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FINITARY AND ARTINIAN-FINITARY GROUPS OVER THE INTEGERS $\mathbb{Z}$

ФІНІТАРНІ ТА АРТИНОВО-ФІНІТАРНІ ГРУПИ НАД ЦІЛИМИ ЧИСЛАМИ $\mathbb{Z}$

In a series of papers we have considered finitary (that is, Noetherian-finitary) and Artinian-finitary groups of automorphisms of arbitrary modules over arbitrary rings. The structural conclusions for these two classes of groups are really very similar, especially over commutative rings. The question arises of the extend to which each class is a subclass of the other.

Here we resolve this question by concentrating just on the ground ring of the integers $\mathbb{Z}$. We show that even over $\mathbb{Z}$ neither of these two classes of groups is contained in the other. On the other hand we show how each group in either class can be built out of groups in the other class. This latter fact helps to explain the structural similarity of the groups in the two classes.

У низці робіт автора розглядались фінітарні (тобто нетерово-фінітарні) і артиново-фінітарні групи автоморфізмів довільних модулів над довільними кільцями. Структурні висновки для цих двох класів груп дуже подібні, особливо у випадку комутативних кілець. Виникає питання про те, в якій мірі один з цих класів є підкласом іншого.

У даній роботі це питання вирішується на прикладі кільця чисел $\mathbb{Z}$. Показано, що навіть для $\mathbb{Z}$ жоден із цих двох класів не міститься в іншому. З іншого боку, показано, як будь-яку групу одного класу можна побудувати з груп іншого, що дає можливість пояснити структурну подібність груп в цих двох класах.

In a series of papers (see [1 – 5]) we have introduced various forms of finitary groups of arbitrary modules over arbitrary rings. Among the most interesting are the Noetherian-finitary (or just finitary for short) and the Artinian-finitary groups. Our structural conclusions in these two cases are not identical, but are too similar to be coincidental, especially over commutative rings. What is the relationship between these two classes of groups? For example, is one a subclass of the other? Here we resolve some of these questions by making a detailed study of these groups, but just over the integers $\mathbb{Z}$. As a result we can say that even over $\mathbb{Z}$ neither of these classes is contained in the other. Notwithstanding this, we also show how, over $\mathbb{Z}$, groups in each of these classes can be pieced together from groups in the other.

We must start by recalling the basic definitions. Let M be a module over the ring R. In [4] we define various generalized finitary groups, of which the following are relevant here.

For α an ordinal or ∞,

$$F_\alpha \mathrm{Aut}_R M = \{ g \in \mathrm{Aut}_R M : M(g-1) \text{ has Krull dimension less than } \alpha \},$$

$$F^\alpha \mathrm{Aut}_R M = \{ g \in \mathrm{Aut}_R M : M(g-1) \text{ has Krull co-dimension less than } \alpha \}.$$

In particular, the Artinian-finitary group of M over R is

$$F_1 \mathrm{Aut}_R M = \{ g \in \mathrm{Aut}_R M : M(g-1) \text{ is Artinian} \}$$

and the Noetherian-finitary group of M over R is

$$F^1 \mathrm{Aut}_R M = \{ g \in \mathrm{Aut}_R M : M(g-1) \text{ is Noetherian} \}.$$

The latter we also call the finitary group of M over R and so denote it by $F\mathrm{Aut}_R M$. There are two further finitary groups we have cause to mention. These are

$$F_{\text{fcs}}\,\text{Aut}_R M = \{ g \in \text{Aut}_R M \colon M(g-1) \text{ has a finite composition series}\}$$

and

$$F_{\widetilde{\mathfrak{F}}}\,\text{Aut}_R M = \{ g \in \text{Aut}_R M \colon M(g-1) \text{ is finite}\}.$$

The basic relationships between these various finitary groups can be found in [4].

Now assume that $R = \mathbb{Z}$, the integers. Any module with Krull dimension has finite uniform dimension. Hence an abelian group M with Krull dimension is an extension of a free abelian group of finite rank by a direct sum of finite number of co-cyclic groups. Thus M has Krull dimension and Krull co-dimension at most 1. Clearly Noetherian, Artinian abelian groups are finite. Thus for any $\mathbb{Z}$-module M we have the following relationships

$$F_{\widetilde{\mathfrak{F}}}\,\text{Aut}_{\mathbb{Z}} M = F_{\text{fcs}}\,\text{Aut}_R M = F\,\text{Aut}_{\mathbb{Z}} M \cap F_1\,\text{Aut}_{\mathbb{Z}} M,$$

$$F_{\text{fcs}}\,\text{Aut}_{\mathbb{Z}} M \le F_1\,\text{Aut}_{\mathbb{Z}} M \le F_2\,\text{Aut}_{\mathbb{Z}} M,$$

$$F_{\text{fcs}}\,\text{Aut}_{\mathbb{Z}} M \le F\,\text{Aut}_{\mathbb{Z}} M = F^1\,\text{Aut}_{\mathbb{Z}} M \le F^2\,\text{Aut}_{\mathbb{Z}} M$$

and

$$F_2\,\text{Aut}_{\mathbb{Z}} M = F_\infty\,\text{Aut}_{\mathbb{Z}} M = F^\infty\,\text{Aut}_{\mathbb{Z}} M \le F^2\,\text{Aut}_{\mathbb{Z}} M.$$

We show below that there are $\mathbb{Z}$-modules M_1 and M^1 such that $F_1\,\text{Aut}_{\mathbb{Z}} M_1$ does not embed into $F\,\text{Aut}_{\mathbb{Z}} M$ for any M, while $F\,\text{Aut}_{\mathbb{Z}} M^1$ does not embed into $F_1\,\text{Aut}_{\mathbb{Z}} M$ for any M. We develop a close relationship between these groups. In particular, there is a vague and ill-defined duality between the finitary and the Artinian-finitary groups, with Noetherian and Artinian interchanging at the module level and finitely generated and periodic interchanging at the group level.

Below we number our results consecutively 1, 2 etc. Where a result can be considered as part of the duality above we label the result 1^1, 2^1 etc. in the finitary case and 1_1, 2_1 etc. in the Artinian-finitary case. The reader should for each relevant i compare i^1 with i_1. Since the duality is incomplete, for a given i^1 there may, for example, be no i_1. Although these two types of finitary groups are incomparable, even over $\mathbb{Z}$, the finitary (that is, the Noetherian-finitary) case does seem the stronger. A positive result i^1 is usually easier to prove that the comparative result i_1, while a negative result i^1 is usually harder to prove that the comparative result i_1, when they both exist.

Let π be any infinite set of odd primes p such that $p \equiv 2 \pmod 3$; note that such sets π do exists by Dirichlet's Theorem. For each p in π let A_p be a divisible abelian p-group of rank 2 and in $GL(2, \mathbb{Z})$ set

$$g = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Regard g as also lying in $\text{Aut}\,A_p \cong GL(2, \mathbb{Z}_p) \ge GL(2, \mathbb{Z})$. Set $A = \times_{p \in \pi} A_p$, the direct product of the A_p, and let G be the split extension $\langle g \rangle A$ of A by $\langle g \rangle$. For each p let E_p be an (additive) Prüfer p^∞-group and set $E = \oplus_{p \in \pi} E_p$.

1^1. *With the notation above the following hold*:

a) *G is periodic and metabelian*;

b) *G embeds into $F\,\mathrm{Aut}_{\mathbb{Z}} M$ for $M = \mathbb{Z}^{(2)} \oplus E$*;

c) *G does not embed into $F_1\,\mathrm{Aut}_{\mathbb{Z}} M$ for any $\mathbb{Z}$-module M.*

Proof. a) Clearly g has order 3. Part a) follows.

b) Regard A as $E \oplus E$, so if $a \in A$, then $a = (a_1, a_2)$ for some a_1 and a_2 in E. Define a map ρ of G into

$$\begin{pmatrix} GL(2, \mathbb{Z}) & E \oplus E \\ 0 & \mathrm{Aut}\, E \end{pmatrix} \cong \mathrm{Aut}\, M$$

by

$$g\rho = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad a\rho = \begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}$$

for $a \in A$. Then ρ in easily seen to be a well-defined embedding of G. Also $M(g\rho - 1) = \mathbb{Z}^{(2)}(g-1)$, which is finitely $\mathbb{Z}$-generated, while $M(a\rho - 1) = \mathbb{Z}a_1 + \mathbb{Z}a_2 \le E$, and so is finite. Therefore G embeds via ρ into $F\,\mathrm{Aut}_{\mathbb{Z}} M$.

c) Note that $3 \notin \pi$. Suppose $G \le F_1\,\mathrm{Aut}_{\mathbb{Z}} M$ for some $\mathbb{Z}$-module M. Then $M(g-1)$ is Artinian. We prove that $M(g-1)$ contains an element of order p for every p in π. This contradiction of π infinite will prove Part c). To accomplish this we need consider only one p in π. To simplify notation set $B = A_p$ and $H = \langle g \rangle B \le G$.

Now $[B, g] = B^{(g-1)}$ is divisible and nontrivial. If $[B, g] < B$, then $B = [B, g] \times C$ for some $C \ne \langle 1 \rangle$ and so g acts reducibly on the $GF(p)$-space $\Omega_1 B = \{ b \in B : b^p = 1 \}$. By Maschke's Theorem (recall $p \ne 3$) g acts diagonally on $\Omega_1 B$. But $GF(p)$ contains no primitive cube root of 1. Thus g centralizes $\Omega_1 B$. By [6], Lemma 3.28, alternatively by Lemma 7 below, the element g centralizes B, which is false. Therefore $[B, g] = B$ and so $\langle g^H \rangle = H$.

Let T be the $\mathbb{Z}$-torsion submodule of M. For each prime q set $T_{q'} = \oplus_{r \ne q} T_r$ and set $M_q = M / T_{q'}$. Now $M_q(b-1)$ is an Artinian q-group and also a B-submodule for any b in B. In particular, $\mathrm{Aut}(M_q(b-1))$ is residually finite, so B centralizes each $M_q(b-1)$ and therefore $[M_q, B, B] = \{0\}$.

Suppose $q \ne p$. Stability theory maps the p-group B into $\mathrm{Hom}(M_q, H_q)$ and the torsion subgroup of the latter is a q-group. Hence $[M_q, B] = \{0\}$ and consequently $[M, B] = \bigcap_{q \ne p} T_{q'} = T_p$. Suppose $M(g-1)$ contains no element of order p. It is also Artinian and so torsion. Therefore $M_p(g-1) = \{0\}$. But then $H = \langle g^H \rangle \ge B$ also centralizes M_p. Consequently $[M, B] = T_{p'}$. But $[M, B] = T_p$ by the above. Thus B centralizes M, which is false. This proves that $M(g-1)$ contains an element of order p and completes the proof of 1^1.

1_1. *Let G be the wreath product of two infinite cyclic groups. The following hold*:

a) *G is finitely generated and metabelian*;

b) *for any prime p, if M is a divisible abelian p-group of rank 2, then G embeds into $F_1\,\mathrm{Aut}_{\mathbb{Z}} M = \mathrm{Aut}_{\mathbb{Z}} M \cong GL(2, \mathbb{Z}_p)$*;

c) *G does not embed into $F\,\mathrm{Aut}_{\mathbb{Z}}M$ for any $\mathbb{Z}$-module M.*

Proof. a) This is clear.

b) Pick any unit u of $\mathbb{Z}_p$ that is transcendental over the rationals $\mathbb{Q}$. Then G is isomorphic to the subgroup

$$\left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

of $GL(2, \mathbb{Z}_p)$.

c) Suppose $G \le F\,\mathrm{Aut}_{\mathbb{Z}}M$ for some $\mathbb{Z}$-module M. By [1], 2.3 c), we may assume that M is finitely $\mathbb{Z}$-generated. Then $\mathrm{Aut}_{\mathbb{Z}}M$ embeds into $GL(n, \mathbb{Z})$ for some integer n and in particular the soluble subgroups of $\mathrm{Aut}_{\mathbb{Z}}M$ are polycyclic. Clearly G is soluble but not polycyclic. The result follows.

From now on let M be some abelian group and let T denote the torsion subgroup of M. Denote the p-primary component of T by T_p and the divisible radical of T by D. Our first positive results, namely 2^1 and 2_1, summarise our starting points; they are extracted from [1] and [5].

2^1. *There is a normal subgroup N of $G = F\,\mathrm{Aut}_{\mathbb{Z}}M$ that is a Fitting group (and hence is locally nilpotent and hyperabelian) such that G/N embeds into $F_1\,\mathrm{Aut}_{\mathbb{Q}\times\mathbb{Z}}M_1$ for some $\mathbb{Q}\times\mathbb{Z}$-module M_1.*

Proof. We apply 2.2 of [1] and its proof and notation. With $N = \bigcap_\sigma C_G(N_\sigma)$, the subgroup N is a Fitting group by [1], 3.2 and 3.7. By [1], 2.2 and its proof, we can embed G/N into a direct product $\times_p FGL(V_p)$, where p runs over 0 and the positive primes of $\mathbb{Z}$ and where V_0 is a vector space over $\mathbb{Q}$ and V_p for $p > 0$ is a vector space over $GF(p)$. Clearly $FGL(V_p) = F_1\,\mathrm{Aut}_{\mathbb{Z}}V_p$ for $p > 0$ and the claim follows with $M_1 = V_0 \oplus \left(\oplus_{p>0} V_p\right)$.

2_1. *There is a locally residually nilpotent normal subgroup N of $G = F_1\,\mathrm{Aut}_{\mathbb{Z}}M$ such that G/N embeds into $F\,\mathrm{Aut}_{\mathbb{Z}}M_1$ for some $\mathbb{Z}$-module M_1.*

Compared with 2^1, the conclusion on N in 2_1 is weaker, while that on G/N is stronger.

Proof. Apply the proof of [5], 2.7, to $\mathbb{Z}$ and M. In the notation of that proof, set $N = \bigcap_{\sigma<\tau} C_G\left(A_{\sigma,1}/M_\sigma\right)$. Each field k_σ of that proof is an image of $\mathbb{Z}$, so $FGL(V_\sigma) = F\,\mathrm{Aut}_{\mathbb{Z}}V_\sigma$. Therefore G/N embeds into $F\,\mathrm{Aut}_{\mathbb{Z}}(\oplus_{\sigma>\tau}V_\sigma)$.

3^1. *If M is finitely generated (that is, if M is $\mathbb{Z}$-Noetherian), there is a positive integer n such that for every prime p, the group $F\,\mathrm{Aut}_{\mathbb{Z}}M$ $(= \mathrm{Aut}_{\mathbb{Z}}M)$ embeds into $GL(n, \mathbb{Z}_p) = F_1\,\mathrm{Aut}_{\mathbb{Z}}M_1$ for M_1 a divisible abelian p-group of rank n.*

For there exists an integer n such that $\mathrm{Aut}_{\mathbb{Z}}M$ embeds into $GL(n, \mathbb{Z}) \le GL(n, \mathbb{Z}_p)$. This simple result 3^1 is true in essence if $\mathbb{Z}$ is replaced by any commutative ring (not so simple, see 14^1 below).

There is no 3_1 analogue to 3^1. Specifically if M is $\mathbb{Z}$-Artinian, then $F_1\,\mathrm{Aut}_{\mathbb{Z}}M$ need not embed into $F\,\mathrm{Aut}_{\mathbb{Z}}M_1$ for any abelian group M_1. The example of 1_1

already shows this. If M is finite, of course $F_1 \operatorname{Aut}_{\mathbb{Z}} M = F \operatorname{Aut}_{\mathbb{Z}} M$. If M is infinite (and still Artinian) and if π denotes the spectrum of the divisible part of M, then π is finite and nonempty and it is easy to show that for some integer n, the group $F_1 \operatorname{Aut}_{\mathbb{Z}} M = (= \operatorname{Aut}_{\mathbb{Z}} M)$ embeds into $\times_{p\in\pi} GL(n, \mathbb{Z}_p)$; thus it embeds into $F \operatorname{Aut}_J J^{(n)}$ for $J = = \prod_{p\in\pi} \mathbb{Z}_p$. This could be interpreted as some sort of weak analogue of 3^1.

4^1. *If* $G = F \operatorname{Aut}_{\mathbb{Z}} M$, *then* $C_G(M/T) = F_{\hat{\aleph}} \operatorname{Aut}_{\mathbb{Z}} M = F_{\text{fcs}} \operatorname{Aut}_{\mathbb{Z}} M \leq F_1 \operatorname{Aut}_{\mathbb{Z}} M$. *In particular,* $G/C_G(T)$ *embeds into* $F \operatorname{Aut}_{\mathbb{Z}} T \leq F_1 \operatorname{Aut}_{\mathbb{Z}} T$ *and if* M *is* $\mathbb{Z}$-*torsion, then* $F \operatorname{Aut}_{\mathbb{Z}} M \leq F_1 \operatorname{Aut}_{\mathbb{Z}} M$.

This is all obvious.

4_1. *Let* $G = F_1 \operatorname{Aut}_{\mathbb{Z}} M$. *Then* $G/C_G(M/D)$ *embeds naturally into* $F \operatorname{Aut}_{\mathbb{Z}}(M/D)$. *In particular, if* $D = \{0\}$, *then* $F_1 \operatorname{Aut}_{\mathbb{Z}} M \leq F \operatorname{Aut}_{\mathbb{Z}} M$.

Proof. For notational simplicity only, assume $D = \{0\}$. If $g \in G$, then $M(g-1)$ is Artinian, torsion and reduced. Therefore $M(g-1)$ is finite and consequently

$$F_1 \operatorname{Aut}_{\mathbb{Z}} M = F_{\hat{\aleph}} \operatorname{Aut}_{\mathbb{Z}} M = F_{\text{fcs}} \operatorname{Aut}_{\mathbb{Z}} M \leq F \operatorname{Aut}_{\mathbb{Z}} M.$$

The general case follows.

In 4^1 and 4_1 we have dealt with the actions of $F \operatorname{Aut}_{\mathbb{Z}} M$ on T and of $F_1 \operatorname{Aut}_{\mathbb{Z}} M$ on M/D. Out next target (8^1 and 8_1) is to analyse the actions of $F \operatorname{Aut}_{\mathbb{Z}} M$ on M/T and of $F_1 \operatorname{Aut}_{\mathbb{Z}} M$ on D. For this we need a couple of lemmas, special cases at least of which are well known.

5. **Lemma.** *Let* X *be a Noetherian module over the commutative ring* R *and let* E *be an injective hull of some irreducible* R-*module* S. *Set* $X^* = \operatorname{Hom}_R(X, E)$. *Then* X^* *is Artinian.*

Proof. Set $\mathfrak{a} = \operatorname{ann}_R(X)$. Then $R/\mathfrak{a}$ is Noetherian. Also $X^* = \operatorname{Hom}_{R/\mathfrak{a}}(X, \operatorname{ann}_E(\mathfrak{a}))$. Assume $X^* \neq \{0\}$. Let $\phi \in X^*$. Then either $X\phi = \{0\}$ or $S \leq X\phi$. The latter implies that $\mathfrak{a}S = \{0\}$. Thus since $X^* \neq \{0\}$, so S is an essential submodule of $\Delta = \operatorname{ann}_E(\mathfrak{a})$. Hence Δ is Artinian as $R/\mathfrak{a}$-module by [7], 4.30, and therefore Δ is also Artinian as R-module.

X is Noetherian and hence finitely generated. For some integer k we have an exact sequence $R^{(k)} \to X \to 0$. Then

$$0 \to X^* \to \operatorname{Hom}_R(R^{(k)}, \Delta)$$

is also exact. But

$$\operatorname{Hom}_R(R^{(k)}, \Delta) \cong \operatorname{Hom}_R(R, \Delta)^{(k)} \cong \Delta^{(k)},$$

and the latter is Artinian. Consequently so too is X^*.

6^1. **Corollary.** *If* X *is any module over the commutative ring* R, *then* $F \operatorname{Aut}_R X$ *is residually an Artinian-finitary group of* R-*module automorphisms.*

Proof. Set $G = F \operatorname{Aut}_R X$ and let S be any irreducible R-module. Denote an injective hull of S by E and put $X^* = \operatorname{Hom}_R(X, E)$. Then G acts on X^* on the left via its right action on X. Let $g \in G$ and set $Y = X(g-1)$. Then Y^* embeds into X^*

via $\psi \mapsto (g-1)\psi$ and then $(g-1)X^* \le Y^*$. By 5 we have that Y^* and hence $(g-1)X^*$ is Artinian. Consequently we have a map ρ_S of G into $F_1 \operatorname{Aut}_R X^*$.

Let $g \in G \setminus \langle 1 \rangle$ and pick x in X with $xg \ne x$. Set $y = x(g-1)$. By [7], 2.24, we can choose the irreducible R-module S such that there is some ϕ in X^* with $y\phi \ne 0$. Then $x((g-1)\phi) \ne 0$ and so $(g-1)X^* \ne \{0\}$. Therefore $g\rho_S \ne 1$. Consequently $\bigcap_S \ker \rho_S = \langle 1 \rangle$ and the corollary follows.

7. **Lemma.** *Let A be a divisible torsion abelian group and let g be an automorphism of A of finite order that fixes every element of A of order twice a prime. Then $g = 1$.*

Proof. If A has finite rank, this is a lemma of Baer, see [6], 3.28. If g fixes every element of A of prime-square order, then $g = 1$ by 1.F.2 of [8]. Clearly, therefore, we may assume that A is a p-group for some odd prime p. We may also assume that g has prime order q and seek a contradiction.

Each factor $A(g-1)^i / A(g-1)^j$ is divisible for $i \le j$, so, for example,

$$\operatorname{Hom}(A/A(g-1), A(g-1)/A(g-1)^2)$$

is torsion-free. Therefore the stability group of the series $A \ge A(g-1) \ge A(g-1)^2$ is torsion-free and hence $A(g-1) = A(g-1)^2$. In particular g cannot centralize $A(g-1)$ and therefore must act faithfully on $A(g-1)$. Thus replacing A by $A(g-1)$ if necessary, we may assume that $A = A(g-1)$.

Set $h = 1 + g + g^2 + \ldots + g^{q-1}$. Then $Ah = A(g-1)h = A(g^q - 1) = \{0\}$. Pick $a \in A$ with $ag \ne a$ and $|a|$ minimal. Necessarily $|a| > p$. Then $p(a(g-1)) = (pa)(g-1) = \{0\}$ by the minimality of a. Hence $b = a(g-1)$ has order p and so is fixed by g. Since $ag = a + b$, this and a simple induction yields that $ag^i = a + ib$ for all $i \ge 1$. Therefore

$$0 = ah = qa + \sum_{1 \le i < q} ib = qa + \frac{q(q-1)b}{2}. \qquad (*)$$

If $p \ne q$, then $pa = 0$ (recall $pb = 0$), which is false. Thus $p = q$. Also p is odd, so p divides $q(q-1)/2$ and $(*)$ becomes $0 = pa$, again a contradiction. The proof of the lemma is complete.

(There is no real need to quote [8], 1. F. 2, at the start of the above proof. If we assume $p = 2$ in the above calculation, then $q = 2$ and $(*)$ yields that $4a = -2b = 0$, a contradiction since here $|a| > 4$.)

8^1. *If $\pi(T) = \{p : T_p \ne \{0\}\}$ is finite, then $F \operatorname{Aut}_{\mathbb{Z}} M$ embeds into $F_1 \operatorname{Aut}_{\mathbb{Z}} M_1$ for some $\mathbb{Z}$-module M_1. Thus in general $F \operatorname{Aut}_{\mathbb{Z}} M$ is residually an Artinian-finitary group of $\mathbb{Z}$-automorphisms (cf. 6^1) and $F \operatorname{Aut}_{\mathbb{Z}}(M/T)$ embeds into $F_1 \operatorname{Aut}_{\mathbb{Z}} M_1$ for some $\mathbb{Z}$-module M_1. In particular if $G = F \operatorname{Aut}_{\mathbb{Z}} M$, then $G/C_G(M/T)$ embeds into $F_1 \operatorname{Aut}_{\mathbb{Z}} M_1$ for some $\mathbb{Z}$-module M_1 and if M is $\mathbb{Z}$-torsion-free, then G embeds into $F_1 \operatorname{Aut}_{\mathbb{Z}} M_1$ for some $\mathbb{Z}$-module M_1.*

In general in 8^1 the group $C_G(T)$ need not embed into some $F \operatorname{Aut}_{\mathbb{Z}} M_1$ by 1^1, cf. 4^1. Also we cannot replace $\pi(T)$ finite by $\pi(T/C_T(G))$ finite, again by the example 1^1.

Proof. Suppose first that T is p-group for some prime p. Then M is residually a p-group; for if $y \in M \setminus \langle 0 \rangle$, then $\langle py \rangle \neq \langle y \rangle$ and there exists a homomorphism ϕ of M into a Prüfer p^{∞}-group P with $(py)\phi = 0 \neq y\phi$, see [7], 2.24. Set $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, P)$. Then $F\mathrm{Aut}_{\mathbb{Z}} M$ acts on M^* on the left via its right action on M. If $g \neq 1$ lies in $F\mathrm{Aut}_{\mathbb{Z}} M$, then $xg \neq x$ for some $x \in M$ and so by the above (take $y = x(g-1)$) there exists some ϕ in M^* with $px(g-1)\phi = 0 \neq x(g-1)\phi$, hence $g\phi \neq \phi$ and $F\mathrm{Aut}_{\mathbb{Z}} M$ acts faithfully on M^*.

For $g \in F\mathrm{Aut}_{\mathbb{Z}} M$, set $X = M(g-1)$. Then X^* is Artinian by the lemma 5 and X^* embeds into M^* via $\psi \mapsto (g-1)\psi$ such that $(g-1)M^* \leq X^*$. Consequently $(g-1)M^*$ is Artinian for any such g and therefore $F\mathrm{Aut}_{\mathbb{Z}} M$ embeds into $F_1\mathrm{Aut}_{\mathbb{Z}} M^*$.

If $\pi(T)$ is just finite, set $T_{p'} = \oplus_{q \neq p} T_q$ and $M_p = M/T_{p'}$. Let P_p be a Prüfer p^{∞}-group and set $(M_p)^* = \mathrm{Hom}_{\mathbb{Z}}(M_p, P_p)$. Then $F\mathrm{Aut}_{\mathbb{Z}} M$ embeds into $\times_{p \in \pi} F\mathrm{Aut}_{\mathbb{Z}}(M_p)$, which embeds into $\times_{p \in \pi} F_1\mathrm{Aut}_{\mathbb{Z}}(M_p)^*$ by the above, which in turn embeds into $F_1\mathrm{Aut}_{\mathbb{Z}}\left(\oplus_{p \in \pi}(M_p)^*\right)$. From this all of 8^1 easily follows.

8_1. *Set* $D_1 = \langle x \in D : 2px = 0$ *for some prime* $p \rangle$. *Let* $G \leq F_1\mathrm{Aut}_{\mathbb{Z}} M$ *be such that* $G/C_G(D)$ *is periodic. Then* $G/C_G(D)$ *embeds into* $F\mathrm{Aut}_{\mathbb{Z}} D_1$. *In particular, if* $G \leq F_1\mathrm{Aut}_{\mathbb{Z}} M$ *is periodic, then* $G/C_G(D)$ *embeds into* $F\mathrm{Aut}_{\mathbb{Z}} D_1$.

The "duality" between 8^1 and 8_1 is imperfect, not least because we have to restrict to periodic groups in 8_1. The symmetry between 9^1 and 9_1 below is also less than ideal.

Proof. By 7 we have that $G/C_G(D)$ embeds into $F_1\mathrm{Aut}_{\mathbb{Z}} D_1$. Clearly $F_1\mathrm{Aut}_{\mathbb{Z}} D_1 = F\mathrm{Aut}_{\mathbb{Z}} D_1$.

9^1. *Set* $G = F\mathrm{Aut}_{\mathbb{Z}} M$. *Then* G *has an abelian normal subgroup* A *such that* G/A *embeds into* $F_1\mathrm{Aut}_{\mathbb{Z}} M_1$ *for some* $\mathbb{Z}$*-module* M_1. *Also* G *is an extension of a subgroup of* $F_1\mathrm{Aut}_{\mathbb{Z}} M$ *by a subgroup of some* $F_1\mathrm{Aut}_{\mathbb{Z}} M_1$.

Proof. By 8^1 the group $G/C_G(M/T)$ embeds into $F_1\mathrm{Aut}_{\mathbb{Z}} M_1$ for some $\mathbb{Z}$-module M_1. Hence $G/(C_G(M/T) \cap C_G(T))$ embeds into $F_1\mathrm{Aut}_{\mathbb{Z}}(M_1 \oplus T)$ by 4^1. Clearly $C_G(M/T) \cap C_G(T)$ embeds into the abelian group $\mathrm{Hom}_{\mathbb{Z}}(M/T, T)$, in fact into the potentially smaller group

$$F\mathrm{Hom}_{\mathbb{Z}}(M/T, T) = \{\phi \in \mathrm{Hom}_{\mathbb{Z}}(M/T, T) : \mathrm{Im}\,\phi \text{ is finite}\}.$$

Finally $C_G(M/T) \leq F_1\mathrm{Aut}_{\mathbb{Z}} M$ by 4^1 again.

9_1. *Set* $G = F_1\mathrm{Aut}_{\mathbb{Z}} M$.

a) *There is an abelian normal subgroup* A *of* G *and a module* L *over the commutative* $J = \prod_{p \text{ prime}} \mathbb{Z}_p$, *such that* G/A *embeds into* $F\mathrm{Aut}_J L$.

b) $C_G(T)$ *is an abelian normal subgroup of* G *and* $G/C_G(T)$ *embeds into* $F_1\mathrm{Aut}_{\mathbb{Z}} T$.

Proof. a) This follows at once from [5], 3.4, except for the identification of the ring J. A check of the construction of the ring S in the proof of [5], 3.4, shows that when $R = \mathbb{Z}$ we may choose $S = J$.

b) Here $[M, G] \le T$, so stability theory shows that $C_G(T)$ can be embedded into the abelian group $\mathrm{Hom}_{\mathbb{Z}}(M/T, T)$. Clearly $G/C_G(T)$ embeds naturally into $F_1 \mathrm{Aut}_{\mathbb{Z}} T$.

10^1. *$F \mathrm{Aut}_{\mathbb{Z}} M$ is locally residually finite and locally an Artinian-finitary automorphism group over $\mathbb{Z}$. More precisely, if G is a finitely generated subgroup of $F \mathrm{Aut}_{\mathbb{Z}} M$ there is an integer n such that for each prime p and divisible abelian p-group M_1 of rank n, the group G embeds into*

$$GL(n, \mathbb{Z}) \le GL(n, \mathbb{Z}_p) \cong F_1 \mathrm{Aut}_{\mathbb{Z}} M_1.$$

Any periodic subgroup of $F \mathrm{Aut}_{\mathbb{Z}} M$ is locally finite.

Proof. Given the finitely generated subgroup G, by [1], 2.3, we may assume that M is finitely $\mathbb{Z}$-generated. Now apply 3^1 (and its proof). Of course $GL(n, \mathbb{Z})$ is residually finite and its periodic subgroups are finite.

10_1. *$F_1 \mathrm{Aut}_{\mathbb{Z}} M$ is locally residually finite. If G is a periodic subgroup of $F_1 \mathrm{Aut}_{\mathbb{Z}} M$, then G is locally finite.*

By 1_1 the group $F_1 \mathrm{Aut}_{\mathbb{Z}} M$ need not be locally a finitary automorphism group over $\mathbb{Z}$. However, the "duality" and 10^1 suggests that the group G in 10_1 should be isomorphic to some finitary group over $\mathbb{Z}$. A trivial consequence of 10_1 is that G is at least locally of this type.

Proof. $F_1 \mathrm{Aut}_{\mathbb{Z}} M$ is locally residually finite by [5], 6.2 c). Suppose $G \le F_1 \mathrm{Aut}_{\mathbb{Z}} M$ is periodic and finitely generated. Then $N = [M, G]$ is Artinian ([4], 2.1 a)). If $\pi = \pi(N)$, then π is finite and for some positive integer m we have the following embeddings:

$$G/C_G(N) \to \times_{p\in\pi} GL(m, \mathbb{Z}_p) \to GL(m|\pi|, \mathbb{C}).$$

Therefore $G/C_G(N)$ is finite. Consequently $C_G(N)$ is also finitely generated. But $C_G(N)$ is abelian, since it stabilizes the series $M \ge N \ge \{0\}$. Therefore $C_G(N)$ is finite and hence G is too.

We now consider residual local finiteness. Here we can handle much of the two cases together.

11. *Set $R = \bigcap_{n\ge1} nM$. If $G = F_\infty \mathrm{Aut}_{\mathbb{Z}} M$, then $G/(C_G(M/R) \cap C_G(T))$ is residually locally finite. If M has finite exponent, then G is locally finite.*

Proof. If $pM = \{0\}$ for some prime p, then $G = FGL(_{GF(p)}M)$, which is locally finite. If $p^n M = \{0\}$, then $G / \bigcap_i C_G(p^{i-1}M/p^iM)$ is locally finite by the previous case, while by stability theory $\bigcap_i C_G(p^{i-1}M/p^iM)$ is nilpotent with finite exponent dividing p^{n-1}. Again G is locally finite. It follows that G is locally finite whenever M has finite exponent.

In general this yields that $G/C_G(M/nM)$ and $G/C_G(\langle x \in T : nx = 0\rangle)$ are locally finite. Since $C_G(M/R) = \bigcap_n C_G(M/nM)$ and $C_G(T) = \bigcap_n C_G(\langle x \in T : nx = 0\rangle)$, the result follows.

11^1. *If* $G = F\,\mathrm{Aut}_{\mathbb{Z}} M$, *then* $C_G(M/R)$ *is abelian (for* R *as in* 11*) and* G *is abelian by residually-locally-finite (in symbols* $G \in \mathfrak{A}R L\mathfrak{F}$*). The group* G *need not be residually locally finite.*

Proof. If $g \in G$, then $M/C_M(g) \cong M(g-1)$, which is finitely $\mathbb{Z}$-generated and hence residually finite. Therefore $R \le \bigcap_{g\in G} C_M(g) = C_M(G)$ and so $C_G(M/R)$ stabilizes the series $M \ge R \ge \{0\}$. Then $C_G(M/R)$ is abelian and $G \in \mathfrak{A}RL\mathfrak{F}$ by 11.

Let $u \in GL(2,\mathbb{Z})$ have infinite order and let A be a divisible abelian p-group of rank 2 for some prime p. Then u acts on A via $GL(2,\mathbb{Z}) \le GL(2,\mathbb{Z}_p) \cong \mathrm{Aut}_{\mathbb{Z}} A$ and we can form the split extension $G = \langle u\rangle A$ of A by $\langle u\rangle$. As in the proof of 1^1(b), the group G can be embedded into $F\,\mathrm{Aut}_{\mathbb{Z}} M$ for $M = \mathbb{Z}^{(2)} \oplus E$, for E a Prüfer p^∞-group.

For any positive integer i the group $[A, u^i]$ is divisible and nontrivial. Also $[A, u^j] \le [A, u^i]$ whenever j is a multiple of i. It follows that there is some positive integer r with $B = [A, u^r] \le [A, u^i]$ for all $i \ge 1$. Suppose N is normal subgroup of G with G/N periodic. Then $u^i \in N$ for some $i \ge 1$ and so $N \ge \langle (u^i)^G\rangle \ge [A, u^i] \ge$ $\ge B$. Then $\langle 1\rangle \ne B \le \bigcap_N N$ and so G cannot be residually periodic. The result follows.

11_1. *If* $G = F_1\,\mathrm{Aut}_{\mathbb{Z}} M$, *then* $C_G(T)$ *is abelian and again* $G \in \mathfrak{A}RL\mathfrak{F}$. *Again the group* G *need not be residually locally finite.*

Proof. Here $[M, G] \le T$, so $C_G(T)$ is abelian, stabilizing the series $M \ge T \ge$ $\ge \{0\}$. Thus $G \in \mathfrak{A}RL\mathfrak{F}$ by 11.

Let u be a unit of $\mathbb{Z}_p$ of infinite order, let A be a Prüfer p^∞-group and let G be the split extension $\langle u\rangle A$ of A by $\langle u\rangle$. By [5], 5.6, the group G embeds into $F_1\,\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z} \oplus A)$.

Suppose N is a normal subgroup of G with G/N periodic. Then $u^i \in N$ for some positive integer i. Now $[A, u^i]$ is divisible, since A is, and nontrivial since $\langle u\rangle$ is infinite. Thus $[A, u^i] = A$ and so $A \le \langle (u^i)^G\rangle \le N$. Therefore G is not residually periodic; indeed its periodic residual is A.

12^1. *If* M/T *is divisible, then* $F\,\mathrm{Aut}_{\mathbb{Z}} M$ *embeds naturally into* $F\,\mathrm{Aut}_{\mathbb{Z}} T \le$ $\le F_1\,\mathrm{Aut}_{\mathbb{Z}} T$. *In particular this holds if* $M = \Pi_p T_p$, *where* T_p *for each prime* p *is an abelian* p*-group.*

Proof. Since M/T is divisible and $M(g-1)$ is finitely generated for each g in $G = F\,\mathrm{Aut}_{\mathbb{Z}} M$, so G centralizes M/T. If $g \in C_G(T)$, then $g-1$ maps M/T onto a finitely generated divisible subgroup of T. Therefore $g - 1 = 0$, so $C_G(T) = \langle 1\rangle$ and the first claim follows. It is easy to see that if $M = \Pi_p T_p$, then M/T is always divisible.

In view of 1_1 the direct analogue of 12^1 is false; in particular $F_1\,\mathrm{Aut}_{\mathbb{Z}}(\Pi_p T_p)$ need not embed into any $F\,\mathrm{Aut}_{\mathbb{Z}} M_1$. However we do have some partial analogues.

12_1. *Suppose* M/T *is divisible; for example, suppose* $M = \Pi_p T_p$, *where* T_p *for each prime* p *is an abelian* p*-group.*

a) *If* $D = \{0\}$ *(equivalently in the special case, if each* T_p *is reduced), then* $F_1 \operatorname{Aut}_{\mathbb{Z}} M$ *embeds naturally into* $F \operatorname{Aut}_{\mathbb{Z}} T$.

b) *If* G *is a periodic subgroup of* $F_1 \operatorname{Aut}_{\mathbb{Z}} M$, *then* G *embeds naturally into* $F_1 \operatorname{Aut}_{\mathbb{Z}} T$.

Proof. a) Let $g \in F_1 \operatorname{Aut}_{\mathbb{Z}} M$ centralize T. Then since M/T is divisible, $M(g-1) \le T$ is divisible and hence $\{0\}$. Thus $F_1 \operatorname{Aut}_{\mathbb{Z}} M$ embeds via restriction into

$$F_1 \operatorname{Aut}_{\mathbb{Z}} T = F_{\hat{\pi}} \operatorname{Aut}_{\mathbb{Z}} T = F \operatorname{Aut}_{\mathbb{Z}} T.$$

b) $\operatorname{Hom}_{\mathbb{Z}}(M/T, T) = \operatorname{Hom}_{\mathbb{Z}}(M/T, D)$, which is $\mathbb{Z}$-torsion-free. Also $F_1 \operatorname{Aut}_{\mathbb{Z}}(M/T) = \langle 1 \rangle$. Therefore G embeds via restriction into $F_1 \operatorname{Aut}_{\mathbb{Z}} T$.

Prüfer groups do not sit easily in either $F \operatorname{Aut}_{\mathbb{Z}} M$ or $F_1 \operatorname{Aut}_{\mathbb{Z}} M$. Again we can handle these two cases simultaneously.

13. *Let* G *be a Prüfer* p^{∞}*-subgroup of* $F_{\infty} \operatorname{Aut}_{\mathbb{Z}} M$ *for some prime* p. *Then* G *centralizes* T *and* M/D *and consequently* G *embeds into* $\operatorname{Hom}_{\mathbb{Z}}(M/T, D)$. *In particular, the same conclusion holds for:*

13^1) $F \operatorname{Aut}_{\mathbb{Z}} M$

and

13_1) $F_1 \operatorname{Aut}_{\mathbb{Z}} M$.

Notwithstanding 13^1 and 13_1, such Prüfer subgroups do arise; for example, see the proofs of 1_1, 11^1 and 11_1.

Proof. Suppose $q^n M = \{0\}$ for some prime q and some positive integer n. Then there is a (normal) subgroup N of G of finite exponent (and hence finite order) with G/N $(\cong G)$ embeddable into $FGL(V)$ for some vector space V over $GF(q)$. If $p = q$, then G is unipotent in $FGL(V)$ and hence is elementary-abelian by residually-finite ([9], Theorem B). Therefore $p \ne q$. Any irreducible abelian subgroup of $FGL(V)$ is finite-dimensional and hence finite. Also the unipotent radical of $G \le$ $\le FGL(V)$ is trivial, so we may assume G to be completely reducible. Thus G is residually finite, which it is not. It follows that M cannot have finite exponent. Hence G centralizes $\langle x \in M : nx = 0 \rangle$ for all $n \ge 1$ and therefore G centralizes T.

Now suppose $T = \{0\}$. Then G embeds into $FGL(\mathbb{Q} \otimes M)$ (either by [4], 2.8.3, or directly). Unipotent subgroups of the latter group are torsion-free. Any homomorphism of G to $GL(n, \mathbb{Q})$ for any integer n is trivial, since periodic subgroups of the latter are finite. Again we obtain a contradiction, namely that $G =$ $= \langle 1 \rangle$. For general $\mathbb{Z}$-modules M, this proves that G centralizes M/T.

Suppose $E \le M$ has finite exponent, where G centralizes M/E and (by the first paragraph of the proof) also E. Then G embeds into the group $\operatorname{Hom}_{\mathbb{Z}}(M/E, E)$ of finite exponent, a contradiction that shows this situation does not arise. A simple transfinite induction, using this fact, shows that G centralizes $M/q^{\alpha}T$ for every prime q and every ordinal α. Thus $[M, G] \le \bigcap_{q,\alpha} q^{\alpha} T = D$. That is, G centralizes M/D, as claimed.

We conclude this paper with a generalization of 3^1 from the integers to an arbitrary commutative ring.

14^1. *Let* M *be a Noetherian module over the commutative ring* R. *Then* $\mathrm{Aut}_R M$ *can be embedded into* $\mathrm{Aut}_R N$ *for some Artinian* R*-module* N.

We have already seen that the dual result to 14^1, where we interchange the words "Noetherian" and "Artinian", is false for $R = \mathbb{Z}$. To prove 14^1, we need the following lemma.

15. *With* R *and* M *as in* 14^1, *suppose we have a series* $\{0\} = M_0 < M_1 < \ldots < M_r = M$, *where* $\mathfrak{p}_i = \mathrm{ann}_R(M_i / M_{i-1})$ *is prime and* M_i / M_{i-1} *is* $R/\mathfrak{p}_i$*-torsion-free for each* $i = 1, 2, \ldots, r$. *Let* $\mathfrak{m}_i$ *be any maximal ideal of* R *containing* $\mathfrak{p}_i$. *Then*

$$\bigcap_{1 \le i \le r} \bigcap_{j \ge 0} \mathfrak{m}_i^j M = \{0\}.$$

Note that a series $\{M_i\}$ for M as in 15 always exists, e. g. see [1], 2.2.

Proof. Now $R/\mathrm{ann}_R M$ is Noetherian and $\mathfrak{p}_i \ge \mathrm{ann}_R M$ for each i. Thus we may assume that R is Noetherian. By the Krull Intersection Theorem we have $\bigcap_j \mathfrak{m}_1^j M_1 = \{0\}$. By the Artin – Rees Lemma for each $j \ge 1$ there is some $k \ge j$ with $\mathfrak{m}_1^k M \cap M_1 \le \mathfrak{m}_1^j M_1$. Therefore $M_1 \bigcap \bigcap_k \mathfrak{m}_1^k M = \{0\}$. By induction on r we have $\bigcap_{i>1} \bigcap_{j \ge 0} \mathfrak{m}_i^k M \le M_1$. The result follows.

Proof of 14^1. We may replace R by $R/\mathrm{ann}_R M$ and assume that R in Noetherian. Let E_i be an injective hull of $R/\mathfrak{m}_i$. By [7], 4.30, each E_i is an Artinian R-module. Set $M^* = \mathrm{Hom}_R(M, \oplus_i E_i)$. Since M is finitely generated, there is an exact sequence $R^{(k)} \to M \to 0$ for some integer k. Hence we have an exact sequence $0 \to M^* \to (R^*)^{(k)}$. Now

$$R^* = \mathrm{Hom}_R(R, \oplus E_i) \cong \oplus E_i.$$

Hence $(R^*)^{(k)}$ is Artinian and consequently so too is M^*.

Clearly $\mathrm{Aut}_R M$ acts on M^* on the left via its right action on M, as indeed does $\mathrm{End}_R M$. Let $\eta \in \mathrm{End}_R M \setminus \{0\}$. Then $x\eta \ne 0$ for some x in M. By 15 there exists an integer i and submodules $U < V$ of M with $x\eta \in V \setminus U$ and $V/U \cong R/\mathfrak{m}_i$. Since E_i is injective, this isomorphism extends to a homomorphism $\phi : M \to E_i$ with $U\phi = \{0\}$. Thus $x\eta\phi \ne 0$ and we have found ϕ in M^* with $\eta\phi \ne 0$. Consequently $\mathrm{End}_R M$ acts faithfully on M^* and therefore so too does $\mathrm{Aut}_R M$. The result follows.

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