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MALLIAVIN CALCULUS FOR FUNCTIONALS WITH GENERALIZED DERIVATIVES AND SOME APPLICATIONS FOR STABLE PROCESSES*

ЧИСЛЕННЯ МАЛЛЯВЕНА ДЛЯ ФУНКЦІОНАЛІВ ІЗ УЗАГАЛЬНЕНИМИ ПОХІДНИМИ ТА ДЕЯКІ ЗАСТОСУВАННЯ ДО СТІЙКИХ ПРОЦЕСІВ

We give the definition of generalized derivative of a functional on a probability space with respect to some formal differentiation. We prove a sufficient condition for the existence of the density of distribution of the functional in terms of its generalized derivative. This result is used to obtain the smoothness of distribution of local time of a stable process.

Введено поняття узагальненої похідної функціонала на ймовірнісному просторі відносно формального диференціювання. Отримано достатню умову існування щільності розподілу функціонала у термінах його узагальненої похідної. Ці результати використано для доведення гладкості розподілу локального часу від стійкого процесу.

1. Introduction. In this paper we give the result which provides the regularity of the distribution of a functional over a space with a smooth probability measure under wide conditions on the existence and nondegeneracy of its derivative.

The well known Malliavin's approach, based on a stochastic calculus of variation and introduced initially in order to obtain the existence of the smooth density for the solution of the SDE driven by Wiener process (see [1]), can be developed for a more wide classes of functionals and probability spaces in the following manner. First, one have to construct some differentiation structure on the given probability space such that the initial probability measure is smooth with respect to it. Then one is to construct the corresponding Sobolev spaces and stochastic (Sobolev) derivative and prove some Malliavin-type theorem which gives the existence of a smooth density for a functional which is infinitely stochastic differentiable and has derivative which is nondegenerated in some sence. And, at last, one have to check that, under some conditions, the obtained theorem can be applied to a certain class of functionals on the initial space, such as integral functionals, solutions of the SDE's, local times etc.

In some cases such a programme can be completely worked out, the most important example is the Malliavin-type calculus for the SDE with jumps which is driven by random Poisson measure with smooth Levy measure. It was started by K. Bichteler (see [2] and bibliography there) and developed by J. Jacod, R. Leandre, J. Picard, T. Komatsu and others (see [3] and bibliography there). But there exists a lot of other cases in which there arise difficulties, caused by nonregularity of the initial differential structure or of the functionals under investigation. Such situation appears for example for time-stretching differentiation w. r. t. Poisson processes with an arbitrary Levy measure constructed in [4] and differentiations w. r. t. stationary and semistable processes constructed in [5]. In the mentioned examples the values of the initial process are not stochastic differentiable by themselves. Therefore the usual technique which gives one opportunity to obtain, say, that in Wiener case the solution of the SDE is infinitely differentiable, does not work and the problem to check for the given functional conditions of the general Malliavin-type theorem (such as theorem 6.1 in [5, p. 55], or theorems 2, 3 in [4]) appears to be very difficult.

In such a situation it is useful to obtain a weaker version of the Malliavin-type theorem which will give the existence of the density (without any statements about its

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smoothness) of the functional under weaker conditions on it. The typical result of this type can be formulated (for a one-dimensional functional) in a following form.

Theorem. *Let functional f has stochastic derivative Df, then*

$$P|_{\{Df \neq 0\}} \circ f^{-1} << \lambda^1.$$

Such a statement was proved with some additional suppositions about the structure of the formal differentiation by the stratification method in [6] and (with weaker conditions) in [7], by the descent method in [5] and in [2] by the Malliavin-type technique with additional condition on the existence of second derivative.

In this paper we prove this result without any additional suppositions about the structure of differentiation or existence of higher derivatives of f. Moreover, we admit the functional f only to have the derivative which is a generalized function in some sence, which is quite new condition in this theory. To demonstrate this result by an example we regard the stable process as the semistable process and use the differentiation from [5] to obtain the smoothness of the distribution of local time of the initial process.

2. Formal differentiation, Sobolev spaces and generalized stochastic derivatives. The notion of the admissible formal differentiation (differentiation rule) on a probability space was introduced in [8] (see also [9]). Let us give the corresponding definition in the form which is convenient for our future discussions.

Let $(\mathscr{X}, \mathscr{B})$ be complete metric space with Borel σ-algebra, H is a real separable Hilbert space. The set $\mathscr{C}$ of bounded measurable functions $\mathscr{X} \to \mathbb{R}$ with the operator ∇_H acting on $\mathscr{C}$ and taking values in the spase of bounded measurable H-valued functions on $\mathscr{X}$ is called a *formal differentiation* (differentiation rule, differentiation) on a space $(\mathscr{X}, \mathscr{B})$, if they satisfy the chain rule

$$\forall f_1, \ldots, f_n \in \mathscr{C}, \quad F \in C_b^1(\mathbb{R}^n): \quad F(f_1, \ldots, f_n) \in \mathscr{C},$$
$$\nabla_H F(f_1, \ldots, f_n) = \sum_{k=1}^{n} F_k'(f_1, \ldots, f_n) \nabla_H f_k. \tag{1}$$

Definition 1. *The formal differentiation $(\nabla_H, \mathscr{C})$ is called to be admissible for a measure μ on $(\mathscr{X}, \mathscr{B})$ if there exists a weak random element ρ in H, such that the following integration by parts formula holds true*

$$\mathrm{E}(\nabla_H f, h)_H = -\mathrm{E}f(\rho, h), \quad h \in H, \quad f \in \mathscr{C}. \tag{2}$$

The element ρ is called the logarithmic derivative of μ w. r. t. $(\nabla_H, \mathscr{C})$.

In the situation of Definition 1 we shall also say that μ is differentiable w. r. t. $(\nabla_H, \mathscr{C})$.

The formal differentiation $(\nabla_H, \mathscr{C})$ is called full w. r. t. measure μ if there exists a countable set $\mathscr{C}_0 \subset \mathscr{C}$ such that $\overline{(\sigma(\mathscr{C}_0))}^{\mu} \supset \mathscr{B}$. Differentiation $(\nabla_H, \mathscr{C})$ is called full if $\sigma(\mathscr{C}_0) = \mathscr{B}$.

Further we suppose that the space $(\mathscr{X}, \mathscr{B}, \mu)$ with the full μ-admissible formal differentiation are fixed. We also suppose that for an arbitrary $p > 0$ the logarithmic derivative has a weak moment of p-th order, i. e.

$$\exists C_p < +\infty: \mathrm{E}|(\rho, h)|^p \le C_p \|h\|_H^p, \quad h \in H.$$

Regard for $p \in (1, +\infty)$ operator ∇_H as the densely defined unbounded operator

$$\nabla_H: L_p(\mathscr{X}, \mu) \ni \mathscr{C} \to L_p(\mathscr{X}, \mu, H).$$

Due to (1) on the functions of the type

$$q = \sum_{k=1}^{n} g_k h_k, \quad g_k \in \mathscr{C}, \quad h_k \in H, \quad k = \overline{1, n} \tag{3}$$

it's adjoint operator is well defined and equals

$$\nabla_H^*(q) = -\sum_{k=1}^{n} [g_k(\rho, h_k) + (\nabla_H g_k, h_k)_H].$$

Thus ∇_H is closable in L_p sence.

Definition 2. *The closure of* ∇_H *in* L_p *sence is called stochastic derivative and denoted by* D_p, *its adjoint* I_p *is called stochastic integral. The domain of* D_p *with the graph norm is called the Sobolev space of the order* $(1, p)$ *over* $\mathscr{X}$ *w. r. t.* μ *and denoted by* $W_p^1(\mathscr{X}, \mu)$.

The higher derivatives D_p^n and corresponding spaces $W_p^n(\mathscr{X}, \mu)$ are defined iteratively. We shall omit the subscript in the notations for D_p, I_p if it should not cause misunderstandings.

Now let us proceed to the definition of the generalized stochastic derivative.

Definition 3. *Let* $f \in \bigcup_{p>1} L_p(\mathscr{X}, \mu)$. *The signed measure* $\mathfrak{D}_h f$ *with finite variation on* $(\mathscr{X}, \mathscr{B})$ *is called to be the weak derivative in the direction* $h \in H$ *of the functional* f *if*

$$\mathrm{E} f I(gh) = \int_{\mathscr{X}} g(x)[\mathfrak{D}_h f](dx), \quad g \in \mathscr{C}.$$

It is obvious that if $f \in W_p^1(\mathscr{X}, \mu)$ then its weak derivative in the direction $h \in H$ is equal

$$[\mathfrak{D}_h f](dx) = (Df(x), h)_H \mu(dx)).$$

It is also easy to see that inverse is not true.

***Example* 1.** Let $\mathscr{X} = H = \mathbb{R}$, $\mu \sim \mathcal{N}(0, 1)$, $\mathscr{C} = C_b^1(\mathbb{R})$ and $\nabla f = f'$. Then function $f(x) = x + \operatorname{sign}(x)$ has the weak derivative $\mathfrak{D} f = \mu + \sqrt{2/\pi}\,\delta_0$. But by the Sobolev lemma $\bigcup_{p>1} W_p^1(\mathbb{R}, \mu) \subset C(\mathbb{R})$, so f is not stochastic differentiable.

The Definition 3 is too wide for our needs, the final definition of the generalized derivative will suppose that it satisfies some analogue of the chain rule.

For a signed measure ν with finite variation on $(\mathscr{X}, \mathscr{B})$ we shall denote by $[\nu]_a$ and $[\nu]_s$ its absolutely continuous and singular parts w. r. t. measure μ respectively.

Definition 4. *Functional* $f \in \bigcup_{p>1} L_p(\mathscr{X}, \mu)$ *has the generalized derivative in direction* $h \in H$ *w. r. t.* ∇_H *if*

a) f *has weak derivative in the direction* h *and for every* $\varphi \in C_b^1(\mathbb{R})$ *functional* $\varphi(f)$ *has weak derivative in the direction* h;

b) *for every* $\varphi \in C_b^1(\mathbb{R})$ $[\mathfrak{D}_h \varphi(f)]_a << [\mathfrak{D}_h f]_a$, $[\mathfrak{D}_h \varphi(f)]_s << [\mathfrak{D}_h f]_s$ *and*

$$\frac{d[\mathfrak{D}_h \varphi(f)]_a}{[\mathfrak{D}_h f]_a}(x) = \varphi'(f(x)), \qquad \left|\frac{d[\mathfrak{D}_h \varphi(f)]_s}{[\mathfrak{D}_h f]_s}(x)\right| \le \sup_{t \in \mathbb{R}} |\varphi'(t)|.$$

***Example* 2.** In the situation of Example 1 it is easy to see that for $\varphi \in C_b^1(\mathbb{R})$ the change of variable formula applied to φ on semi-axis $(-\infty, 0]$ and $[0, +\infty)$ gives that

$$[\mathcal{D}f](dx) = \varphi'(x+\text{sign}(x))\mu(dx) + [\varphi(1)-\varphi(0)]\sqrt{\frac{2}{\pi}}\delta_0(dx)$$

and thus f has the generalized derivative w. r. t. ∇.

3. The main theorem. Further we suppose that the space $\mathcal{X}$ with the full differentiation rule $(\nabla_H, \mathcal{C})$ and the measure μ which is differentiable w. r. t. $(\nabla_H, \mathcal{C})$ are fixed.

Denote for $f \in \bigcup_{p>1} L_p(\mathcal{X}, \mu)$ by $\mathcal{N}_f$ the set of $h \in H$ such that f has generalized derivative in the direction h and choose some sequence $\{h_k\} \subset \mathcal{N}_f$ which is dense in $\mathcal{N}_f$ in the $\|\cdot\|_H$-norm. Denote

$$\Delta_f = \bigcup_k \left\{[\mathcal{D}_{h_k} f]_a \neq 0\right\},$$

one can see that the set Δ_f does not depend $(\text{mod}\,\mu)$ on the choice of the sequence $\{h_k\}$.

Theorem 1. *Let* $\mathcal{N}_f \neq \varnothing$, *then*

$$\mu|_{\Delta_f} \circ f^{-1} << \lambda^1.$$

Remarks. **1.** In the case $f \in W_p^1(\mathcal{X}, \mu)$ Theorem 1 gives that

$$\mu|_{\{Df \neq 0\}} \circ f^{-1} << \lambda^1.$$

2. In the situation of the Examples 1, 2 the statement of the theorem can be obtained as a corollary of changing of variables formula: one has only to apply this formula to the intervals $(-\infty, 0)$ and $(0, +\infty)$, which are the intervals of smoothness of the initial function f. However, if we take on the same place the function $f_1(x) = x + \sum_{q_i \le x} p_i$ where $p_i > 0$, $\sum_i p_i = 1$ and $\{q_i\} = \mathbb{Q}$, then f_1 will not have any intervals of smoothness. Therefore, even in one-dimensional case Theorem 1 can not be obtained immediately from the changing of variables formula and needs some additional technique to be proved.

To prove the theorem we need the following auxiliary result. Let $h \in H$ be fixed, f has the generalized derivative in the direction h, denote by $\{\mu_x, x \in \mathcal{X}\}$ the conditional probability of μ with respect to $\sigma(f)$ (it exists because $(\nabla_H, \mathcal{C})$ is full). Denote by U_h^f the set of all $x \in \mathcal{X}$ such that $\mu_x\left(\left\{\frac{d[\mathcal{D}_h]_a}{d\mu} \neq 0\right\}\right) > 0$.

Lemma 1.

$$\mu|_{U_h^f} \circ f^{-1} << \lambda^1.$$

Proof. Denote by $Y_f \subset \mathcal{X}$ such set that $\text{var}[\mathcal{D}_h f]_a(Y_f) = \text{var}[\mathcal{D}_h f]_a(\mathcal{X})$, $\text{var}[\mathcal{D}_h f]_s(Y_f) = 0$. Fix the sequence $\{g_n\} \subset \mathcal{C}$ such that $|g_n| \le 1$ and $g_n \to 1\!\!1_{Y_f}$, $n \to \infty$ $(\text{mod}\ \text{var}[\mathcal{D}_h f]_a + \text{var}[\mathcal{D}_h f])$. Let $g \in \mathcal{C}_0$ be fixed, denote

$$\rho_n^g(t) = \mathrm{E}[I(gg_n h)|f=t], \qquad \zeta_n^g(t) = \mathrm{E}\left[gg_n \frac{d[\mathcal{D}_h]_a}{d\mu}|f=t\right], \qquad t \in \mathbb{R}.$$

Denote by μ^f the image of the measure μ under the function f and by μ_a^f and μ_s^f its absolutely continuous and singular components w. r. t. λ^1 respectively.

It follows from Definition 4 that for every $\varphi \in C_b^1(\mathbb{R})$

$$\left|\int_{\mathbb{R}} \left[\varphi(t)\rho_n^g(t) - \varphi'(t)\zeta_n^g(t)\right]\mu^f(dt)\right| \le \sup_{t\in\mathbb{R}} |\varphi'(t)| \, \|g\|_\infty \int_{\mathfrak{X}} |g_n(x)| \operatorname{var}[\mathfrak{D}_h f]_s(dx). \quad (4)$$

Fix the set $O_f \subset \mathbb{R}$ such that $\mu_s^f(O_f) = \mu_s^f(\mathbb{R})$, $\mu_a^f(O_f) = 0$ and choose the sequence $\{\varphi_m\} \subset C_b^1(\mathbb{R})$ such that

1) $\varphi_m(0) = 0$, $\max(|\varphi_m|, |\varphi_m'|) \le 1$;

2) $\varphi_m' \to \operatorname{sign}(\zeta_n) \mathbb{1}_{O_f}$, $m \to \infty$ $(\operatorname{mod} \lambda^1 + \mu_s^f)$.

Then $\varphi_m \to 0$, $m \to +\infty$ uniformly on every finite interval, and passing to the limit by $m \to \infty$ in (4) one has that

$$\int_{\mathbb{R}} |\zeta_n^g(t)| \mu_s^f(dt) \le \|g\|_\infty \int_{\mathfrak{X}} |g_n(x)| \operatorname{var}[\mathfrak{D}_h f]_s(dx).$$

Now, taking $n \to \infty$ we have that

$$\zeta_n(t) \to \zeta(t) \equiv \mathrm{E}\left[g \frac{d[\mathfrak{D}_h]_a}{d\mu} \,|\, f = t\right]$$

for μ^f-almost all $t \in \mathbb{R}$ and

$$\int_{\mathfrak{X}} |g_n(x)| \operatorname{var}[\mathfrak{D}_h f]_s(dx) \to 0, \quad n \to \infty$$

and therefore

$$\int_{\mathbb{R}} |\zeta^g(t)| \mu_s^f(dt) = 0.$$

This implies that for the set $U_{g,h}^f \equiv \left\{\mathrm{E}\left[g \frac{d[\mathfrak{D}_h f]_a}{d\mu} \,|\, f\right] \ne 0\right\}$

$$\mu|_{U_{g,h}^f} \circ f^{-1} \ll \lambda^1.$$

This gives the needed statement because

$$U_h^f = \bigcup_{g\in\mathscr{C}_0} U_{g,h}^f.$$

Lemma is proved.

Proof of the theorem. Due to the result of the Lemma 1 it is sufficient to prove that for any fixed $h \in \mathcal{N}_f$

$$\left\{\frac{d[\mathfrak{D}_h f]_a}{d\mu} \ne 0\right\} \subset U_h^f \quad (\operatorname{mod} \mu).$$

As far as $U_h^f \in \sigma(f)$ there exists a sequence of continuous functions with compact support on $\mathbb{R}$ $\{\varphi_n\}$, $|\varphi_n| \le 1$, such that $\varphi_n(f) \to 1 - \mathbb{1}_{U_h^f}$, $n \to \infty$ a. s. Then for $g \in \mathscr{C}_0$ one has

$$\mathrm{E}\left[1 - \mathbb{1}_{U_h^f}\right] \frac{d[\mathfrak{D}_h f]_a}{d\mu} g = \lim_{n\to\infty} \mathrm{E}\varphi_n(f) \frac{d[\mathfrak{D}_h f]_a}{d\mu} g =$$

$$= \lim_{n\to\infty} \int_{\mathfrak{X}} f_n(f(x)) \left[\int_{\mathfrak{X}} \frac{d[\mathfrak{D}_h f]_a}{d\mu} g \, d\mu_x\right] \mu(dx) = \int_{\mathfrak{X}\setminus U_h^f} \left[\int_{\mathfrak{X}} \frac{d[\mathfrak{D}_h f]_a}{d\mu} g \, d\mu_x\right] \mu(dx) = 0$$

by the definition of the set U_h^f. Therefore $1\!\!1_{\mathscr{X}\setminus U_h^f}\cdot\dfrac{d[\mathscr{D}_h f]_a}{d\mu}=0$ that gives the needed statement. Theorem is proved.

***Remark* 3.** As one can see, we proved the statement which formally is even stronger than claimed in the theorem, namely the set Δ_f was changed by the bigger set $\bigcup_k U_{h_k}^f$. In the case $\mathscr{X}=H=\mathbb{R}^d$, $\nabla_H f=(f_1',\ldots,f_d')$ and $f\in C^1(\mathbb{R})$ one can use the Sard theorem and the implicit function theorem to show that this sets a. s. coincide. The question whether this sets coincide in the general case is still open.

4. The generalized derivative and the distribution density for a local time of a stable process. Let $\{\xi(t),\, t\geq 0\}$ be a one-dimensional stable process with independent increments of the order $\alpha\in(0,2]$, i. e. the characteristic function of $\xi(t)$ is equal to

$$\varphi_t(\lambda)\equiv\exp\left\{iat\lambda-ct|\lambda|^{\alpha}\left[1-i\beta\frac{\mu}{|\mu|}\operatorname{tg}\frac{\pi}{2}\alpha\right]\right\},\qquad c>0,\qquad |\beta|\leq 1.$$

Denote by μ the distribution of ξ in $\mathscr{X}=D([0,+\infty))$.

Let $\alpha\in(1,2]$. It is known (see [10]) that then for every point $x\in\mathbb{R}$ there exists the local time $\{\nu_t^x,\, t\geq 0\}$ of the process ξ at the point x. The following properties of the family $\{\nu^x\}$ can be easily obtained from the general properties of Markov W-functionals:

1) ν^x is nondecreasing function by t and its set of the points of growth is a subset of $\{t\,|\,\xi(t)=x\}$;

2) $\nu_t^x = L_1-\lim\limits_{\varepsilon\to 0}\dfrac{1}{2\varepsilon}\displaystyle\int_0^t 1\!\!1_{[x-\varepsilon,\,x+\varepsilon]}(\xi(s))\,ds;$

3) $\nu_t^{x_n}\to\nu_t^x$ in L_1 when $x_n\to x$.

Let the process ξ be centered, i. e. $a=0$. Then process ξ is semistable of the order $\gamma=1/\alpha$ in the sence that for every $u\in\mathbb{R}$ process $T_u\xi\equiv\left\{e^{\gamma u}\xi(te^{-u}),\, t\geq 0\right\}$ has the same distribution with ξ. Further we use the differential structure corresponding to semistable processes (see [5, p. 48]) to construct the formal differentiation ∇_H which is full and admissible w. r. t. μ and ν_t^x has a generalized derivative w. r. t. ∇_H.

Denote by $\mathscr{C}$ the set of functionals of the form

$$f=F\left(\int_0^{t_1}c_1(\xi(s))\,ds,\ldots,\int_0^{t_n}c_n(\xi(s))\,ds\right),\qquad F\in C_b^1(\mathbb{R}^n),\qquad c_k\in C_b^1(\mathbb{R}).\qquad (5)$$

It easy to see that for every functional (5) there exists the limit in the L_p sence (for every p)

$$\nabla f\equiv\lim_{u\to 0}\frac{f(T_u\xi)-f(\xi)}{u}=\sum_{j=1}^{n}F_j'\left(\int_0^{t_1}c_1(\xi(s))\,ds,\ldots,\int_0^{t_n}c_n(\xi(s))\,ds\right)\times$$

$$\times\left[\int_0^{t_j}\left\{c_j(\xi(s))+\gamma\xi(s)c_j'(\xi(s))\right\}ds-t_jc_j\big(\xi(t_j)\big)\right].$$

Obviously, the pair $(\nabla, \mathscr{C})$ is the full formal differentiation on $\mathscr{X}$ (now the Hilbert space H from the general definition coincides with $\mathbb{R}^1$ and is ommited in notations). As far as $\mathrm{E}f(\xi) = \mathrm{E}f(T_u\xi)$ for every $u \in \mathbb{R}$, $f \in \mathscr{C}$, this differentiation is admissible w. r. t. μ with logarithmic derivative $\rho \equiv 0$. Note that the fact that the process ξ is not differentiable in L_p sence implies that $\xi(T) \notin W_p^1$, $p > 1$, $T > 0$.

Let $T > 0$ be fixed.

Theorem 2. *The functional* ν_T^0 *has generalized derivative* $\mathscr{D}\nu_T^0$ *w. r. t.* μ *and*

1) $\dfrac{d[\mathscr{D}_{\nu_T^0}]_a}{d\mu} = (1-\gamma)\nu_T^0$;

2) *the measure* $[\mathscr{D}\nu_T^0]_s$ *coincide with the weak limit of the measures* $\mu_\varepsilon(\cdot) \equiv \equiv -\mu(\cdot \cap \{|\xi(T)| < \varepsilon\})/\varepsilon$, $\varepsilon \to 0$ *and is equal* $-Tp_T(0)\mu^T$ *where* p_T *is the distribution density for* $\xi(T)$ *and* μ^T *is the distribution of* ξ *under condition* $\{\xi(T) = 0\}$.

Proof. Let $\varepsilon > 0$ be fixed, denote

$$f_\varepsilon = \frac{1}{2\varepsilon}\int_0^T \mathbb{1}_{|\xi(s)|<\varepsilon}\, ds.$$

One has that for $u \in \mathbb{R}$

$$f_\varepsilon(T_u\xi) = \frac{1}{2\varepsilon}\int_0^{e^{-u}T} \mathbb{1}_{|\xi(s)|<e^{-\gamma u}\varepsilon}\, ds$$

and

$$\frac{f_\varepsilon(T_u\xi) - f_\varepsilon(T_{-u}\xi)}{2u} \xrightarrow[u\to 0]{L_1} f_\varepsilon(\xi) - \frac{\gamma}{2}\left[\nu_T^\varepsilon + \nu_T^{-\varepsilon}\right] - \frac{1}{2\varepsilon}T\,\mathbb{1}_{|\xi(T)|<\varepsilon}.$$

Let $g \in \mathscr{C}$ be fixed, one has $\mathrm{E}f_\varepsilon(T_u\xi)g(T_u\xi) = \mathrm{E}f_\varepsilon g$, $u \in \mathbb{R}$. Therefore

$$\mathrm{E}f_\varepsilon(T_u\xi)\frac{g(T_u\xi) - g(T_{-u}\xi)}{2u} = -\mathrm{E}g(T_u\xi)\frac{f_\varepsilon(T_u\xi) - f_\varepsilon(T_{-u}\xi)}{2u}. \tag{6}$$

The right-hand side of (6) converges to

$$-\mathrm{E}g\left\{f_\varepsilon - \frac{\gamma}{2}\left[\nu_T^\varepsilon + \nu_T^{-\varepsilon}\right] - \frac{1}{2\varepsilon}T\,\mathbb{1}_{|\xi(T)|<\varepsilon}\right\}$$

when $u \to 0$ by the Lebesgue theorem of the majorised convergence. It is easy to check that

$$f_\varepsilon(T_u\xi) \to f_\varepsilon, \qquad \frac{g(T_u\xi) - g(T_{-u}\xi)}{2u} \to \nabla g, \qquad u \to 0$$

in the mean square sence. Therefore the left-hand side of (6) converges to $\mathrm{E}f_\varepsilon \nabla g = -\mathrm{E}f_\varepsilon I(g)$, and the functional f_ε has the weak derivative $\mathscr{D}f_\varepsilon$ with

$$\frac{d\mathscr{D}f_\varepsilon}{d\mu} = \left\{f_\varepsilon - \frac{\gamma}{2}\left[\nu_T^\varepsilon + \nu_T^{-\varepsilon}\right] - \frac{1}{2\varepsilon}T\,\mathbb{1}_{|\xi(T)|<\varepsilon}\right\}.$$

Analogously, for every $\varphi \in C_b^1$ the functional $\varphi(f_\varepsilon)$ has the weak derivative $\mathcal{D}\varphi(f_\varepsilon)$ with

$$\frac{d\mathcal{D}\varphi(f_\varepsilon)}{d\mu} = \varphi'(f_\varepsilon)\left\{f_\varepsilon - \frac{\gamma}{2}\left[\nu_T^\varepsilon + \nu_T^{-\varepsilon}\right] - \frac{1}{2\varepsilon}T\,\mathbb{1}_{|\xi(T)|<\varepsilon}\right\}.$$

Our next step is to show that the family $\{\mathcal{D}\varphi(f_\varepsilon), \varepsilon > 0\}$ has the weak limit $\mathcal{D}\varphi(\nu_T^0)$ as $\varepsilon \to 0$. Note that as far as every functional from $\mathcal{C}$ is continuous in the metrics of $D([0, +\infty))$, this will give us that $\varphi(\nu_T^0)$ has the weak derivative equal to $\mathcal{D}\varphi(\nu_T^0)$.

By the Lebesgue theorem and properties of the local time ν

$$\varphi'(f_\varepsilon)\left\{f_\varepsilon - \frac{\gamma}{2}\left[\nu_T^\varepsilon + \nu_T^{-\varepsilon}\right]\right\} \to (1-\gamma)\varphi'(\nu_T^0)\nu_T^0 \quad \text{in} \quad L_1(\mu),\ \varepsilon \to 0.$$

Thus the measures

$$d\kappa_\varepsilon^{\varphi,1} \equiv \varphi'(f_\varepsilon)\left\{f_\varepsilon - \frac{\gamma}{2}\left[\nu_T^\varepsilon + \nu_T^{-\varepsilon}\right]\right\}d\mu$$

converge as $\varepsilon \to 0$ to the measure $d\kappa_0^{\varphi,1} \equiv (1-\gamma)\varphi'(\nu_T^0)\nu_T^0 d\mu$ in variation.

Consider the family $d\kappa_\varepsilon^{\varphi,2} \equiv \frac{1}{2\varepsilon}T\varphi'(f_\varepsilon)\mathbb{1}_{|\xi(T)|<\varepsilon}d\mu$, $\varepsilon > 0$ and prove that it is weakly compact. As far as μ is the distribution of the process with independent increments and the density $\frac{d\kappa_\varepsilon^{\varphi,2}}{d\mu}$ is $\sigma\{\xi(\tau), \tau \le T\}$-measurable, it is sufficient to prove compactness of the family $d\kappa_\varepsilon^{\varphi,2}|_{D([0,T])}$. Due to [11, p. 179] to prove such a compactness it is sufficient to verify that there exist such $\delta > 0$, $b > 0$, $C < +\infty$ that for every triple $0 \le t_1 < t_2 < t_3 \le T$ with $\Delta = t_3 - t_1 < \frac{T}{3}$

$$\int\limits_{D([0,+\infty))} |\xi(t_2) - \xi(t_1)|^b |\xi(t_3) - \xi(t_2)|^b \operatorname{var}[\kappa_\varepsilon^{\varphi,2}](d\xi) \le C\Delta^{1+\delta}. \tag{7}$$

Suppose first that $t_3 \le \frac{2T}{3}$. Then

$$\int\limits_{D([0,+\infty))} |\xi(t_2) - \xi(t_1)|^b |\xi(t_3) - \xi(t_2)|^b \operatorname{var}[\kappa_\varepsilon^{\varphi,2}](d\xi) \le$$

$$\le \frac{T\|\varphi'\|_\infty}{2\varepsilon} \int\limits_{|x_1+x_2+x_3+x_4|<\varepsilon} p_{t_1}(x_1)p_{t_2-t_1}(x_2)p_{t_3-t_2}(x_3)p_{T-t_3}(x_4)\,dx_1dx_2dx_3dx_4,$$

where

$$p_t(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-ix\lambda}\varphi_t(\lambda)\,d\lambda$$

is the density of $\xi(t)$. We have

$$\frac{1}{2\varepsilon}\int\limits_{|x_1+x_2+x_3+x_4|<\varepsilon} |x_2|^b|x_3|^b p_{t_1}(x_1)p_{t_2-t_1}(x_2)p_{t_3-t_2}(x_3)p_{T-t_3}(x_4)\,dx_1dx_2dx_3dx_4 =$$

$$= \int_{\mathbb{R}^3} |x_2|^b |x_3|^b \frac{\int_{-x_1+x_2+x_3-\varepsilon}^{-x_1+x_2+x_3+\varepsilon} p_{T-t_3}(x_4)\,dx_4}{2\varepsilon} p_{t_1}(x_1) p_{t_2-t_1}(x_2) p_{t_3-t_2}(x_3)\,dx_1 dx_2 dx_3 \le$$

$$\le \left[\sup_{t\in[T/3,T],\, x\in\mathbb{R}} p_t(x)\right]\left[\int_{\mathbb{R}} |x|^b p_1(x)\,dx\right]^2 |t_2-t_1|^{b/\alpha}|t_3-t_2|^{b/\alpha}, \quad b<\alpha.$$

Now, taking $b \in (\alpha/2, \alpha)$, we obtain (7) with

$$C = T\|\varphi'\|_\infty \left[\sup_{t\in[T/3,T],\, x\in\mathbb{R}} p_t(x)\right]\left[\int_{\mathbb{R}} |x|\, p_1(x)\,dx\right]^2.$$

Now let $t_3 > \frac{2T}{3}$, then $t_1 > \frac{T}{3}$ and the same estimation can be obtained using inequality

$$\frac{1}{2\varepsilon}\int_{|x_1+x_2+x_3+x_4|<\varepsilon} |x_2|^b |x_3|^b p_{t_1}(x_1) p_{t_2-t_1}(x_2) p_{t_3-t_2}(x_3) p_{T-t_3}(x_4)\,dx_1 dx_2 dx_3 dx_4 =$$

$$= \int_{\mathbb{R}^3} |x_2|^b |x_3|^b \frac{\int_{-x_2+x_3+x_4-\varepsilon}^{-x_2+x_3+x_4+\varepsilon} p_{t_1}(x_1)\,dx_1}{2\varepsilon} p_{t_2-t_1}(x_2) p_{t_3-t_2}(x_3) p_{T-t_3}(x_4)\,dx_2 dx_3 dx_4 \le$$

$$\le \left[\sup_{t\in[T/3,T],\, x\in\mathbb{R}} p_t(x)\right]\left[\int_{\mathbb{R}} |x|^b p_1(x)\,dx\right]^2 |t_2-t_1|^{b/\alpha}|t_3-t_2|^{b/\alpha}, \quad b<\alpha.$$

Therefore the family $\{\kappa_\varepsilon^{\varphi,2}, \varepsilon>0\}$ is weakly compact and consequently has the weak limit point. As far as for every $g \in \mathscr{C}$

$$\lim_{\varepsilon\to 0} \int_{D([0,+\infty))} g(\xi)\kappa_\varepsilon^{\varphi,2}(d\xi) = \lim_{\varepsilon\to 0} \int_{D([0,+\infty))} g(\xi)\kappa_\varepsilon^{\varphi,1}(d\xi) - E\varphi(\nu_T^0) I(g),$$

and $\mathscr{C}$ generates the Borel σ-algebra in $D([0,+\infty))$, this limit point is unique and there exists the weak limit

$$\kappa_0^{\varphi,2} = \lim_{\varepsilon\to 0}\kappa_\varepsilon^{\varphi,2}.$$

This gives that $\varphi(\nu_0^T)$ has the weak derivative w. r. t. ∇ equal to

$$\mathscr{D}\varphi(\nu_T^0) = \kappa_0^{\varphi,1} - \kappa_0^{\varphi,2}.$$

Analogously, there exists

$$\kappa_0^1 = \lim_{\varepsilon\to 0}\kappa_\varepsilon^1, \qquad \kappa_0^2 = \lim_{\varepsilon\to 0}\kappa_\varepsilon^2,$$

with

$$d\kappa_\varepsilon^1 \equiv \left\{f_\varepsilon - \frac{\gamma}{2}\left[\nu_T^\varepsilon + \nu_T^{-\varepsilon}\right]\right\}d\mu, \qquad d\kappa_\varepsilon^2 \equiv \frac{1}{2\varepsilon}T\,\mathbb{1}_{|\xi(T)|<\varepsilon}\,d\mu, \qquad \varepsilon > 0,$$

and ν_T^0 has the weak derivative

$$\mathcal{D}\nu_T^0 = \kappa_0^1 - \kappa_0^2.$$

In the same way with (7) one can obtain the estimate

$$\int\limits_{D([0,+\infty))} |\xi(T)-\xi(t)|^b \kappa_\varepsilon^2(d\xi) \le C(T-t)^\delta,$$

which will imply that the process ξ is stochastically continuous at the point T with respect to measure κ_0^2. As far as this process has the left limit at T a. s. with respect to this measure, this gives that trajectories of this process are a. s continuous at T with respect to κ_0^2. Then by Theorem 5.1, Ch. 1 [11] for every fixed $\varepsilon_0 > 0$

$$\kappa_0^2\{|\xi(T)| > \varepsilon_0\} \le \limsup_{\varepsilon\to 0} \kappa_\varepsilon^2\{\xi(T) > \varepsilon_0\} = 0,$$

and therefore $\kappa_0^2\{\xi(T) > 0\} = 0$, i.e. $\kappa_0^2 \perp \mu$. Thus

$$[\mathcal{D}\nu_0^T]_a = \kappa_0^1, \qquad [\mathcal{D}\nu_0^T]_s = -\kappa_0^2.$$

As far as

$$\left|\frac{d\kappa_\varepsilon^{\varphi,2}}{d\kappa_\varepsilon^2}\right| = |\varphi'(f_\varepsilon)| \le \|\varphi'\|_\infty, \qquad \varepsilon > 0,$$

we have that $\kappa_0^{\varphi,2} << \kappa_0^2$, $\left[\mathcal{D}\varphi(\nu_T^0)\right]_a = \kappa_0^{\varphi,1}$, $\left[\mathcal{D}\varphi(\nu_T^0)\right]_s = -\kappa_0^{\varphi,2}$ and

$$\left|\frac{d\kappa_0^{\varphi,2}}{d\kappa_0^2}\right| \le \|\varphi'\|_\infty.$$

This together with the equality

$$\frac{d\kappa_0^{\varphi,1}}{d\mu} = (1-\gamma)\nu_T^0\varphi'(\nu_T^0) = \varphi'(\nu_T^0)\frac{d\kappa_0^{\varphi,1}}{d\kappa_0^1}$$

gives the needed statement. Theorem is proved.

Now we can use Theorem 1 to obtain the existence of the density of ν_T^0.

Theorem 3. *The distribution of* ν_T^0 *has a density.*

Proof. It follows from Theorems 1, 2 that $\mu\big|_{\{\nu_T^0\neq 0\}}\circ\left[\nu_T^0\right]^{-1} << \lambda^1$. Therefore to prove the theorem we need to show that $\mu\left\{\nu_T^0 = 0\right\} = 0$.

Suppose that it is not true and for some $T > 0$ $a_T \equiv \mu\left\{\nu_T^0 = 0\right\} > 0$. Then for $t \le$ $\le T$ $a_t \ge a_T$, and the Laplace transform of ν_0^t is uniformly detached from zero on $[0, T]$:

$$u(\lambda, t) \equiv \mathrm{E}\exp[-\lambda\nu_t^0] \ge a_T, \qquad \lambda \ge 0. \tag{8}$$

Due to formula (3) from [10] this Laplace transform satisfies equation

$$u(\lambda, t) = 1 - \lambda\int_0^t u(\lambda, t-s)p_s(0)\,ds. \tag{9}$$

Due to (8) the integral in the right-hand side of (9) is not less than $a_T \int_0^t p_s(0)\,ds \in \mathbb{R}^+$ for $t \in [0, T]$. This gives that for such t $u(\lambda, t) \to -\infty$, $\lambda \to +\infty$, and this is impossible. Thus our supposition is false and $\mu\{\nu_T^0 = 0\} = 0$, $T \in \mathbb{R}^+$. Theorem is proved.

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