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THE APPLICATION OF NUMERICAL-ANALYTIC METHOD FOR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A PARAMETER

ЗАСТОСУВАННЯ ЧИСЕЛЬНО-АНАЛІТИЧНОГО МЕТОДУ ДО СИСТЕМ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ПАРАМЕТРОМ

The numerical-analytic method is applied for systems of differential equations with a parameter under assumptions that corresponding functions satisfy the Lipschitz conditions in matrix notation. Some existence results are also obtained for problems with deviations of an argument.

Чисельно-аналітичний метод застосовується до систем диференціальних рівнянь з параметром у припущенні, що відповідні функції задовольняють умови Ліпшиця в матричних позначеннях. Доведено також деякі теореми існування для задач з відхиленнями аргументу.

1. Introduction. Problems with a parameter have been considered for many years. Some of them appeared as mathematical models of physical systems. Conditions which guarantee the existence of solutions are important analysis theorems. Existence results are usually obtained by using fixed point theorems (Banach, Schauder), by the method of successive approximations or by constructing monotone iterations combined with lower and upper solutions. The comparison method seems to be interesting for finding constructive conditions in order to establish the existence / uniqueness results (see, for example, [1]). A useful approach is also to apply Samoilenko's numerical-analytic method described for example in [2]. The application of this technique to differential problems with integral boundary conditions can be found, for example, in papers [3–10] (see also [11]). In this paper we are going to apply this method for systems of differential equations with a parameter of the form

$$x'(t) = f(t, x(t), \lambda), \qquad t \in J = [0, T],$$
$$\lambda = g(x(T), \lambda) \tag{1}$$

together with the integral condition

$$Ax(0) + B\int_0^T x(s)\,ds = d. \tag{2}$$

Here $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$. The matrices $A_{p\times p}$, $B_{p\times p}$, and $d_{p\times 1}$ are constant. Assume that B^{-1} exist.

The numerical-analytic method combined with the comparison one is used to formulate corresponding results under the assumption that f and g satisfy the Lipschitz conditions (with respect to the last two variables) in matrix notation. The aim of the present paper is to discuss the conditions under which the solution exists and it is the limit of successive approximations and Seidel's iterations too. More general differential problems with deviations are also considered and corresponding existence results are given in the last section of this paper.

2. Assumptions. Put

$$\mathcal{L}f(t, x, \lambda) = n\left(1 - \frac{t}{T}\right)\int_0^t f(s, x(s), \lambda)\,ds - \frac{t}{T}\int_t^T f(s, x(s), \lambda)\,ds,$$

$$\Omega(t,u) = \left(1-\frac{t}{T}\right)\int_0^t u(s)\,ds + \frac{t}{T}\int_t^T u(s)\,ds.$$

Indeed, $\mathcal{L}f(0,x,\lambda) = \mathcal{L}f(T,x,\lambda) = O_{p\times 1}$. According to the numerical-analytic method, find the vector δ such that $x(t) = \bar{k}_0 + \mathcal{L}f(t,x,\lambda) + \delta t$ satisfies condition (2) ($\bar{k}_0$ is a certain vector of initial data). Substituting it to problem (1) we have the following auxiliary problem

$$x(t) = \bar{k}_0 + \mathcal{L}f(t,x,\lambda) - \frac{2t}{T^2}\int_0^T \mathcal{L}f(s,x,\lambda)\,ds + tB_1(\bar{k}_0) \equiv F(t,x,\lambda;\bar{k}_0), \quad t\in J,$$
$$\lambda = g(x(T),\lambda) \tag{3}$$

and

$$T\int_0^T f(s,x(s),\lambda)\,ds + 2\int_0^T \mathcal{L}f(s,x,\lambda)\,ds = T^2 B_1(\bar{k}_0)$$

with $B_1(\bar{k}_0) = \frac{2}{T^2}B^{-1}[d-(A+BT)\bar{k}_0]$ assuming that $\det(B)\neq 0$. Indeed, $F(0,x,\lambda;\bar{k}_0) = \bar{k}_0$, so $x(0) = \bar{k}_0$. Note that if B is a zero matrix, then the numerical-analytic method can not be used.

Let us introduce the following assumptions.

Assumption H_1. 1^0. *There are matrices $K_{p\times p}$, $L_{p\times q}$ with nonnegative entries such that*

$$|f(t,x,y)-f(t,\bar{x},\bar{y})| \le K|x-\bar{x}| + L|y-\bar{y}|$$

for all $t\in J$, $x,\bar{x}\in\mathbb{R}^p$, $y,\bar{y}\in\mathbb{R}^q$.

2^0. *There are matrices $M_{q\times p}$, $N_{q\times q}$ with nonnegative entries, $\rho(N)<1$ and such that*

$$|g(x,y)-g(\bar{x},\bar{y})| \le M|x-\bar{x}| + N|y-\bar{y}|$$

for all $x,\bar{x}\in\mathbb{R}^p$, $y,\bar{y}\in\mathbb{R}^q$. Here $|\cdot|$ denotes the absolute value of the vector, so $|x| = (|x_1|,\ldots,|x_p|)^T$ or $|y| = (|y_1|,\ldots,|y_q|)^T$. Moreover, $\rho(N)$ denotes the spectral radius of the matrix N.

Assumption H_2. *For any nonnegative function $h\in C(J,\mathbb{R}_+^p)$ there exists a unique solution $u\in C(J,\mathbb{R}_+^p)$ of the comparison equation*

$$K\Omega(t,u) + D(t)\int_0^T \Omega(s,u)\,ds + h(t) = u(t), \quad t\in J, \tag{4}$$

where

$$D(t) = k\frac{2t}{T^2} + L\frac{4t}{T}\left(\frac{4}{3}-\frac{t}{T}\right)(I-N)^{-1}M(I-ZM)^{-1}K \quad \text{with} \quad Z = L\frac{2T}{3}(I-N)^{-1}$$

assuming that $(I-N)^{-1}$ and $(I-ZM)^{-1}$ exist and are nonnegative.

Note that $D(T) = \frac{2}{T}[K+ZM(I-ZM)^{-1}K] = \frac{2}{T}(I-ZM)^{-1}K$. It means that if u is a solution of equation (4), then

$$u(T) = \frac{2}{T}(I - ZM)^{-1} K \int_0^T \Omega(s,u)ds + h(T).$$

Put

$$\Omega_1(t, u, v) = K\left[\Omega(t,u) + \frac{2t}{T^2}\int_0^T \Omega(s,u)ds\right] + 2t\left(\frac{4}{3} - \frac{t}{T}\right)Lv.$$

Then, by Assumption $H_1(1^0)$, for $t \in J$, we have

$$|\mathcal{L}f(t,x,y) - \mathcal{L}(t,\bar{x},\bar{y})| \le K\Omega(t,|x-\bar{x}|) + 2t\left(1-\frac{t}{T}\right)L|y-\bar{y}|,$$

$$|F(t,x,y;\bar{k}_0) - F(t,\bar{x},\bar{y};\bar{k}_0)| \le |\mathcal{L}f(t,x,y) - \mathcal{L}(t,\bar{x},\bar{y})| + \tag{5}$$

$$+ \frac{2t}{T^2}\int_0^T |\mathcal{L}f(s,x,y) - \mathcal{L}f(s,\bar{x},\bar{y})|ds \le \Omega_1(t,|x-\bar{x}|,|y-\bar{y}|).$$

3. Lemmas. For $n = 0, 1, \ldots$, let us define the sequences $\{u_n, w_n\}$ by formulas

$$u_0(t) = u(t), \quad t \in J,$$

$$u_{n+1}(t) = \Omega_1(t, u_n, w_n), \quad t \in J,$$

$$w_0 = (I - N)^{-1}[Mu_0(T) + r],$$

$$w_{n+1}(t) = Mu_n(T) + Nw_n.$$

Here u denotes the solution of (4) with

$$h(t) = \frac{3t}{T}\left(\frac{4}{3} - \frac{t}{T}\right)[(I - ZM)^{-1}Zr + ZM(I - ZM)^{-1}R(T)] + R(t)$$

for $r = |g(x_0(T), \lambda_0) - \lambda_0|$, $R(t) = |F(t, x_0, \lambda_0; \bar{k}_0) - x_0(t)|$.

To obtain a solution of problem (3) we shall first establish some properties for sequences $\{u_n, w_n\}$. They are given in the next two lemmas.

Lemma 1. *Let Assumptions H_1 and H_2 be satisfied. Assume that $\rho(ZM) < 1$. Then, for $n = 0, 1, \ldots$*

$$u_{n+1}(t) \le u_n(t) \le u_0(t) \text{ for } t \in J \text{ and } w_{n+1} \le w_n \le w_0.$$

Moreover, the sequences $\{u_n, w_n\}$ converge uniformly to zero functions, so $u_n(t) \to 0$, $t \in J$, $w_n \to 0$ if $n \to \infty$.

Proof. Note that the matrix $(I - N)^{-1}$ exists and its entries are nonnegative because of the condition $\rho(N) < 1$. Note that $h(T) = (I - ZM)^{-1}[Zr + R(T)]$, so

$$Mu_0(T) + r = M\left[\frac{2}{T}(I - ZM)^{-1}K\int_0^T \Omega(s,u_0)ds + h(T)\right] + r =$$

$$= M(I - ZM)^{-1}\left[K\frac{2}{T}\int_0^T \Omega(s,u_0)ds + Zr + R(T)\right] + r.$$

Hence

$$u_1(t) = K\left[\Omega(t,u_0) + \frac{2t}{T^2}\int_0^T \Omega(s,u_0)\,ds\right] + 2Lt\left(\frac{4}{3}-\frac{t}{T}\right)(I-N)^{-1}[Mu_0(T)+r] =$$

$$= \left\{K\frac{2t}{T^2} + L\frac{4t}{T}\left(\frac{4}{3}-\frac{t}{T}\right)(I-N)^{-1}M(I-ZM)^{-1}K\right\}\int_0^T \Omega(s,u_0)\,ds +$$

$$+ K\Omega(t,u_0) + 2Lt\left(\frac{4}{3}-\frac{t}{T}\right)(I-N)^{-1}\{M(I-ZM)^{-1}[Zr+R(T)]+r\} \le$$

$$\le K\Omega(t,u_0) + D(t)\int_0^T \Omega(s,u_0)\,ds + h(t) = u_0(t), \quad t \in J,$$

$$w_1 = Mu_0(T) + N(I-N)^{-1}[Mu_0(T)+r] \le w_0.$$

By induction in n, we are able to prove that

$$u_{n+1}(t) \le u_n(t), \quad t \in J, \quad w_{n+1} \le w_n, \quad n = 0, 1, \ldots.$$

Now, if $n \to \infty$, then $u_n \to u$, $w_n \to w$, where the pair (u, w) is a solution of the system

$$u(t) = K\left[\Omega(t,u) + \frac{2t}{T^2}\int_0^T \Omega(s,u)\,ds\right] + 2Lt\left(\frac{4}{3}-\frac{t}{T}\right)w, \quad t \in J,$$

$$w = Mu(T) + Nw.$$

Hence $w = (I-N)^{-1}Mu(T)$, so

$$K\Omega(t,u) + 2K\frac{t}{T^2}\int_0^T \Omega(s,u)\,ds + 2Lt\left(\frac{4}{3}-\frac{t}{T}\right)(I-N)^{-1}Mu(T) = u(t), \quad t \in J.$$

Finding from this $u(T)$, we finally obtain $K\Omega(t,u) + D(T)\int_0^T \Omega(s,u)\,ds = u(t)$, $t \in J$. By Assumption H_2, $u(t) = 0$ on J and then $w = 0$. The proof is complete.

Lemma 2. *Assume that* $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$, *and* $A_{p\times p}$, $B_{p\times p}$ *and* $d_{p\times 1}$ *are given constant matrices. Assume that* $\det(B) \ne 0$. *Let Assumptions* H_1 *and* H_2 *be satisfied and* $\rho(ZM) < 1$. *Then we have the estimates*

$$|x_n(t) - x_0(t)| \le u_0(t), \quad t \in J, \quad |\lambda_n - \lambda_0| \le w_0,$$
$$|x_{n+k}(t) - x_k(t)| \le u_k(t), \quad t \in J, \quad |\lambda_{n+k} - \lambda_k| \le w_k, \tag{6}$$

where $x_0 \in C^1(J, \mathbb{R}^p)$, $\lambda_0 \in \mathbb{R}^q$ *and*

$$x_{n+1}(t) = F(t, x_n, \lambda_n; \bar{k}_0), \quad \lambda_{n+1} = g(x_n(T), \lambda_n). \tag{7}$$

Moreover,

$$Ax_{n+1}(0) + B\int_0^T x_{n+1}(s)\,ds = d, \quad n = 0, 1, \ldots.$$

Proof. Indeed,

$$|x_1(t) - x_0(t)| = R(t) \le h(t) \le u_0(t), \quad t \in J,$$

$$|\lambda_1 - \lambda_0| = r \le [N(I-N)^{-1} + I]r \le w_0.$$

Assume that

$$|x_k(t) - x_0(t)| \le u_0(t), \quad t \in J, \quad |\lambda_k - \lambda_0| \le w_0$$

for some $k \ge 0$. Then, by (5), we have

$$|x_{k+1}(t) - x_0(t)| \le |F(t, x_k, \lambda_k; \bar{k}_0) - F(t, x_0, \lambda_0; \bar{k}_0)| + R(t) \le$$

$$\le \Omega_1(t, u_0, w_0) + R(t) = u_0(t), \quad t \in J,$$

$$|\lambda_{k+1} - \lambda_0| \le |g(x_k(T), \lambda_k) - g(x_0(T), \lambda_0)| + r \le Mu_0(T) + Nw_0 + r = w_0.$$

Hence, by mathematical induction, we have

$$|x_n(t) - x_0(t)| \le u_0(t), \quad t \in J, \quad |\lambda_n - \lambda_0| \le w_0$$

for $n = 0, 1, \ldots$. Basing on the above, let us assume that

$$|x_{n+k}(t) - x_k(t)| \le u_k(t), \quad t \in J, \quad |\lambda_{n+k} - \lambda_k| \le w_k$$

for all n and some $k \ge 0$. Then, by (5), we see that

$$|x_{n+k+1}(t) - x_{k+1}(t)| = |F(t, x_{n+k}, \lambda_{n+k}; \bar{k}_0) - F(t, x_k, \lambda_k; \bar{k}_0)| \le$$

$$\le \Omega_1(t, u_k, w_k) = u_{k+1}(t), \quad t \in J,$$

$$|\lambda_{n+k+1} - \lambda_{k+1}| = |g(x_{n+k}(T), \lambda_{n+k}) - g(x_k(T), \lambda_k)| \le Mu_k(T) + Nw_k = w_{k+1}.$$

Hence, by mathematical induction, the estimates (6) hold. It is quite simple to verify that x_{n+1} satisfies integral condition (2) for any $n = 0, 1, \ldots$. It ends the proof.

4. Existence results. Combining Lemmas 1 and 2 we have following theorem.

Theorem 1. *Assume that* $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$, *and* $A_{p\times p}$, $B_{p\times p}$ *and* $d_{p\times 1}$ *are given constant matrices. Assume that* $\det(B) \ne 0$. *Let Assumption* H_1 *and* H_2 *be satisfied and* $\rho(ZM) < 1$. *Then, for every* $\bar{k}_0 \in \mathbb{R}^p$, *there exists a unique solution* $(\bar{x}, \bar{\lambda})$ *of problem* (3) *where* $x_n(t) \to \bar{x}(t)$, $t \in J$, $\lambda_n \to \bar{\lambda}$ *as* $n \to \infty$ *and we have the estimates*

$$|x_n(t) - \bar{x}(t)| \le u_n(t), \quad t \in J, \quad |\lambda_n - \bar{\lambda}| \le w_n.$$

Moreover, $(\bar{x}, \bar{\lambda})$ *is the solution of problem* (1), (2) *iff*

$$T\int_0^T f(s, \bar{x}(s), \bar{\lambda})\,ds + 2\int_0^T \mathcal{L}f(s, \bar{x}, \bar{\lambda})\,ds = T^2 B_1(\bar{k}_0).$$

We see that under the assumption $\rho(N) < 1$, theoretically we may solve the second equation of (1) with respect to λ, so $\lambda = \mathcal{B}(x(T))$ and then substituting it to the first equation of (1) we have only equation for x. This approach is useless since the mapping $\mathcal{B}$ is not know explicitly, so we can not find the approximate solution (x_n, λ_n) of problem (3). It is a reason to consider problem (1) as a differential-algebraic one.

***Remark* 1.** Note that equation (4) has a unique solution $u \in C(J, R_+^p)$ if we assume that

$$\rho(W) < 1 \quad \text{for} \quad W = \frac{8}{9}T(I - ZM)^{-1}K.$$

In place of iterations (7), it is sometimes convenient to use Seidel's method described by

$$\begin{aligned} \tilde{x}_{n+1}(t) &= F(t,\tilde{x}_n,\tilde{\lambda}_n;\bar{k}_0), \\ \tilde{\lambda}_{n+1} &= g(\tilde{x}_{n+1}(T),\tilde{\lambda}_n), \end{aligned} \quad \text{or} \quad \begin{aligned} \bar{\lambda}_{n+1} &= g(\bar{x}_n(T),\bar{\lambda}_n), \\ \bar{x}_{n+1}(t) &= F(t,\bar{x}_n,\bar{\lambda}_{n+1};\bar{k}_0) \end{aligned} \tag{8}$$

for $t \in J$, and $n = 0, 1, \ldots$.

For $t \in J$, and $n = 0, 1, \ldots$, let us define the sequences:

$$\begin{aligned} \tilde{u}_0(t) &= u_0(t), \quad \tilde{w}_0 = w_0, & \bar{u}_0(t) &= u_0(t), \quad \bar{w}_0 = w_0, \\ \tilde{u}_{n+1}(t) &= \Omega_1(t,\tilde{u}_n,\tilde{w}_n), & \bar{w}_{n+1} &= M\bar{u}_n(T) + N\bar{w}_n, \\ \tilde{w}_{n+1} &= M\tilde{u}_{n+1}(T) + N\tilde{w}_n, & \bar{u}_{n+1}(t) &= \Omega_1(t,\bar{u}_n,\bar{w}_{n+1}). \end{aligned}$$

Now, we are able to show the following result by mathematical induction.

Lemma 3. *Let Assumption H_1 and H_2 hold. Assume that $\rho(ZM) < 1$. Then*

$$\bar{u}_n(t) \le u_n(t), \quad t \in J, \quad \bar{w}_n \le w_n, \quad n = 0, 1, \ldots,$$

$$\tilde{u}_n(t) \le u_n(t), \quad t \in J, \quad \tilde{w}_n \le w_n, \quad n = 0, 1, \ldots,$$

and $\bar{u}_n(t) \to 0$, $\bar{w}_n \to 0$, $\tilde{u}_n(t) \to 0$, $\tilde{w}_n \to 0$ if $n \to \infty$.

The simple consequence of Lemma 3 is the following theorem.

Theorem 2. *Assume that all assumptions of Theorem 1 are satisfied. Then, $\bar{x}_n(t) \to \bar{x}(t)$, $\bar{\lambda}_n \to \bar{\lambda}$, $\tilde{x}_n(t) \to \bar{x}(t)$, $\tilde{\lambda}_n \to \bar{\lambda}$, $t \in J$ as $n \to \infty$ for $\bar{x}_0(t) = \tilde{x}_0(t) = x_0(t)$, $\bar{\lambda}_0 = \tilde{\lambda}_0 = \lambda_0$, $t \in J$. Moreover, we have the estimates*

$$|\bar{x}_n(t) - \bar{x}(t)| \le \bar{u}_n(t), \quad t \in J, \quad |\bar{\lambda}_n - \bar{\lambda}| \le \bar{w}_n,$$

$$|\tilde{x}_n(t) - \bar{x}(t)| \le \tilde{u}_n(t), \quad t \in J, \quad |\tilde{\lambda}_n - \bar{\lambda}| \le \tilde{w}_n$$

for $n = 0, 1, \ldots$.

Note that iterations (7) and (8) converge to $(\bar{x}, \bar{\lambda})$ under the same conditions but basing on Lemma 3 we see that the error estimates for (8) are better in comparing with the corresponding estimates for (7). This notice is important since $\{x_n, \lambda_n\}$, $\{\bar{x}_n, \bar{\lambda}_n\}$ and $\{\tilde{x}_n, \tilde{\lambda}_n\}$ are approximate solutions of problem (3).

Theorem 3. *Assume that all assumptions of Theorem 1 are satisfied. Then*

$$|\Delta(t,\bar{x},\bar{\lambda};\bar{k}_0) - \Delta(t,x_n,\lambda_n;\bar{k}_0)| \le Q(t,u_n,w_n),$$

$$|\Delta(t,\bar{x},\bar{\lambda};\bar{k}_0) - \Delta(t,\bar{x}_n,\bar{\lambda}_n;\bar{k}_0)| \le Q(t,\bar{u}_n,\bar{w}_n),$$

$$|\Delta(t,\bar{x},\bar{\lambda};\bar{k}_0) - \Delta(t,\tilde{x}_n,\tilde{\lambda}_n;\bar{k}_0)| \le Q(t,\tilde{u}_n,\tilde{w}_n)$$

for $t \in J$, $n = 0, 1, \ldots$, where

$$\Delta(t,x,\lambda;\bar{k}_0) = T\int_0^T f(s,x,\lambda)\,ds + 2\int_0^T \mathcal{L}f(s,x,\lambda)\,ds - T^2 B_1(\bar{k}_0),$$

$$Q(t,u,w) = K\int_0^T [Tu(s) + 2\Omega(s,u)]\,ds + \frac{5}{3}LT^2 w.$$

5. Differential-algebraic systems with deviations. Let $\alpha \in C(J, J)$. Let us consider the following problem

$$x'(t) = f(t, x(\alpha(t)), \lambda), \quad t \in J = [0, T],$$
$$\lambda = g(x(T)), \lambda) \tag{9}$$

with condition (2), where $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$. Let $\det(B) \neq \neq 0$. According to the numerical-analytic method find the vector δ such that

$$x(t) = \bar{k}_0 + \mathcal{P}z(t) + \delta t \quad \text{with} \quad \mathcal{P}z(t) = \left(1 - \frac{t}{T}\right)\int_0^t z(s)ds - \frac{t}{T}\int_t^T z(s)ds$$

satisfies condition (2). Then, this and (9) give the following auxiliary problem

$$z(t) = f(t, \bar{k}_0 + \mathcal{P}z(\alpha(t)) - \frac{2\alpha(t)}{T^2}\int_0^T \mathcal{P}z(s)ds + \alpha(t)B_1(\bar{k}_0), \lambda) \equiv \mathcal{F}(t, z, \lambda; \bar{k}_0),$$
$$\lambda = g(\bar{k}_0 + TB_1(\bar{k}_0) - \frac{2}{T}\int_0^T \mathcal{P}z(s)ds, \lambda) \equiv \mathcal{G}(z, \lambda; \bar{k}_0) \tag{10}$$

and

$$T\int_0^T z(s)ds + 2\int_0^T \mathcal{P}z(s)ds = T^2 B_1(\bar{k}_0),$$

where B_1 is defined as in Section 2.

Now, let us define the sequences $\{z_n, \gamma_n\}$ by formulas

$$z_{n+1}(t) = \mathcal{F}(t, z_n, \gamma_n; \bar{k}_0), \quad t \in J, \quad z_0 \in C(J, \mathbb{R}^p),$$
$$\gamma_{n+1} = \mathcal{G}(z_n, \gamma_n; \bar{k}_0), \quad \gamma_0 \in \mathbb{R}^q, \quad n = 0, 1, \ldots . \tag{11}$$

Basing on Assumption H_1, we have the following properties

$$|\mathcal{F}(t, z, \lambda; \bar{k}_0) - \mathcal{F}(t, w, \gamma; \bar{k}_0)| \leq$$
$$\leq K\left[\Omega(\alpha(t), |z - w|) + \frac{2\alpha(t)}{T^2}\int_0^T \Omega(s, |z - w|)ds\right] + L|\lambda - \gamma|,$$

$$|\mathcal{G}(z, \lambda; \bar{k}_0) - \mathcal{G}(w, \gamma; \bar{k}_0)| \leq \frac{2}{T}M\int_0^T \Omega(s, |z - w|)ds + N|\lambda - \gamma|$$

because $|\mathcal{P}z(t) - \mathcal{P}w(t)| \leq \Omega(t, |z - w|)$.

Assumption H_3. *For any nonnegative function $H \in C(J, \mathbb{R}^p_+)$ there exists a unique solution $v \in C(J, \mathbb{R}^p_+)$ of the comparison equation*

$$v(t) = K\Omega(\alpha(t), v) + Q(t)\int_0^T \Omega(s, v)ds + H(t)$$

with $Q(t) = \frac{2}{T}\left[\frac{\alpha(t)}{T}K + L(I - N)^{-1}M\right]$.

For $n = 0, 1, \ldots,$ let us define the sequences $\{u_n, w_n\}$ by relations

$$u_0(t) = v(t), \quad t \in J,$$
$$u_{n+1}(t) = K\left[\Omega(\alpha(t), u_n) + \frac{2\alpha(t)}{T^2}\int_0^T \Omega(s, u_n)ds\right] + Lw_n, \quad t \in J, \tag{12}$$

$$w_0 = (I - N)^{-1}\left[M\frac{2}{T}\int_0^T \Omega(s, u_0)ds + r\right],$$
$$w_{n+1} = \frac{2}{T}M\int_0^T \Omega(s, u_n)ds + Nw_n, \tag{13}$$

where v is defined as in Assumption H_3 with

$$H(t) = L(I-N)^{-1}r + R(t),$$
$$R(t) = |\mathcal{F}(t, z_0, \gamma_0; \bar{k}_0) - z_0(t)|, \quad r = |\mathcal{G}(z_0, \gamma_0; \bar{k}_0) - \gamma_0|.$$

Lemma 4. *Assume that* $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$, $\alpha \in C(J, J)$. *Moreover,* $A_{p\times p}$, $B_{p\times p}$ *and* $d_{p\times 1}$ *are given constant matrices. Assume that* $\det(B) \neq 0$. *Let Assumptions* H_1 *and* H_3 *be satisfied. Then the sequences* $\{u_n, w_n\}$ *of the form* (12), (13) *are nonincreasing and converge uniformly to zero functions if* $n \to \infty$. *Moreover, we can show that*

$$|z_n(t) - z_0(t)| \le u_0(t), \quad |\gamma_n - \gamma_0| \le w_0,$$
$$|z_{n+k}(t) - z_k(t)| \le u_k(t), \quad |\gamma_{n+k} - \gamma_k| \le w_k \tag{14}$$

for $t \in J$ *and* $n = 0, 1, \ldots,$ *where* z_n *and* γ_n *are defined by* (11).

Proof. By mathematical induction, it is simple to show that

$$u_{n+1}(t) \le u_n(t) \le u_0(t), \quad t \in J, \quad w_{n+1} \le w_n \le w_0, \quad n = 0, 1, \ldots.$$

Hence $u_n \to 0$, $w_n \to 0$, by Assumption H_3. Now we are going to show (14). Note that

$$|z_1(t) - z_0(t)| = R(t) \le u_0(t),$$
$$|\gamma_1 - \gamma_0| = r \le [(I-N)(I-N)^{-1} + N(I-N)^{-1}]r = (I-N)^{-1}r \le w_0.$$

If we assume that $|z_k(t) - z_0(t)| \le u_0(t)$, $|\gamma_k - \gamma_0| \le w_0$, $t \in J$ for some $k \ge 1$, then, using the estimations for $\mathcal{F}$ and $\mathcal{G}$, we see that

$$|z_{k+1}(t) - z_0(t)| \le |\mathcal{F}(t, z_k, \gamma_k; \bar{k}_0) - \mathcal{F}(t, z_0, \gamma_0; \bar{k}_0)| + R(t) \le$$
$$\le K\left[\Omega(\alpha(t), u_0) + \frac{2\alpha(t)}{T^2}\int_0^T \Omega(s, u_0)ds\right] + Lw_0 + R(t) = u_0(t),$$
$$|\gamma_{k+1} - \gamma_0| \le |\mathcal{G}(t, z_k, \gamma_k; \bar{k}_0) - \mathcal{G}(t, z_0, \gamma_0; \bar{k}_0)| + r \le$$
$$\le \frac{2}{T}M\int_0^T \Omega(s, u_0)ds + Nw_0 + r = w_0.$$

Hence, by mathematical induction, we have

$$|z_n(t) - z_0(t)| \le u_0(t), \quad |\gamma_n - \gamma_0| \le w_0, \quad t \in J, \quad n = 0, 1, \ldots .$$

The rest of estimates (14) can be proved by the similar argument. It ends the proof.

Theorem 4. *Assume that all assumptions of Lemma 4 are satisfied. Then, for every* $\bar{k}_0 \in \mathbb{R}^p$, *system* (11) *of sequences* $\{z_n, \gamma_n\}$ *converges to the unique solution* $(\bar{z}, \bar{\gamma})$ *of problem* (10), *so* $z_n(t) \to \bar{z}(t)$, $\gamma_n \to \bar{\gamma}$ *for* $t \in J$ *if* $n \to \infty$ *and we have the error estimates*

$$|z_n(t) - \bar{z}(t)| \le u_n(t), \quad t \in J, \quad |\gamma_n - \bar{\gamma}| \le w_n, \quad n = 0, 1, \ldots$$

Here u_n, w_n *are defined by formulas* (12), (13). *Moreover,* $(\bar{x}, \bar{\gamma})$ *with* $\bar{x} = \bar{k}_0 \int_0^t \bar{z}(s)\,ds$ *is the solution of problem* (9) *with condition* (2) *iff*

$$2\int_0^T \mathcal{P}\bar{z}(s)\,ds + T\int_0^T \bar{z}(s)\,ds = T^2 B_1(\bar{k}_0).$$

***Remark* 2.** Note that Assumption H_3 holds if we assume that $\rho(W) < 1$, where

$$W = 2K \max_{t \in J}\left\{\alpha(t)\left[\frac{3}{4} - \frac{\alpha(t)}{T}\right]\right\} + \frac{2}{3}TL(I-N)^{-1}M.$$

Similarly as before to find a solution $(\bar{z}, \bar{\gamma})$ of problem (10) we can apply Seidel's method. It means that we can formulate the following theorem.

Theorem 5. *Let all ssumptions of Lemma 4 be satisfied. Then the results of Theorem 4 hold and* $\bar{z}_n(t) \to \bar{z}(t)$, $\tilde{z}_n(t) \to \bar{z}_n(t)$, $\bar{\gamma}_n \to \bar{\gamma}$, $\tilde{\gamma}_n \to \bar{\gamma}$, *where* $\{\bar{z}_n, \bar{\gamma}_n\}$ *and* $\{\tilde{z}_n, \tilde{\gamma}_n\}$ *are defined by*

$$\bar{z}_{n+1}(t) = \mathcal{F}(t, \bar{z}_n, \bar{\gamma}_n; \bar{k}_0), \qquad \tilde{\gamma}_{n+1} = \mathcal{G}(\tilde{z}_n(T), \tilde{\gamma}_n; \bar{k}_0),$$

$$\bar{y}_{n+1}(t) = \mathcal{G}(\bar{z}_{n+1}(T), \bar{\gamma}_n; \bar{k}_0), \qquad \tilde{z}_{n+1}(t) = \mathcal{F}(t, \tilde{z}_n, \tilde{\gamma}_{n+1}; \bar{k}_0)$$

for $t \in J$, $n = 0, 1, \ldots$ *with* $\bar{z}_0(t) = \tilde{z}_0(t) = z_0(t)$, $t \in J$ *and* $\bar{\gamma}_0 = \tilde{\gamma}_0 = \gamma_0$.

Moreover, we have the error estimates

$$|\bar{z}_n(t) - \bar{z}(t)| \le \bar{u}_n(t), \qquad |\bar{\gamma}_n - \bar{\gamma}| \le \bar{w}_n,$$

$$|\tilde{z}_n(t) - \bar{z}(t)| \le \tilde{u}_n(t), \qquad |\tilde{\gamma}_n - \bar{\gamma}| \le \tilde{w}_n$$

for $t \in J$, $n = 0, 1, \ldots$, *where*

$$\bar{u}_0(t) = u_0(t), \qquad \bar{w}_0(t) = w_0(t),$$

$$\bar{u}_{n+1}(t) = K\left[\Omega(\alpha(t), \bar{u}_n) + \frac{2\alpha(t)}{T^2}\int_0^T \Omega(s, \bar{u}_n)\,ds\right] + L\bar{w}_n,$$

$$\bar{w}_{n+1} = \frac{2}{T}M\int_0^T \Omega(s, \bar{u}_{n+1})\,ds + N\bar{w}_n,$$

$$\tilde{u}_0(t) = u_0(t), \qquad \tilde{w}_0 = w_0,$$

$$\bar{w}_{n+1} = \frac{2}{T} M \int_0^T \Omega(s, \bar{u}_n) ds + N \bar{w}_n,$$

$$\bar{u}_{n+1}(t) = K \left[\Omega(\alpha(t), \bar{u}_n) + \frac{2\alpha(t)}{T^2} \int_0^T \Omega(s, \bar{u}_n) ds \right] + L \bar{w}_{n+1}$$

with u_0, w_0 *defined as in relations* (12) *and* (13), *respectively.*

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