, s) = \#_{\{Z_n(\omega)\}}(s), \qquad L_{\mathbb{Z}}(\omega, s) = l_{\{Z_n(\omega)\}}(s),$$

for $\omega \in \Omega$, $s \ge 0$.

From Proposition 9.1 we obtain the following result.

Theorem 9.1. *Let a continuous increasing unbounded function $g(\cdot)$ be POV and let $\mathbb{Z}$ be a random sequence such that $\lim_{n\to\infty} Z_n(\omega) = \infty$ a. s. Then the following statements hold:*

(a) *if* $\lim_{n\to\infty} \frac{Z_n(\omega)}{g(n)} = a$ *a. s. for some* $a \in (0, \infty)$, *then*

$$\lim_{s\to\infty} \frac{F_{\mathbb{Z}}(\omega, s)}{g^{-1}(s/a)} = \lim_{s\to\infty} \frac{M_{\mathbb{Z}}(\omega, s)}{g^{-1}(s/a)} = \lim_{s\to\infty} \frac{N_{\mathbb{Z}}(\omega, s)}{g^{-1}(s/a)} = \lim_{s\to\infty} \frac{L_{\mathbb{Z}}(\omega, s)}{g^{-1}(s/a)} = 1 \quad a. s.;$$

(b) *if* $\lim_{n\to\infty} \frac{Z_n(\omega)}{g(n)} = \infty$ *a. s., then*

$$\lim_{s\to\infty} \frac{F_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{M_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{N_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{L_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = 0 \quad a. s.;$$

(c) *if* $\lim_{n\to\infty} \frac{Z_n(\omega)}{g(n)} = 0$ *a. s., then*

$$\lim_{s\to\infty} \frac{F_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{M_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{N_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{L_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \infty \quad a. s.$$

***Corollary* 9.1.** *Let* $\mathbb{Z}$ *be a random sequence such that* $\lim_{n\to\infty} Z_n(\omega) = \infty$ *a. s., and let an increasing continuous RV function* $g(\cdot)$ *be a positive index* α. *If* $\lim_{n\to\infty} \frac{Z_n(\omega)}{g(n)} = a$ *a. s. for some* $a \in [0, \infty]$, *then*

$$\lim_{s\to\infty} \frac{F_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{M_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{T_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \lim_{s\to\infty} \frac{L_{\mathbb{Z}}(\omega, s)}{g^{-1}(s)} = \left(\frac{1}{a}\right)^{1/\alpha} \quad a. s.$$

Here we assume that $(1/\infty) = 0$ *and* $(1/0) = \infty$.

***Example* 9.1.** Let $X_n = (X_n(\omega), \omega \in \Omega)$, $n = \ldots, -1, 0, +1, \ldots$, be a strictly stationary sequence of random variables such that $\mathrm{E}|X_0| < \infty$ and $\mathrm{E}[X_0 / \mathcal{F}] = a > 0$ a. s., where $\mathcal{F}$ is the σ-algebra of shift invariant events.

Put $Z_0 = 0$, $Z_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$, $n \geq 1$. Then

$$\lim_{n\to\infty} \frac{Z_n(\omega)}{n} = a(\omega) \quad \text{a. s.}$$

and by Corollary 9.1

$$\lim_{s\to\infty} \frac{F_{\mathbb{Z}}(\omega, s)}{s} = \lim_{s\to\infty} \frac{M_{\mathbb{Z}}(\omega, s)}{s} = \lim_{s\to\infty} \frac{T_{\mathbb{Z}}(\omega, s)}{s} = \lim_{s\to\infty} \frac{L_{\mathbb{Z}}(\omega, s)}{s} = \frac{1}{a(\omega)} \quad \text{a. s.}$$

***Example* 9.2.** This is an extension of Example 3.2 in [1]. We use the same notation as in Example 9.1. Further let $b(\cdot)$ be a positive continuous function such that

(i) $b(\cdot)$ is PRV,

(ii) $t / b(t)$ is increasing and unbounded,

(iii) $\limsup_{t\to\infty} \frac{b(ct)}{b(t)} < c$ for all $c > 1$.

Put $Z_0 = 0$, and

$$Z_n(\omega) = \frac{1}{b(n)} \sum_{k=1}^{n} X_k(\omega), \qquad n \geq 1.$$

Then

$$\lim_{t \to \infty} \frac{Z_n(\omega)}{(n/b(n))} = a \quad \text{a. s.}$$

The first-passage time for this scheme

$$M(t) + 1 = \inf\{n: Z_n > t\} = \inf\{n: S_n > tb(n)\}$$

is of some statistical importance and plays a key role in what is called the nonlinear renewal theory.

It is clear that the function $g(t) = t/b(t)$, $t > 0$, is continuous increasing unbounded and moreover this function is POV. Thus by Theorem 9.1

$$\lim_{s \to \infty} \frac{F_Z(\omega, s)}{g^{-1}(s/a)} = \lim_{s \to \infty} \frac{M_Z(\omega, s)}{g^{-1}(s/a)} = \lim_{s \to \infty} \frac{N_Z(\omega, s)}{g^{-1}(s/a)} = \lim_{s \to \infty} \frac{L_Z(\omega, s)}{g^{-1}(s/a)} = 1 \quad \text{a. s.}$$

More examples for the discrete scheme can be found in [1].

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