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GROUPS WITH VARIOUS MINIMAL CONDITIONS ON SUBGROUPS

ГРУПИ З РІЗНИМИ УМОВАМИ МІНІМАЛЬНОСТІ ДЛЯ ПІДГРУП

We briefly review some recent works on groups with the weak minimal condition on nonnilpotent subgroups. We also study the class of groups with the weak minimal condition on non-(soluble of derived length d) subgroups.

Наведено короткий огляд деяких недавніх робіт, присвячених групам із слабкою умовою мінімальності для ненільпотентних підгруп. Також вивчаються групи із слабкою умовою мінімальності для підгруп, які не є розв'язними ступеня d.

1. Introduction. A group G is said to have the minimal condition on subgroups if every nonempty set S of subgroups of G contains a subgroup H such that if $K \leq H$ and $K \in S$ then $K = H$, equivalently of course, every descending chain of subgroups of G terminates in finitely many steps. Groups with the minimal condition on subgroups have played an influential role in the theory of infinite groups and the study of such groups, and groups with related properties, motivated much of the research of Professor S. N. Černikov (see, for example, [1–3]). Indeed, as is well-know, a group G is called a *Černikov group* (or *extremal group* as Černikov called such a group) if it is abelian-by-finite and satisfies the minimal condition on subgroups. Černikov showed in [1] (Theorems 3 and 4) that a soluble group with the minimal condition is an extremal group. Černikov's work was studied by the likes of R. Baer (see, for example, [4], Satz 2), O. J. Schmidt (see [5], Theorem 9) and numerous others.

Since Černikov's original investigations, groups with minimality conditions of many types have been studied. In this paper we discuss recent research on such issues. If $\mathcal{P}$ is a subgroup-theoretical property or class of groups then the group G satisfies min-$\mathcal{P}$, the minimal condition on $\mathcal{P}$-subgroups, if it has no infinite descending chain of $\mathcal{P}$-subgroups. Černikov [3] showed that a locally soluble group with min-sn, the minimal condition on subnormal subgroups, is a Černikov group and in [2, 3] he showed that a locally soluble group with min-ab, the minimal condition on abelian subgroups, is also Černikov. This work paved the way for the celebrated result of Šunkov, Kegel and Wehrfritz (see [6] and [7]) which asserts that a locally finite group with min-ab is Černikov.

We let $\overline{\mathcal{P}}$ denote the class of all groups that either are not-$\mathcal{P}$ groups or are trivial. S. N. Černikov discussed groups with the minimal condition on non-abelian subgroups in [8, 9]. Motivated by such work, Zaicev [10, 11] and Baer [12] defined the *weak minimal condition* on $\mathcal{P}$-subgroups, min-∞-$\mathcal{P}$. A group G satisfies min-∞-$\mathcal{P}$ if it has no infinite strictly descending chain of $\mathcal{P}$-subgroups $H_1 > H_2 > \ldots$ with eath index $|H_i : H_{i+1}|$ infinite. Zaicev himself considered the class of groups satisfying the weak minimal condition on non-abelian subgroups in [13] where he showed that this condition is equivalent to the condition min-∞, at least for locally soluble-by-finite groups. By virtue of the result in [14] such a group is then a soluble-by-finite minimax group.

We denote the class of nilpotent groups by $\mathfrak{N}$ and the class of soluble groups of

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derived length at most d by $\mathfrak{S}_d$. In [15], [16] and [17] we have obtained results that are similar to the above-mentioned results of Zaicev for groups in the classes min-$\overline{\mathfrak{S}_d}$, min-$\overline{\mathfrak{N}}$ and min-∞-$\overline{\mathfrak{N}}$. In [15] we discussed the class of locally graded groups with min-$\overline{\mathfrak{S}_d}$. (Recall that a group G is *locally graded* if every nontrivial finitely generated subgroup of G has a nontrivial finite image.) One reason for restricting attention to locally graded groups is to avoid having to worry about exotic groups such as the Tarski monsters. The main result of [15] gives the precise structure of locally graded groups with min-$\overline{\mathfrak{S}_d}$. In Section 4 below we obtain an analogous result for the class of locally finite groups with min-∞-$\overline{\mathfrak{S}_d}$.

In [16] it was shown that a locally graded group with min-$\overline{\mathfrak{N}}$ that is not locally finite is necessarily nilpotent. At the same time we obtained the structure of locally finite groups with min-∞-$\overline{\mathfrak{N}}$. We state this result next.

Theorem A. *If G is a locally finite group that is neither Černikov nor nilpotent, then G has min-∞-$\overline{\mathfrak{N}}$ if and only if G contains a normal nilpotent p-subgroup K, for some prime p, such that G/K is Černikov and every nonnilpotent subgroup of G that has infinite rank contains K*

The subgroup K, mentioned above, is the Černikov residual of G; that is, $K = \bigcap\{N \triangleleft G : G/N \text{ is Černikov}\}$. Of course a group with all proper subgroups nilpotent satisfies min-∞-$\overline{\mathfrak{N}}$. Our proof of Theorem A therefore requires an important recent result of Asar [18] which says that a locally nilpotent p-group with all proper subgroups nilpotent-by-Černikov is itself nilpotent-by-Černikov. Asar's theorem can be used to show that a locally graded group with all proper subgroups nilpotent is necessarily soluble (see also [19] in this context).

In [17] we extended Theorem A to locally soluble-by-finite groups satisfying min-∞-$\overline{\mathfrak{N}}$ by obtaining the following result.

Theorem B. *Let G be a locally soluble-by-finite group and let K denote the minimax residual of G. Then G satisfies min-∞-$\overline{\mathfrak{N}}$ if and only if one of the following holds:*

(i) *G is nilpotent;*

(ii) *G is minimax, or*

(iii) *G is locally nilpotent, K is nilpotent. G/K is minimax and every nonnilpotent nonminimax subgroup of G contains K.*

Here by the minimax residual of a group G we mean the intersection of all normal subgroups K of G such that G/K is minimax. In [17] we showed that the minimax residual K is periodic and involves just finitely many primes. In Proposition 1 below we show that, as in the locally finite case, K is actually a p-group, for some prime p. We let $\mathfrak{N}_c$ denote the class of nilpotent groups of class at most c. In Section 2 we also classify the locally graded groups with min-∞-$\overline{\mathfrak{N}_c}$, a fairly straightforward matter in view of the results of [17].

The proof of Theorem B given in [17] depends heavily upon results of Zaicev [13, 14]. In contrast with the min-$\overline{\mathfrak{N}}$ case, the structure of locally graded groups with min-∞ is but one difficulty that is to be encountered when one tries to extend the results of [17], and Theorem B in particular, to the class of all locally graded groups. For example, it is unknown whether there exists an infinite finitely generated residually finite p-group, for some prime p, with all proper subgroups either finite or of finite index.

Although the theorem of Asar mentioned above provides a great deal of information about groups in which every proper subgroup is nilpotent, little seems to be known about groups all of whose proper subgroups are soluble. In Section 3 we obtain some preliminary results concerning such groups, but shed little light on the question of whether an infinite locally graded group with all proper subgroups soluble must be soluble. Clearly, a finite group with all proper subgroups soluble need not be soluble, and even groups whose proper subgroups are all abelian need not be soluble, as the so-called Tarski monsters illustrate (although the Tarski groups are not locally graded of course). Clearly, then, the class consisting of all groups all of whose proper subgroups are soluble is difficult to understand.

Placing a bound on the derived lengths of the proper subgroups of a group yields a more successful theory. For example, Zaicev showed in [20] that an infinite soluble group with all proper subgroups soluble of derived length at most d is itself soluble of derived length at most d, and in [15] an analogous result was obtained for locally graded such groups. Not surprisingly, groups with all proper subgroups soluble of derived length at most d arise naturally in the study of groups with min-$\overline{\mathfrak{S}_d}$.

An obvious class of groups to consider next is the class of groups with min-∞-$\overline{\mathfrak{S}_d}$. Immediately we are faced with the existence of groups that are not soluble of derived length d but all of whose subgroups of infinite index are soluble of derived length at most d; certainly the Tarski monsters are examples of groups of this kind. Moreover, there are a number of reasons for suspecting that there exist finitely generated soluble groups of derived length d such that every subgroup of infinite index and of infinite rank has derived length less than d. The authors intend to discuss such groups in a subsequent work. Even locally nilpotent groups with min-∞-$\overline{\mathfrak{S}_d}$ need not be minimax or soluble of derived length d; this is established by Theorem 5 which is proved in Section 4.

Theorem 5 shows that there is a huge difference between the class of groups with min-∞-$\overline{\mathfrak{S}_d}$ and the subclass of groups with min-$\overline{\mathfrak{S}_d}$, for the main result of [15] implies that locally soluble groups with min-$\overline{\mathfrak{S}_d}$ are soluble of derived length at most d or Černikov. As one might hope, nilpotent groups with min-∞-$\overline{\mathfrak{S}_d}$ are better behaved as we show in Theorem 3. In Corollary 1 we show that radical groups with min-∞-$\overline{\mathfrak{S}_d}$ are soluble, and in Theorem 4 we give a general structure theorem for locally soluble-by-finite groups with min-∞-$\overline{\mathfrak{S}_d}$.

2. Groups with min-∞-$\overline{\mathfrak{N}}$. In this section we obtain a couple of results concerning the structure of groups with min-∞-$\overline{\mathfrak{N}}$.

Lemma 1. *Let G be locally soluble-by-finite group and suppose that G has min-∞-$\overline{\mathfrak{N}_c}$. Then either $G \in \mathfrak{N}_c$ or G is minimax.*

Proof. By Theorem A of [17], G is either locally nilpotent or minimax, so we may assume that G is locally nilpotent but not minimax. Let T denote the torsion subgroup of G and note that T is locally finite. By a result of Ostylovskii [21], either $T \in \mathfrak{N}_c$ or T is Černikov. If $T \notin \mathfrak{N}_c$ then G/T satisfies min-∞. Thus G/T is minimax and, since T is Černikov, it follows that G is also minimax. Hence we may assume that $T \in \mathfrak{N}_c$. Now G/T satisfies min-∞-$\overline{\mathfrak{N}}$ so, by [22], G/T is nilpotent. Hence G is soluble.

Let F be a finitely generated non-$\mathfrak{N}_c$-subgroup of G. By [17] (Lemma 13), G is a Baer group and so F is a subnormal subgroup of G and consequently there exists a finite series $F = F_0 \triangleleft F_1 \triangleleft \ldots \triangleleft F_n = G$. Since G is not minimax, but F is minimax, there exists a least integer $i \geq 0$ such that F_i is minimax but F_{i+1} is not minimax. However $F_i \notin \mathfrak{N}_c$ so F_{i+1}/F_i must have min-∞ and is therefore minimax. Clearly this implies that F_{i+1} is minimax, a contradiction, and the result follows.

As mentioned in the Introduction, the following result strengthens Theorem B by improving our understanding of the structure of the minimax residual of a locally nilpotent group with min-∞-$\overline{\mathfrak{N}}$.

Proposition 1. *Let G be a locally nilpotent group that satisfies* min-∞-$\overline{\mathfrak{N}}$, *and suppose that G is neither nilpotent nor minimax. Let K denote the minimax residual of G. Then K is a p-group for some prime p.*

Proof. Suppose that G is a counterexample to the proposition. We know from [17] (Proposition 3) that K is a π-group for some finite set of primes π. Suppose that K is not a p-group and let p, q be distinct primes in π. Let U be the $\{p, q\}'$-component of K. Then G/U is not minimax, by the definition of K, and since all nilpotent images of G are minimax, by [17] (Lemma 1), we see that G/U is not nilpotent, so that G/U is also a counterexample to the lemma. Thus we may set $U = 1$ and write $K = P \times Q$, where P, Q are the p, q-components, respectively, of K.

Let T be the torsion subgroup of G. By Theorem B G/K is a minimax group. Hence G/T is minimax, and therefore nilpotent, since $K \leq T$. Now T is not minimax and if T is nonnilpotent then Theorem A shows that K is a p-group since K is the Černikov residual of T. Thus T is nilpotent and Lemma 13 of [17] shows that G is a Baer group. There is a finitely generated subgroup F of G such that the isolator $I_{G/T}(FT/T)$ is G/T, and there is a finite subnormal series from the nilpotent subgroup FT to G the factors of which are Černikov r_i-groups for certain primes r_i. We now proceed by induction on the minimal length of such a series, the induction hypothesis being that groups with a shorter such series are either nilpotent, or minimax, or have minimax residual an r-group for some prime r. Firstly we note that if J is a subgroup of G that contains T then the minimax residual of J is contained in K, since J/K is minimax. By induction, then, there exists $H \lhd G$ with $T \leq H$ and G/H a Černikov r-group for some prime r, such that either H is nilpotent or H is minimax or the minimax residual of H is a p-group or a q-group. Since G/T is minimax T is not minimax, so H is not minimax. If H is not nilpotent then, denoting by L the minimax residual of H, we thus have H/L minimax (by [17], Theorem B (i)) and without loss of generality L is a p-group. But this implies that Q is minimax and hence that G/P is minimax contradicting the definition of K.

Thus H is nilpotent. Without loss of generality $r \neq p$. Then the r-component V/Q of T/Q is certainly Černikov, and G/V is an r-free locally nilpotent group having a nilpotent subgroup H/V whose r-isolator in G/V is itself G/V. Thus G/V is nilpotent and therefore minimax, which gives G/Q minimax, contradicting the definition of K. This completes the proof.

3. Groups with all proper subgroups soluble. In this section we prove our result concerning groups with all proper subgroups soluble. It is easy to see that an uncountable group of this kind is soluble. We have the following reduction result.

Theorem 1. *Let G be a periodic locally graded group with all the proper subgroups of G soluble. Then one of the following holds:*

(i) *G is soluble;*

(ii) *G is finite;*

(iii) *G is a perfect locally nilpotent p-group for some prime p.*

Proof. Let G be as stated and suppose that G is infinite and insoluble (and therefore countable). Suppose also that $H \leq G$ is finitely generated and insoluble. Then $H = G$ and, since G is locally graded, G contains a normal subgroup N such that G/N is finite and nontrivial. Let d denote the derived length of N and k the order of G/N. It is easy to see that the derived length of every proper subgroup of G is at most $d + k$ and Lemma 2.1 of [15] now yields a contradiction. Hence G is locally soluble.

Since G is countable we can now write $G = \bigcup_{i \geq 1} G_i$, where G_i is a finite soluble group and $G_i \le G_{i+1}$ for each $i \ge 1$. Suppose that G is not a p-group and let $p' = \pi(G)\setminus\{p\}$ where, as usual, $\pi(G)$ is the set of primes q such that G contains an element of order q. Using results of P. Hall (see, for example, [23], 9.1.7) we may write $G_i = S_iT_i$ for each i, where S_i is a Sylow p-subgroup of G_i and T_i is a Hall p'-subgroup of G_i. Furthermore, by Hall's theorems again, we may assume after choosing the subgroups S_i and T_i appropriately, that $S_i \le S_{i+1}$ and $T_i \le T_{i+1}$, for each i. Now $G = ST$, where $S = \bigcup_{i\ge1} S_i$ and $T = \bigcup_{i\ge1} T_i$. Moreover, $\pi(G)$ is the *disjoint* union of $\pi(S) = p$ and $\pi(T) = p'$ and so $S, T \lneq G$. Consequently, S and T are soluble, of derived lengths s and t respectively, say. Hence, for each i, S_i and T_i are of derived lenght at most s and t, respectively. Therefore, by a theorem of Kazarin [24] (Theorem 4), the derived length of G_i is at most $2st + s + t$ for all i. It follows that G is soluble of derived length at most $2st + s + t$, which is a contradiction. We deduce that G is a p-group for some prime p and hence, since G is locally soluble, that G is locally nilpotent. If $G \ne G'$ we have that G', and therefore G, is soluble, a contradiction. Thus G is perfect and the result follows.

4. Groups with min-∞-$\overline{\mathfrak{S}_d}$. Finally we consider groups with min-∞-$\overline{\mathfrak{S}_d}$. Locally finite groups of this type are particularly easy to handle. The following result is analogous to Theorem 1 of [10]. We recall that a subgroup K of a group G is said to be *finitely separated from* G if there exists a normal subgroup A of finite index in G such that $G \ne AK$.

Lemma 2. *Let G be a locally finite group and suppose that G satisfies* min-∞-$\overline{\mathfrak{S}_d}$. *Then G satisfies* min-$\overline{\mathfrak{S}_d}$.

Proof. Suppose that G is not soluble of derived length d and let $H_1 \ge H_2 \ge \ldots$ be a descending chain of non-$\mathfrak{S}_d$ subgroups. Then there is an integer n such that $|H_i : H_{i+1}|$ is finite for all $i \ge n$. By [17] (Corollary 1) there is a finite subgroup $K \notin \mathfrak{S}_d$ such that K is not finitely separated from H_n. Thus, if $L \triangleleft H_n$ and $|H_n : L|$ is finite, we have that $H_n = LK$. In particular $|H_n : L| \le K$. Hence every subgroup of finite index in H_n has bounded index and it follows that the chain $H_n \ge H_{n+1} \ge \ldots$ terminates. Thus G has min-$\overline{\mathfrak{S}_d}$, as required.

As an immediate corollary we have the following extension of Theorem 3.5 of [15].

Theorem 2. *Let G be a locally finite group satisfying* min-∞-$\overline{\mathfrak{S}_d}$. *Then one of the following holds:*

(i) *G is soluble of derived length at most d;*

(ii) *G is Černikov, or*

(iii) *G has a normal subgroup N such that G/N is s soluble Černikov group and N is isomorphic to one of the groups $SL(2, K)$, $PSL(2, K)$, ${}^2B_2(K)$ o r ${}^2G_2(K)$, for some locally finite field K satisfying the minimal condition on subfields.*

We now consider more general groups with min-∞-$\overline{\mathfrak{S}_d}$. First we have the following useful result.

Lemma 3. *Let G be a group satisfying* min-∞-$\overline{\mathfrak{S}_d}$ *and suppose that G/G' is not minimax. Then $G \in \mathfrak{S}_d$.*

Proof. Let G be as stated and suppose, for a contradiction, that $G \notin \mathfrak{S}_d$. Then there exists a finitely generated subgroup F of G such that $F \notin \mathfrak{S}_d$. Clearly $G/G'F$ is not minimax. Observe that if A is an abelian group that is not minimax, then there exists $B \le A$ with neither B nor A/B minimax. By applying this observation repeatedly, in an obvious way, to the subgroups of G/FG', we easily obtain the desired contradiction the proof.

We let min-∞-n denote the weak minimal condition on normal subgroups.

Lemma 4. *Let G be a group satisfying* min-∞-$\overline{\mathfrak{I}_d}$. *Suppose that G contains a non-$\mathfrak{I}_d$ subgroup H satisfying* min-∞. *Then G satisfies* min-∞-n.

Proof. Let $N_1 \geq N_2 \geq \ldots$ be a descending chain of normal subgroups of G. Since H has min-∞ there exists a natural number n such that $|H \cap N_i : H \cap N_{i+1}|$ is finite for all $i \geq n$. Now $HN_1 \geq HN_2 \geq \ldots$ is a descending chain of non-$\mathfrak{I}_d$ subgroups of G and so there is a natural number m such that $|HN_i : HN_{i+1}|$ is finite for all $i \geq m$. It follows easily that $|N_i : N_{i+1}|$ is finite for all $i \geq \max(m, n)$ and the proof is complete.

For nilpotent groups of the type under discussion we have the following pleasing structure theorem.

Theorem 3. *Let G be a nilpotent group satisfying* min-∞-$\overline{\mathfrak{I}_d}$. *Then either G is minimax or G is soluble of derived length at most d.*

Proof. Suppose that $G \notin \mathfrak{I}_d$, so that G contains a finitely generated subgroup K such that $K \notin \mathfrak{I}_d$. Since K is nilpotent and finitely generated, it is minimax, and Lemma 4 now shows that G has min-∞-n. We deduce that G/G' is minimax and, since G is nilpotent, it follows that G is minimax, (see, for instance, the Corollary to Theorem 2.26 in [25]). The proof is complete.

Lemma 5. *Let G be a locally nilpotent group with* min-∞-$\overline{\mathfrak{I}_d}$. *Then G is soluble.*

Proof. Let T be the torsion subgroup of G. Since T is locally finite, Lemma 2 shows that T satisfies min-$\overline{\mathfrak{I}_d}$, and results from [15] now establish that either $T \in \mathfrak{I}_d$ or T is minimax. It follows that T is soluble.

To show that G/T is also soluble, and thereby complete the proof, it suffices to assume that G is countable and torsionfree. If G has finite rank, then G is nilpotent (and hence soluble), so suppose that G does not have finite rank. By the arguments of [22], there exists $H \leq G$ with $I_G(H) = G$, where $I_G(H)$ denotes the isolator of H in G, and an infinite descending chain of subgroups from G to H, each of which has infinite index in its predecessor. Clearly this implies that $H \in \mathfrak{I}_d$ and since $I_G(H) = G$, it follows from a result of P. Hall [26] (Theorem 4.6) that $G \in \mathfrak{I}_d$.

Corollary 1. *Let G be a radical group satisfying* min-∞-$\overline{\mathfrak{I}_d}$. *Then G is soluble.*

Proof. It is easy to show, by using induction and Lemma 5, that G is soluble if its upper Hirsch – Plotkin series terminates after finitely many steps. Suppose then, for a contradiction, that G is not soluble and note that every normal $\mathfrak{I}_d$-subgroup of G is contained in the $(d+1)$-st term of the upper Hirsch – Plotkin series of G, H say. Since the upper Hirsch – Plotkin series of H terminates after finitely many steps it follows that H is soluble. Now, if $H \in \mathfrak{I}_d$, H lies in the d-th term of the upper Hirsch – Plotkin series of G and so that series terminates after d steps, a contradicition. Hence $H \notin \mathfrak{I}_d$ and we deduce that G/H has min-∞. A result of Zaicev [10] now shows that G/H is soluble, and the result follows easily.

Next we obtain our main structure theorem for locally soluble-by-finite groups with min-∞-$\overline{\mathfrak{I}_d}$.

Theorem 4. *Let G be a locally soluble-by-finite group with* min-∞-$\overline{\mathfrak{I}_d}$. *Then either G is soluble-by-finite or G contains a perfect normal subgroup K that is $\mathfrak{I}_d$-by-simple and such that G/K is a (soluble minimax)-by-finite group.*

Proof. Let K denote the minimax residual of G.

We first claim that G/K is soluble-by-finite. To establish this claim it clearly suffices to assume that $K = 1$; thus we may assume that G is residually minimax. If $G \in \mathfrak{S}_d$ then there is nothing to prove. So suppose there exists a finitely generated subgroup F of G such that $F \notin \mathfrak{S}_d$. We may choose F to be not finitely separated from G, by [17] (Corollary 1). Now G has a system of normal subgroups N_λ, for $\lambda \in \Lambda$, with each G/N_λ soluble-by-finite and minimax, and $\bigcap_{\lambda\in\Lambda} N_\lambda = 1$. Let R_λ/N_λ denote the finite residual of G/N_λ. By [25] (Theorem 10.33), R_λ/N_λ is abelian, so if $R = \bigcap_{\lambda\in\Lambda} R_\lambda$ then $R' \leq \bigcap_{\lambda\in\Lambda} N_\lambda = 1$ and R is abelian. Factoring we may assume that G is residually finite, so there is a system of normal subgroups $\{V_i : i \in I\}$ with each G/V_i finite and $\bigcap_{i\in I} V_i = 1$. Now F is soluble-by-finite and so there exists $J \lhd F$ with F/J finite and $J^{(r)} = 1$ for some natural number r. Now $FV_i = G$ for each $i \in I$ and so JV_i has finite index in G, bounded by $|F/J|$. Without loss of generality we may assume that $|G : JV_1|$ is the maximum of all the indices $|G : JV_1|$, and intersecting each V_i with V_1 we may thus assume that $|G : JV_i| = |G : JV_1|$ for all $i \in I$ and hence $JV_i = JV_1$ for all $i \in I$. Then

$$(JV_1)^{(r)} = \bigcap_{i\in I}(JV_1)^{(r)} \leq \bigcap_{i\in I} V_i = 1,$$

so G is soluble-by-finite, as claimed. Thus G/K is soluble-by-finite, as claimed.

Now suppose that G is not soluble-by-finite. By the above argument, K is not soluble and so G/K is minimax (using min-∞ on G/K). By Lemma 3 above, K/K' is minimax and therefore trivial, since K is the minimax residual of G. Indeed, if L denotes the minimax residual of K then, as above, K/L is soluble-by-finite and therefore has a solube radical M/L of finite index in K/L. Since $M \lhd G$ we have G/M finite-by-minimax and hence minimax, so $M = K$, which gives K/L soluble and hence trivial. So K has no nontrivial minimax images. Let N denote the iterated Hirsch – Plotkin radical of K. If N is not soluble of derived length at most d then K/N has min-∞ and is therefore minimax, hence trivial. But a radical group with min-∞-$\overline{\mathfrak{S}_d}$ is soluble and since K is perfect we have that K is trivial and hence G is soluble-by-finite, a contradiction. Thus $N \in \mathfrak{S}_d$.

It remains to show that K/N is simple. For this we may assume that $N = 1$, so K has no nontrivial normal soluble subgroups. Thus, if U is any nontrivial normal subgroup of K then K/U has min-∞ and is therefore minimax. As we have seen, K has no nontrivial minimax images and so $U = K$ and K is simple.

Theorem 5. *For each integer $d \geq 2$ there exists a locally nilpotent group G of infinite rank that is not of derived length d in which every subgroup of infinite index is either finitely generated nilpotent or of derived length d. In particular, G satisfies the condition min-∞-$\mathfrak{S}_d$ and G is neither minimax nor soluble of derived length at most d.*

Proof. We let $k = 2^{d-1}$ and p be a fixed prime. Let $F = \langle x, y_1, \ldots, y_k | R\rangle$, where R is the set of relations: $[y_i, y_j] = 1$ for all i, j, $\gamma_{2^d+1}(F) = 1$ and $(F^{(d)})^p = 1$. Let $\overline{F}$ be the free nilpotent group of nilpotency class 2^d with generators x, $y_1, \ldots, y_k$, so that F is an image of $\overline{F}$. In the group $\overline{F}$ there are "derived length d commutators" that are of infinite order, modulo the normal closure of $\langle [y_i, y_j] | 1 \leq i, j \leq k\rangle$. The commutator of weight 2^d which is itself a weight 2^{d-1} commutator with entries $[x, y_i]$ for all i is such an example (the form of the commutator is exactly that of the one needed to define the variety $\mathfrak{S}_d$, see [27] (Section 3.3)). Thus $F^{(d)} \neq 1$, since all we have done by adding the law $(F^{(d)})^p = 1$ is factor by the p-th power of a free abelian subgroup.

Next we claim that $H = \langle F', y_1, \ldots, y_k, x^p \rangle \in \mathfrak{S}_d$. To see this, we need to show that all "derived length d commutators" with entries in H are trivial, but, since $F^{(d)}$ is centrainly central in F (since $\gamma_{2^d+1}(F) = 1$) we need only check such commutators all of whose entries are in $\{x^p, y_1, \ldots, y_k\} \cup F'$. Note also that we can ignore F' here, since (again) all commutators of weight $2^d + 1$ are trivial. Let σ be a "derived length d commutator" in $\{x^p, y_1, \ldots, y_k\}$. If no x^p appears as an entry then, for some i and j, the element $[y_i, y_j]$ appears in the expression for σ and so $\sigma = 1$, but if x^p appears then σ is a p-th power of a "derived length d commutator" with entries in F, and so $\sigma = 1$ again. The claim follows.

Since F is a finitely generated nilpotent group that is not soluble of derived length d there is a subgroup F^* of F such that $F^* \notin \mathfrak{S}_d$ but every infinite index subgroup of F^* belongs to $\mathfrak{S}_d$. Let $A = \underset{i \geq 1}{\mathrm{Dr}} \langle a_i \rangle$ be an infinite elementary abelian p-group. Define an action of F^* on A via $a^\alpha = a$ for all $a \in A$, $\alpha \in \langle F', y_1, \ldots, y_k \rangle = C$, say, and $[a_i, x] = a_{i-1}$ for all $i \geq 1$, with $a_0 = 1$.

Set $G = A \rtimes F^*$, which is clearly a locally nilpotent group of infinite rank that is not in $\mathfrak{S}_d$. Let K be a subgroup of infinite index in G. If $K \cap A$ is finite then K is a finitely generated nilpotent group and is therefore minimax. Suppose, therefore, that $K \cap A$ is infinite. We claim that $K \in \mathfrak{S}_d$.

If $K \leq CA = C \times A$ then $K \in \mathfrak{S}_d$, since $C \leq H$. If $K \not\leq CA$ then there is a least $r > 0$ such that $x^r ca \in K$, for some $c \in C$, $a \in A$; also $KCA = \langle x^r \rangle CA$ in this case. If p divides r then $K \leq HA$ and $K' \leq H'[A, H] \leq H'A$, $K'' \leq H'' \leq [A, H'] = H''$, since $H' \leq F' \leq C$, which centralizes A. Thus $K^{(d)} \leq H^{(d)} = 1$ and $K \in \mathfrak{S}_d$ in this case. Suppose that $p \nmid r$. Since $K \cap A$ is centralized by CA and normalized by K, it is normalized by $\langle x^r \rangle$ and hence by $\langle x \rangle$, using the fact that G is locally nilpotent and $p \nmid r$. However this implies that $K \cap A = A$, since $K \cap A$ is infinite. Thus $A \leq K$ and

$$K = K \cap AF^* = A(K \cap F^*)$$

which implies that $K \cap F^*$ has infinite index in F^* and is therefore soluble of derived length at most d. As before,

$$K' = [A, K \cap F^*](K \cap F^*)' \leq A(K \cap F^*)'$$

and

$$K'' \leq [A, (K \cap F^*)'](K \cap F^*)'' \leq (K \cap F^*)'',$$

again using $[A, F'] = 1$. Hence $K^{(d)} \leq (K \cap F^*)^{(d)} = 1$ and $K \in \mathfrak{S}_d$, as claimed.

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