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MORPHISMS OF BALL'S STRUCTURES OF GROUPS AND GRAPHS

МОРФІЗМИ КУЛЬОВИХ СТРУКТУР ГРУП ТА ГРАФІВ

We introduce and study two kinds of morphisms between ball's structures related to groups and graphs.

Введено і досліджено два типи морфізмів між кульовими структурами, пов'язаними з групами та графами.

1. Introduction and main results. Following [1] by *ball's structure* we mean a triple $\mathbf{B} = (X, P, B)$, where X, P are the nonempty sets and, for any $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of X, which is called a ball of radius α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$.

Let $\mathbf{B_1} = (X_1, P_1, B_1)$ and $\mathbf{B_2} = (X_2, P_2, B_2)$ be the ball structures. We say that a mapping f of X_1 onto X_2 is a $\succ$-mapping of $\mathbf{B_1}$ onto $\mathbf{B_2}$ if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that

$$B_2(f(x), \beta) \subseteq f(B_1(x, \alpha))$$

for every $x \in X_1$. If there exists a $\succ$-mapping of $\mathbf{B_1}$ onto $\mathbf{B_2}$, we write $\mathbf{B_1} \succ \mathbf{B_2}$.

An injective mapping $f: X_1 \to X_2$ is called a $\prec$-mapping of B_1 into B_2 if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$. If there exists a $\prec$-mapping of $\mathbf{B_1}$ into $\mathbf{B_2}$, we write $\mathbf{B_1} \prec \mathbf{B_2}$.

A bijection $f: X_1 \to X_2$ is called an *isomorphism* between the ball's structures $\mathbf{B_1}$ and $\mathbf{B_2}$ if f is a $\succ$-mapping and f is a $\prec$-mapping.

Let $Gr = (V, E)$ be a graph with a set of vertices V and a set of edges E, $E \subseteq V \times V$, $E^{-1} = E$, $E^{-1} = \{(y, x): (x, y) \in E\}$ and $(x, x) \notin E$ for every $x \in X$. If $x, y \in V$ belong to the distinct connected components of Gr, put $d(x, y) = \infty$. Otherwise, denote by $d(x, y)$ the length of the shotest path between x and y. Given any $x \in V$ and $n \in \omega$, put $B(x, n) = \{y \in V: d(x, y) \le n\}$. A ball's structure (V, ω, B) is denoted by $\mathbf{B}(Gr)$.

For every natural number n, put $[n] = \{1, 2, \ldots, n\}$ and denote by I_n the graph $([n], E_n)$, where $E_n = \{(1,2), (2,3), \ldots, (n-1, n)\}$. Denote by I the graph $(\mathbf{N}, E)$, where $E = \{(n, n+1): n \in \mathbf{N}\}$.

Let G be a group with the identity e and let Fin be a family of all finite subsets of G containing e. Given any $x \in G$, $F \in Fin$, put $B(x, F) = Fx$. A ball's structure (G, Fin, B) is denoted by $\mathbf{B}(G)$.

Assume that a group G is generated by a finite subset S, $S = S^{-1}$. By Cayley graph Cay of group G with a pregiven finite set of generators we understand a graph (G, E), where $(x, y) \in E$ if and only if $x \ne y$ and $x = sy$ for some $s \in S$. Note that the identity mapping $f: G \to G$ is an isomorphism between the ball's structures $\mathbf{B}(G)$ and $\mathbf{B}(Cay)$.

Theorem 1. *For every infinite connected graph Gr, the following statements hold*

$$\mathbf{B}(Gr) \succ \mathbf{B}(I), \quad \mathbf{B}(I) \prec \mathbf{B}(Gr).$$

A group G is called locally finite if every finite subset of G generates a finite subgroup.

Theorem 2. *For every infinite group G, the following statements are equivalent:*

1) $\mathbf{B}(G) \succ \mathbf{B}(I)$;

2) $\mathbf{B}(I) \prec \mathbf{B}(Gr)$;

3) *G is not locally finite.*

Theorem 3. *Let G be an infinite group. Then $\mathbf{B}(G) \succ \mathbf{B}(I)$ if and only if either G has an infinite cyclic subgroup of finite index or G is a countable locally finite group.*

Theorem 4. *Let G_1, G_2 be the countable locally finite groups. Then $\mathbf{B}(G_1) \succ \succ \mathbf{B}(G_2)$ and $\mathbf{B}(G_1) \prec \mathbf{B}(G_2)$.*

Theorem 5. *Let G_1, G_2 be the countable locally finite groups. Then $\mathbf{B}(G_1)$ and $\mathbf{B}(G_2)$ are isomorphic if and only if, for every finite subgroup F of G_1, there exists a finite subgroup H of G_2 such that $|F|$ is a divisor of $|H|$, and, for every finite subgroup H of G_2, there exists a finite subgroup F of G_1 such that $|H|$ is a divisor of $|F|$.*

2. Dominating mappings of graphs. Let $Gr_1 = (V_1, E_1)$, $Gr_2 = (V_2, E_2)$ be the graphs, $k \in \mathbf{N}$ and let f be a mapping of V_1 onto V_2. We say that f is a k-*dominating* mapping of Gr_1 onto Gr_2 if

$$B(f(x), 1) \subseteq f(B(x, k))$$

for every $x \in V_1$. The following Lemma 1 states that a mapping $f: V_1 \to V_2$ is a $\succ$-mapping of $\mathbf{B}(Gr_1)$ onto $\mathbf{B}(Gr_2)$ if and only if f is a k-dominating mapping of Gr_1 onto Gr_2 for some $k \in \mathbf{N}$.

Lemma 1. *Let $Gr_1 = (V_1, E_1)$, $Gr_2 = (V_2, E_2)$ be the graphs and let f be a k-dominating mapping. Then $B(f(x), m) \subseteq f(B(x, km))$ for all $x \in V_1$, $m \in \omega$.*

Proof. Fix any $x \in V_1$ and $m \in \omega$. Take any $y \in B(f(x), m)$ and choose the elements $y_1, y_2, \ldots, y_n$, $n \le m$ from $B(f(x), m)$ such that $y_1 = f(x)$, $y_n = y$ and $(y_i, y_{i+1}) \in E_2$ for every $i \le n-1$. By assumption of Lemma 1, we can choose the elements $x_1, x_2, \ldots, x_n$ from V_1 such that $x = x_1$, $f(x_1) = y_1$, $f(x_2) = y_2, \ldots$ $\ldots, f(x_n) = y_n$ and $d(x_i, x_{i+1}) \le k$ for every $i \le n-1$. Since $y = y_n$, $n \le m$, then $y \in f(B(x, km))$.

Lemma 2. *Let $Gr_1 = (V_1, E_1)$, $Gr_2 = (V_2, E_2)$ be the graphs. Suppose that there exists a mapping $g: \omega \to \omega$ such that $|B(x, m)| \le g(m)$ for all $x \in V_1$, $m \in \in \omega$. If $\mathbf{B}(Gr_1) \succ \mathbf{B}(Gr_2)$, then there exists $k \in \omega$ such that $|B(y, m)| \le g(km)$ for all $y \in V_2$, $m \in \omega$.*

Proof. Let $f: V_1 \to V_2$ be a $\succ$-mapping. Choose $k \in \omega$ such that $B(f(x), 1) \subseteq \subseteq f(B(x, k))$ for every $x \in V_1$. By Lemma 1, $B(f(x), m) \subseteq f(B(x, km))$ for every $x \in V_1$. Hence, $|B(f(x), m)| \le g(km)$ for all $x \in V_1$, $m \in \omega$. Since f maps V_1 onto V_2, then $|B(y, m)| \le g(km)$ for all $y \in V_2$, $m \in \omega$.

Lemma 3. *Let $Gr = (V, E)$ be a finite connected graph, $|V| = n$, $n \ge 2$ and let $x, y \in V$, $(x, y) \in E$. Then there exists a 3-dominating bijection f of Gr onto I_n such that $f(x) = 1$, $f(y) = n$.*

Proof. We proceed by induction on n. For $n = 2$, Gr is isomorphic to I_2, so the statement is trivial. Let $n > 2$. Replacing Gr by its spannig tree, we may suppose that Gr itself is a tree. Consider two cases.

Case 1. $|B(x, 1)| = 2$, so $B(x, 1) = \{x, y\}$. Delete the vertex x and the edge (x, y) from Gr. Then we have got the tree $Gr' = (V', E')$, where $V' = V \setminus \{x\}$, $E' = E \setminus \{(x, y)\}$. Take a graph I'_{n-1} with the set of vertices $\{2, 3, \ldots, n\}$ and the set of edges $\{(2,3),(3,4),\ldots,(n-1,n)\}$. Since $|V'|\geq 2$, then there exists $z\in V'$ such that $(y,z)\in E'$. By assumption of induction, there exists a bijection $f'\colon V'\to\{2,\ldots,n\}$, which is a 3-dominating mapping of Gr' onto I'_{n-1} and $f'(y)=n$, $f'(z)=2$. Define a bijection $f\to[n]$ by the rule $f(x)=1$ and $f(v)=f'(v)$ for each $v\in V'$. Since the distance between x and z in Gr is equal to 2, then f is a 3-dominating mapping of Gr onto I_n.

Case 2. $|B(x,1)|>2$. If $|B(y,1)|=1$, then we can apply Case 1 with the pair y, x instead of x, y. Thus, we may assume that $|B(x,1)|>1$, $|B(y,1)|>1$. Delete the edge (x,y) from Gr. Then we have two trees $Gr_1=(V_1,E_1)$, $Gr_2=(V_2,E_2)$, with $x\in V_1$, $y\in V_2$. Let $|V_1|=k$, $|V_2|=m$. Then $k\geq 2$, $m\geq 2$ and $k+m=n$.

Denote by I'_k a graph with the set of vertices $\{1,2,\ldots,k\}$ and the set of edges $\{(1,2),(2,3),\ldots,(k-1,k)\}$. Take any element $x'\in V_1$ with $(x,x')\in E_1$. By inductive assumption, there exists a bijection $f'\colon V_1\to\{1,2,\ldots,k\}$, which is a 3-dominating mapping of Gr_1 onto I'_k such that $f'(x)=1$, $f'(x')=k$.

Denote by I'_m a graph with the set of vertices $\{k+1,k+2,\ldots,k+n\}$ and the set of edges $\{(k+1,k+2),(k+2,k+3),\ldots,(n-1,n)\}$. Take any element $y'\in V_2$ with $(y,y')\in E_2$. By inductive assumption, there exists a bijection $f''\colon V_2\to\{k+1,k+2,\ldots,n\}$, which is a 3-dominating mapping of Gr_2 onto I'_m such that $f''(y')=k+1$, $f''(y)=n$.

Define a bijection $f\colon V\to[n]$ by the rule $f(v)=f'(v)$ for all $v\in V_1$ and $f(v)=f''(v)$ for all $v\in V_2$. Since the distance between x' and y' in Gr is equal to 3, then f is a 3-dominating mapping of Gr onto I_n. By construction of f, $f(x)=1$, $f(y)=n$.

Lemma 4. *Let $Gr=(V,E)$ be a graph. Then $\mathbf{B}(I)\succ\mathbf{B}(Gr)$ if and only if there exist a partition $V=\bigcup_{i\in\omega}V_i$ and a natural m such that $|V_i|\leq m$ and $B(x,1)\cap V_j=\varnothing$ for all $x\in V_i$, $i\in\omega$ and $j>i+1$.*

Proof. Suppose that $\mathbf{B}(I)\succ\mathbf{B}(Gr)$ and fix a $\succ$-mapping $f\colon\mathbf{N}\to V$. Choose a natural number k such that $B(f(y),1)\subseteq f(B(y,k))$ for every $y\in\mathbf{N}$. Put $m=2k+1$ and partition $\mathbf{N}$ into consecutive segments $A_0,A_1,\ldots$ of length m. Put $V_0=f(A_0)$, $V_1=f(A_1)\setminus V_0$, $V_2=f(A_2)\setminus(V_1\cup V_2)$, $\ldots$. Clearly, $|V_i|\leq m$ and $V=\bigcup_{i\in\omega}V_i$. Fix $i\in\omega$ and take any $x\in V_i$. Pick $a\in A_i$ with $f(a)=x$. Then

$$B(x,1)\;=\;B(f(a),1)\;\subseteq\;f(B(a,k))\;\subseteq\;f(A_{i-1}\cup A_i\cup A_{i+1}).$$

Hence, $B(x,1)\cap V_j=\varnothing$ for every $j>i+1$.

Now assume that there exist a partition $V=\bigcup_{i\in\omega}V_i$ and $m\in\mathbf{N}$ satisfying the assumption of Lemma 4. Define a bijection $f\colon\mathbf{N}\to V$ such that, if $(a,b)\in\mathbf{N}$, $a<b$ and $f(a)\in V_i$, $f(b)\in V_j$, then $i\leq j$. Fix $i\in\omega$ and take any $x\in V_i$. Pick $a\in\mathbf{N}$ with $f(a)=x$. Then

$$B(f(a),1)\;=\;B(x,1)\;\subseteq\;V_{i-1}\cup V_i\cup V_{i+1}.$$

Hence, $B(f(a),1)\subseteq f(B(a,2m))$. It follows that f is a $2m$-dominating mapping. By Lemma 1, f is a $\succ$-mapping of $\mathbf{B}(I)$ onto $\mathbf{B}(Gr)$.

Let $\mathbf{B}=(X,P,B)$ be a ball's structure and let $\alpha\in P$. An injective sequence $\langle x_n\rangle_{n\in\omega}$ of elements of X is called on α-ray if $x_{n+1}\in B(x_n,\alpha)$ for every $n\in\omega$.

Lemma 5. *Let $\mathbf{B}=(X,P,B)$ be a ball's structure, $\alpha\in P$. If $\mathbf{B}(I)\succ\mathbf{B}$, then every disjoint family of α-rays in X is finite.*

Proof. Let $f\colon\mathbf{N}\to X$ be a $\succ$-mapping. Choose $m\in\omega$ such that

$$B(f(y), \alpha) \subseteq f(B(y, m)) \qquad (*)$$

for every $y \in \mathbf{N}$. Let $\langle x_n \rangle_{n\in\omega}$ be an α-ray. Pick $y_0 \in \mathbf{N}$ with $f(y_0) = x_0$. Using $(*)$, construct inductively a sequence $\langle y_n \rangle_{n\in\omega}$ in $\mathbf{N}$ such that $f(y_n) = x_n$ and $|y_{n+1} - y_n| \le m$ for every $n \in \omega$. Since the sequence $\langle y_n \rangle_{n\in\omega}$ is injective, then every segment $[a, b] \in \mathbf{N}$ of length m with $a \ge y_0$ contains a point c such that $f(c) \in \{x_n : n \in \omega\}$. It follows that every disjoint family of α-rays in X is of cardinality $\le n$.

3. Ball's structures of direct products. Let $\langle k_i \rangle_{i\in\omega}$ be a sequence of natural numbers, $[k_i] = \{1, 2, \ldots, k_i\}$, $i \in \omega$. By *direct product* $X = \otimes_{i\in\omega}[k_i]$ we mean a set of all vectors $x = (x(0), x(1), \ldots, x(i), \ldots)$ such that $x(i) \in [k_i]$ and $x(i) = 1$ for all but finitely many $i \in \omega$. Given any $x \in X$ and $m \in \omega$, put

$$B(x, m) = \{y \in X : y(i) = x(i) \quad \text{for all} \quad i \ge m\}.$$

A ball's structure (X, ω, B) will be denoted by $\mathbf{B}(\langle x_i \rangle_{i\in\omega})$.

Lemma 6. *Let $\langle k_i \rangle_{i\in\omega}$ and $\langle m_i \rangle_{i\in\omega}$ be the sequence of natural numbers such that $k_i \ge m_i$ for every $i \in \omega$. Then*

$$\mathbf{B}(\langle k_i \rangle_{i\in\omega}) \succ \mathbf{B}(\langle m_i \rangle_{i\in\omega}), \qquad \mathbf{B}(\langle m_i \rangle_{i\in\omega}) \prec \mathbf{B}(\langle k_i \rangle_{i\in\omega}).$$

Proof. For every $i \in \omega$, fix any mapping f_i of $[k_i]$ onto $[m_i]$. Define a mapping $f : \otimes_{i\in\omega}[k_i] \to \otimes_{i\in\omega}[m_i]$ by the rule

$$f(x(0), x(1), \ldots, x(i), \ldots) = (f_0(x(0)), f_1(x(1)), \ldots, f_i(x(i)), \ldots).$$

Then $B(f(x), m) = f(B(x, m))$ for any $x \in \otimes_{i\in\omega}[k_i]$, $m \in \omega$. Hence, f is a $\succ$-mapping.

For every $i \in \omega$, fix any injective mapping $g_i : [m_i] \to [k_i]$. Define a mapping $g : \otimes_{i\in\omega}[m_i] \to \otimes_{i\in\omega}[k_i]$ by the rule

$$g(y(0), y(1), \ldots, y(i), \ldots) = (g_0(y(0)), g_1(y(1)), \ldots, g_i(y(i)), \ldots).$$

Then $g(B(y, m)) \subseteq B(g(y), m)$ for all $y \in \otimes_{i\in\omega}[m_i]$, $m \in \omega$. Hence, g is a $\prec$-mapping.

Lemma 7. *Let $\langle k_i \rangle_{i\in\omega}$ be a sequence of natural numbers and let $g : \omega \to \omega$ be a nondecreasing mapping. Put $m_0 = k_0 k_1 \ldots k_{g(0)}$ and, for every $i \in \omega$, $m_{i+1} = k_{g(i)+1} k_{g(i)+2} \ldots k_{g(i+1)}$. Then the ball's structures $\mathbf{B}(\langle k_i \rangle_{i\in\omega})$ and $\mathbf{B}(\langle m_i \rangle_{i\in\omega})$ are isomorphic.*

Proof. Fix any bijection $f_0 : [m_0] \to [k_0] \times \ldots \times [k_{g(0)}]$ and, for every $i \in \omega$, $i > 0$, fix any bijection

$$f_i : [m_i] \to [k_{g(i)+1}] \times \ldots \times [k_{g(i+1)}].$$

Define a bijection $f : \otimes_{i\in\omega}[m_i] \to \otimes_{i\in\omega}[k_i]$ by rule $f(x(0), x(1), \ldots, x(i), \ldots) = (f_0(x(0)), f_1(x(1)), \ldots, f_i(x(i)), \ldots)$.

Since $f(B(x, m)) = B(f(x), g(0) + g(1) + \ldots + g(m-1))$ for every $x \in \otimes_{i\in\omega}[m_i]$ and every natural number m, then f is an isomorphism.

Lemma 8. *Let $\langle k_i \rangle_{i\in\omega}$ and $\langle m_i \rangle_{i\in\omega}$ be the sequences of natural numbers such that $k_i > 1$, $m_i > 1$ for each $i \in \omega$. Then $\mathbf{B}(\langle k_i \rangle_{i\in\omega}) \succ \mathbf{B}(\langle m_i \rangle_{i\in\omega})$ and $\mathbf{B}(\langle m_i \rangle_{i\in\omega}) \prec \mathbf{B}(\langle k_i \rangle_{i\in\omega})$.*

Proof. By Lemma 7, there exist a sequence $\langle K_i \rangle_{i\in\omega}$ of natural numbers such that $\mathbf{B}(\langle k_i \rangle_{i\in\omega})$ and $\mathbf{B}(\langle K_i \rangle_{i\in\omega})$ are isomorphic and $K_i \ge m_i$ for each $i \in \omega$. By Lemma 6,

$$\mathbf{B}(\langle K_i\rangle_{i\in\omega}) \succ \mathbf{B}(\langle m_i\rangle_{i\in\omega}), \quad \mathbf{B}(\langle m_i\rangle_{i\in\omega}) \prec \mathbf{B}(\langle K_i\rangle_{i\in\omega}).$$

Lemma 9. *Let $\langle k_i\rangle_{i\in\omega}$ and $\langle m_i\rangle_{i\in\omega}$ be the sequences of natural numbers. For every $i\in\omega$,* put *$K_i = k_0k_1\dots k_i$, $M_i = m_0m_1\dots m_i$. Then the ball's structures $\mathbf{B}(\langle k_i\rangle_{i\in\omega})$ and $\mathbf{B}(\langle m_i\rangle_{i\in\omega})$ are isomorphic if and only if, for every $k\in\omega$, there exist l, $m\in\omega$ such that $K_k \mid M_l$ and $M_k \mid K_m$.*

Proof. Assume that these ball's structures are isomorphic and fix an isomorphism $f\colon \otimes_{i\in\omega}[k_i]\to\otimes_{i\in\omega}[m_i]$. Since f is a $\prec$-mapping then there exists $l\in\omega$ such that $f(B(x,k))\subseteq B(f(x),l)$ for every $x\in\otimes_{i\in\omega}[k_i]$. Fix any $a\in\otimes_{i\in\omega}[m_i]$ and take any $x\in\otimes_{i\in\omega}[k_i]$ with $f(x)\in B(a,l)$. Then $B(f(x),l)=B(a,l)$ and $f(B(x,k))\subseteq B(a,l)$. It follows that $f^{-1}(B(a,l))$ is a disjoint union of the ball's of radius k. Note that every ball of radius l in $\otimes_{i\in\omega}[m_i]$ is of cardinality M_l and every ball of radius k in $\otimes_{i\in\omega}[k_i]$ is cardinality K_k. Hence, $K_k \mid M_l$. To find the number m, it suffices to repeat this argument for isomorphism f^{-1}.

Now assume that, for every $k\in\omega$, there exist $l, m\in\omega$ such that $K_k\mid M_l$ and $M_k \mid K_m$. Applying Lemma 7, we may suppose that

$$K_0\mid M_0,\ M_0\mid K_1,\ K_1\mid M_1,\ M_1\mid K_2,\ K_2\mid M_2,\ \dots .$$

Put $s_0=K_0$, $s_1=\dfrac{M_0}{K_0}$, $s_2=\dfrac{K_1}{M_0}$, $s_3=\dfrac{M_1}{K_1}$, $s_4=\dfrac{K_2}{M_1}$, $\dots$.

By Lemma 7, $\mathbf{B}(\langle k_i\rangle_{i\in\omega})$ is isomorphic to $\mathbf{B}(\langle s_i\rangle_{i\in\omega})$.

Lemma 10. *Let a group G be a union of an increasing chain of its subgroup $G_0\subset G_1\subset\dots\subset G_i\subset\dots$, $G_0=\{e\}$, e is the identity of G. Let $k_i=|G_{i+1}:G_i|$, $i\in\omega$. Then $\mathbf{B}(G)$ is isomorphic to $\mathbf{B}(\langle k_i\rangle_{i\in\omega})$.*

Proof. For every $i\in\omega$, decompose G_{i+1} on the right cosets by the subgroup G_i and choose some subset X_i of representatives of cosets such that $e\in X_i$. Thus, $G_{i+1}=G_iX_i$. Take any element $g\in G$ and choose the minimal subgroup G_{m+1} with $g\in G_{m+1}$. For $g=e$ we choose G_1. Then $g=g_{m-1}x_m$, $g_{m-1}\in G_m$, $x_m\in X_m$. Since $g_{m-1}\in G_m$, then $g_{m-1}=g_{m-2}x_{m-1}$ for some $g_{m-2}\in G_{m-1}$, $x_{m-1}\in X_{m-1}$. After $m+1$ steps we obtain the representation

$$g = x_0x_1\dots x_{m-1}x_m, \quad x_0\in X_0, \quad x_1\in X_1, \ \dots, \quad x_m\in X_m.$$

Note that this representation is unique. For every $i\in\omega$, fix any bijection $f_i\colon X_i\to[k_i]$ such that $f_i(e)=1$. Define a bijection $f\colon G\to\otimes_{i\in\omega}[k_i]$ by the rule $f(g)=(f_0(x_0), f_1(x_1),\dots,f_m(x_m),1,1,\dots)$.

Since every finite subset of G is contained in some subgroup G_i, then f is an isomorphism between $\mathbf{B}(G)$ and $\mathbf{B}(\langle k_i\rangle_{i\in\omega})$.

Lemma 11. *Let G be a direct sum $G=\oplus_\omega \mathbf{Z}_2$ of ω copies of the group $\mathbf{Z}_2=\{0,1\}$. Then $\mathbf{B}(I)\succ\mathbf{B}(G)$.*

Proof. We may assume that $I=(\omega,E)$, $E=\{(i,i+1)\colon i\in\omega\}$. For every $k\in\omega$, take a binary representation of k

$$k = a_02^0 + a_12^1 + \dots + a_n2^n.$$

Define a mapping $f\colon\omega\to G$ by the rule

$$f(k) = (a_0,a_1,\dots,a_n,0,0,0,\dots).$$

Observe that $B(g,m)\subseteq[g-2^{m+1}, g+2^{m+1}]$ for any $g\in G$, $m\in\omega$. *It follows* that f is a $\succ$-mapping of $\mathbf{B}(I)$ onto $\mathbf{B}(G)$.

4. Proof of theorems. ***Proof of Theorem* 1.** Let $Gr = (V, E)$. To prove the first statement $\mathbf{B}(Gr) \succ \mathbf{B}(I)$, we construct a 3-dominating mapping $f: V \to \mathbf{N}$. Replacing Gr by its spanning tree, we may suppose that Gr itself is a tree. Fix any point $x \in V$, put $f(x) = 1$ and consider the set $S = \{y \in V : (x, y) \in E\}$. Deleting the vertex x and the edges (x, y), $y \in S$, we obtain a disjoint family $\{T_y : y \in S\}$ of trees with $y \in T_y$. Denote $S_0 = \{y \in S : T_y$ is finite$\}$ and consider three cases.

Case 1. $S_0 = 0$. Put $f(y) = 2$ for all $y \in S$ and, in what follows, we shall map every tree T_y onto the subset $\{2, 3, \ldots\}$ of $\mathbf{N}$.

Case 2. S_0 is nonempty and finite. Let $S_0 = \{y_1, y_2, \ldots, y_n\}$ and let V_1, V_2, … …, V_n be the sets of vertices of $T_{y_1}, T_{y_2}, \ldots, T_{y_n}$. Let $|V_1| = m_1$, $|V_2| = m_2$, … …, $|V_n| = m_n$. If $|V_i| \geq 2$, take any $y_i' \in V_i$ with $(y_i, y_i') \in E$. If $|V_i| = 1$, put $y_i = y_i'$. By Lemma 3, there exist the 3-dominating mappings

$$f_1: V_i \to \{2, 3, \ldots, m_1 + 1\}, \quad f_1(y_1') = 2, \quad f_1(y_1) = m_1 + 1,$$

$$f_2: V_2 \to \{m_1 + 2, m_1 + 3, \ldots, m_1 + m_2 + 1\}, \quad f_2(y_2') = m_1 + 2, \quad f_2(y_2) = m_1 + m_2 + 1,$$

. . . .

$$f_n: V_n \to \{m_1 + m_2 + \ldots + m_{n-1} + 2, \ldots, m_1 + m_2 + \ldots + m_{n-1} + m_n + 1\},$$

$$f_n(y_n') = m_1 + m_2 + \ldots + m_{n-1} + 2, \quad f_n(y_n) = m_1 + m_2 + \ldots + m_n + 1.$$

Define a mapping

$$f: V_1 \cup V_2 \cup \ldots \cup V_n \to \{2, 3, \ldots, m_1 + m_2 + \ldots + m_{n-1} + m_n + 1\}$$

by the rule $f_i(v) = f(v)$ if and only if $v \in V_i$. Since the distance between y_i and y_{i+1}' in G is ≤ 3, then f is a 3-dominating mapping of tree T with the set of vertices $\{x\} \cup V_1 \cup V_2 \cup \ldots \cup V_n$ onto $\{1, 2, \ldots, m_1 + m_2 + \ldots + m_{n-1} + m_n + 1\}$. Put $f(y) = m_1 + m_2 + \ldots + m_{n-1} + m_n + 1$ for every $y \in S \setminus S_0$, delete T and, in what follows, we shall map every tree T_y, $y \in S \setminus S_0$ onto the subset $\{m_1 + \ldots + m_n + 1, m_1 + \ldots + m_n + 2, \ldots\}$ of $\mathbf{N}$.

Case 3. S_0 is infinite. Partition S_0 into countable subsets and take any subset S' of the partition. Let $S' = \{y_1, y_2, \ldots, y_n, \ldots\}$ and let $V_1, V_2, \ldots, V_n, \ldots$ be the sets of vertices $T_{y_1}, T_{y_2}, \ldots, T_{y_n}, \ldots$. Denote by T' the subtree of Gr with the set of vertices $\{x\} \cup V_1 \cup V_2 \cup \ldots \cup V_n \cup \ldots$. Using the arguments of Case 2, define a 3-dominating mapping f of T' onto $\mathbf{N}$. Delete T' from Gr. If $S \setminus S_0 = \varnothing$, then we have got the mapping $f: V \to \mathbf{N}$. Otherwise, put $f(y) = 1$ for every $y \in S \setminus S_0$ and, in what follows, we shall map every tree T_y, $y \in S \setminus S_0$ onto $\mathbf{N}$.

Repeating this procedure, we extend f onto V.

The second statement $\mathbf{B}(I) \prec \mathbf{B}(Gr)$ is much more easy. Suppose that the graph Gr is locally finite, i. e. every $B(x, 1)$, $x \in V$ is finite. By König Lemma, there exists a 1-ray $\langle x_n \rangle_{n \in \mathbf{N}}$ in Gr. Put $f(n) = x_n$, $n \in \mathbf{N}$ and note that $f(B(x, m)) \subseteq B(f(x), m)$ for all $x \in \mathbf{N}$, $m \in \omega$. Hence, f is a $\prec$-mapping. If Gr is not locally finite, fix any vertex $v \in V$ with an infinite ball $B(v, 1)$. Choose any countable subset $\{y_n : \mathbf{N}\}$ from $B(v, 1) \setminus \{v\}$. Put $f(n) = y_n$, $n \in \mathbf{N}$. Since $f(B(x, m)) \subseteq B(f(x), 2)$ for all $x \in \mathbf{N}$, $m \in \omega$, then f is a $\prec$-mapping.

***Proof of Theorem* 2.** $1 \Rightarrow 3$. Suppose that $\mathbf{B}(G) \succ \mathbf{B}(I)$, but G is locally finite. Fix a $\prec$-mapping $f: G \to \mathbf{N}$ and choose a finite subgroup H of G such that $B(f(g), 1) \subseteq f(B(g, H))$ for every $g \in G$. Fix any element $g_0 \in G$ and take a maximal natural number $m \in f(B(g_0, H))$. Choose $g_1 \in B(g_0, H)$ with $f(g_1) = m$.

Since H is a subgroup, then $B(g_1, H) = B(g_0, H)$. Hence, $B(f(g_1), 1) \subseteq f(B(g_0, H))$ and $m+1 \in f(B(g_0, H))$, a contradiction with the choice of m.

$2 \Rightarrow 3$. Suppose that $\mathbf{B}(I) \prec \mathbf{B}(G)$, but G is locally finite. Fix a $\prec$-mapping $f: \mathbf{N} \to G$ and choose a finite subgroup H of G such that $f(B(n,1)) \subseteq \subseteq B(f(n), H)$ for every $n \in \mathbf{N}$. Choose a maximal number $m \in f^{-1}(B(f(1), H))$. Since $f(m) \in B(f(1), H)$ and H is a subgroup, then $B(f(m), H) = B(f(1), H)$. Since $f(B(m, 1)) \subseteq B(f(m), H)$, then $m+1 \in B(f(1), H)$, a contradiction with the choice of m.

$3 \Rightarrow 1$. Choose an infinite finitely generated subgroup G' of G. Partition G onto right cosets by G' and fix some set X of representatives of cosets. Thus, $G = G'X$ and every element $g \in G$ has an unique representation of the form $g = g'x$, $g' \in G'$, $x \in X$. Define a mapping $f': G \to G'$ by the rule $f'(g) = g'$. Clearly, f' is a $\succ$-mapping of $\mathbf{B}(G)$ onto $\mathbf{B}(G')$. Identify $\mathbf{B}(G')$ with the ball's structure $\mathbf{B}(Cay)$ of Cayley graph Cay of G'. By Teorem 1, there exists a $\succ$-mapping f'' of $\mathbf{B}(Cay)$ onto $\mathbf{B}(I)$. Then $f = f''f'$ is a $\succ$-mapping of $\mathbf{B}(G)$ onto $\mathbf{B}(I)$.

$3 \Rightarrow 2$. Choose an infinite finitely generated subgroup G' of G. Identify $\mathbf{B}(G')$ with the ball's structure $\mathbf{B}(Cay)$.

***Proof of Theorem* 3.** Assume that $\mathbf{B}(I) \succ \mathbf{B}(Gr)$ and consider two cases.

Case 1. G has an element g of infinite order. Let C be the subgroup generated by g, e be the identity of G. Put $\alpha = \{e, g\}$ and observe that, for every $x \in G$, the sequence $\langle g^n x \rangle_{n \in \omega}$ is an α-ray in $\mathbf{B}(G)$. By Lemma 5, C is a subgroup of infinite index.

Case 2. G is a torsion group. Suppose that G is not locally finite and choose a finite subset F of G which generates an infinite subgroup G'. Partition G onto right cosets by G' and pick some set X of representatives of cosets. Thus, $G = G'X$ and every element $g \in G$ has an unique representation $g = g'x$, $g' \in G'$, $x \in X$. Define a mapping $f: G \to G'$ by the rule $f(g) = g'$. Clearly, f is a $\succ$-mapping of $\mathbf{B}(G)$ onto $\mathbf{B}(G')$. Hence, $\mathbf{B}(I) \succ \mathbf{B}(G')$. Identify $\mathbf{B}(G')$ with the ball's structure $\mathbf{B}(Cay)$ of its Cayley graph. By Lemma 2, G' is a group of linear growth. By Gromov Theorem [2], G' has a nilpotent subgroup H of finite index. Since H is finitely generated torsion nilpotent group, then H is finite. Hence, G' is finite, a contradiction.

Let G be a countable locally finite group. By Lemma 10, there exists a sequence $\langle k_n \rangle_{n \in \omega}$ of natural numbers such that $\mathbf{B}(G)$ is isomorphic to $\mathbf{B}(\langle k_n \rangle_{n \in \omega})$. Let H be a direct sum of ω copies of $\mathbf{Z}_2$. By Lemma 8, $\mathbf{B}(H) \succ \mathbf{B}(\langle k_n \rangle_{n \in \omega})$. By Lemma 11, $\mathbf{B}(I) \succ \mathbf{B}(H)$. Hence, $\mathbf{B}(I) \succ \mathbf{B}(G)$.

Now let G be a finite extension of an infinite cyclic group C generated by element g. We may suppose that C is an invariant subgroup, so $x^{-1}gx \in \{g, g^{-1}\}$ for every element $x \in G$. Partition G into right cosets by C and choose a set of representatives $H = \{h_1, h_2, \ldots, h_n\}$ such that $H = H^{-1}$. For $i, j \in \{1, 2, \ldots, n\}$, pick $a(i,j) \in \mathbf{Z}$ such that $h_i h_j \in g^{a(i,j)} H$. Put $a = \max\{|a(i,j)| + 1 : i, j \in \{1, 2, \ldots, n\}\}$. Consider the Cayley graph Cay of group G determined by the set $H \cup \{g, g^{-1}\}$ of generators. Put $V_0 = \{g^k H : |k| \le a\}$, $V_1 = \{g^k H : a < |k| \le 2a\}$, $V_2 = \{g^k H : 2a < |k| \le 3a\}, \ldots$.

By Lemma 4, $\mathbf{B}(I) \succ \mathbf{B}(Cay)$.

***Proof of Theorem* 4.** Apply Lemma 10 and Lemma 8.

***Proof of Theorem* 5.** Choose a sequence $F_0 \subset F_1 \subset \ldots \subset F_i \subset \ldots$ of finite subgroups of G_1 such that $G_1 = \bigcup_{i\in\omega} F_i$ and F_0 is the identity subgroup of G_1. Put $k_i = |F_{i+1} : F_i|$, $i \in \omega$. Choose a sequence $H_0 \subset H_1 \subset \ldots \subset H_i \subset \ldots$ of finite subgroups of G_2 such that $G_2 = \bigcup_{i\in\omega} H_i$ and H_0 is the identity subgroup of G_2. Put $m_i = |H_{i+1} : H_i|$, $i \in \omega$. By Lemma 10, $\mathbf{B}(G_1)$, $\mathbf{B}(\langle k_i \rangle_{i\in\omega})$ and $\mathbf{B}(G_2)$, $\mathbf{B}(\langle m_i \rangle_{i\in\omega})$ are the isomorphic pairs of ball's structures. Note that every finite subgroup F of G_1 is contained in some subgroup F_k, and every finite subgroup of G_2 is contained in some subgroup H_m. Then we can apply Lemma 9.

5. Comments and open problem. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be the ball's structures. Suppose that $\mathbf{B}_1 \succ \mathbf{B}_2$ (resp. $\mathbf{B}_1 \prec \mathbf{B}_2$). Is $\mathbf{B}_2 \prec \mathbf{B}_1$ (resp. $\mathbf{B}_2 \succ \mathbf{B}_1$)? Let G be a countable locally finite group. By Theorem 3, $\mathbf{B}(I) \succ \mathbf{B}(G)$. By Theorem 2, the relation $\mathbf{B}(G) \prec \mathbf{B}(I)$ is not true. Thus, the answer to the first question is negative. The following example gives a negative answer to the second question. Consider a complete graph Gr with the set of vertices ω. Attach a copy of graph $I(m+1)$ to each vertex $m \in \omega$. Denote the resulting graph by G'. Clearly, $\mathbf{B}(Gr) \prec \mathbf{B}(Gr')$. It is easy to see that the relation $\mathbf{B}(Gr') \succ \mathbf{B}(Gr)$ is false.

Let $\mathbf{B} = (X, P, B)$ be an arbitrary ball's structure. Consider a complete graph Gr with the set of vertices X. Then the identity mapping $f: X \to X$ is a $\succ$-mapping of $\mathbf{B}(Gr)$ onto $\mathbf{B}$. In particular, for every group G, there exists a graph Gr such that $\mathbf{B}(Gr) \succ \mathbf{B}(G)$. Now suppose that a group of G is countable and denote by F' a free group of countable rank. Note that a homomorphism f' of F' onto G is a $\succ$-mapping of $\mathbf{B}(F')$ onto $\mathbf{B}(G)$. Embed F' into a free group F of rank 2 and note that $\mathbf{B}(F) \succ \mathbf{B}(F')$. Identify $\mathbf{B}(F)$ with $\mathbf{B}(Cay)$ where Cay is a Cayley graph of F. Since Cay is a connected locally finite graph and $\mathbf{B}(Cay) \succ \mathbf{B}(G)$, then we have proved the following statement.

There exists a countable connected locally finite graph Gr such that $\mathbf{B}(Gr) \succ \succ \mathbf{B}(G)$ for every countable group G.

On the other hand, if Gr is a locally finite graph, G is an infinite group and $\mathbf{B}(Gr) \succ \mathbf{B}(G)$, then Gr is connected and, consequently, Gr and G are countable.

Let G be a group and let Gr be a graph such that the ball's structures $\mathbf{B}(G)$ and $\mathbf{B}(Gr)$ are isomorphic. We may assume that $Gr = (G, E)$ and the identity mapping $f: G \to G$ is an isomorphism. Since f is a $\succ$-mapping, then Gr is connected. Since f is a $\succ$-mapping, then there exists a finite subset H of G such that $B(x, 1) \subseteq Hx$ for every $x \in G$ and we have got the following statement.

For every group G, $\mathbf{B}(G)$ is isomorphic to $\mathbf{B}(Gr)$ for some graph Gr if and only if G is finitely generated.

By *weighted graph* we mean a graph $Gr = (V, E)$ and a function $w: E \to \mathbf{N}$, which assigns a weight $w(y)$ to each edge $y \in E$. A length of a path $x_1, x_2, \ldots, x_n$ in a weighted graph is a sum of weights of consecutive edges $(x_1, x_2), (x_2, x_3), \ldots$ $\ldots, (x_{n-1}, x_n)$. Put $d(x, x) = 0$ for each $x \in V$. If $x, x' \in V$ belong to the distinct connected component of Gr put $d(x, x') = \infty$. Otherwise, denote by $d(x, x')$ a length of the shortest path between x and x'. Put $B(x, m) = \{x' \in V : d(x, x') \le m\}$, $m \in \omega$. A ball's structure (V, ω, B) is called a ball's structure of weighted graph (V, E, w).

Let $G = \{g_n : n \in \omega\}$ be a countable group with the identity g_0 and let $Gr = (G, E)$ be a complete graph. Partition $G = G_0 \cup G_1 \cup G_2$ such that $G_0 = \{g \in G : g^2 = g_0\}$, $G_1^{-1} = G_2$.

Define any bijection $f: G_0 \cup G_1 \to \mathbf{N}$ and extend it to G_2 by the rule $f(g) = f(g^{-1})$ for every $g \in G_2$. Given any edge $y = (g_1, g_2)$, put $w(y) = f(g_2 g_1^{-1})$. It is easy to check that the ball's structure $\mathbf{B}(G)$ is isomorphic to the ball's structure $\mathbf{B}(Gr)$ of weighted graph Gr.

Let $\mathbf{Z}$ be a group of integers. By Theorem 2, $\mathbf{B}(\mathbf{Z}) \succ \mathbf{B}(I)$ and, by Theorem 3, $\mathbf{B}(I) \succ \mathbf{B}(\mathbf{Z})$. Let $f: \mathbf{N} \to \mathbf{Z}$ be a bijection. Show that, for every $m \in \mathbf{N}$, there exists $n \in \mathbf{N}$ such that $f(n) - f(n+1) > m$. Choose $k \in \mathbf{N}$ with $f[1,k] = [0,m]$ and take a minimal number $n \in \mathbf{N}$ such that $n > k$, $f(n) > m$, $f(n+1) < 0$. Then $f(n) - f(n+1) > m$. It follows that the ball's structure $\mathbf{B}(I)$ and $\mathbf{B}(\mathbf{Z})$ is not isomorphic. Which groups has a ball's structures isomorphic to $\mathbf{B}(\mathbf{Z})$. A slight modification of proof of Theorem 3 gives the following answer.

For a group G, $\mathbf{B}(G)$ is isomorphic to $\mathbf{B}(\mathbf{Z})$ if and only if G is a finite extension of an infinite cyclic group.

***Question* 1.** Characterize the finite connected graphs which admit a 1-dominating bijection onto I_n? The same question for a 2-dominating bijection.

***Question* 2.** Let G_1, G_2 be the infinite locally finite groups of the same cardinality. Is $\mathbf{B}(G_1) \succ \mathbf{B}(G_2)$? Is $\mathbf{B}(G_1) \prec \mathbf{B}(G_2)$?

It follows from Theorem 5 that there are exactly continuum classes of countable groups with isomorphic ball's structures.

***Question* 3.** Let α be an infinite cardinal. How many classes of groups of cardinality α with isomorphic ball's structures?

1. *Protasov I. V.* Combinatorial size of subsets of groups and graphs // Алгебраїчні структури та їх застосування: Зб. наук. пр. – Київ: Ін-т математики НАН України, 2002. – С. 76 – 85.
2. *Gromov M.* Groups of polinomial growth and expanding maps // Publ. Math. Inst. Haut. Etud. Sant. – 1981. – **53**. – P. 53 – 73.

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