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V. P. Shunkov (Inst. Comput. Model. Sib. Div. Rus. Acad. Sci., Krasnoyarsk, Russia)

ON PLACEMENT OR PRIME ORDER ELEMENTS IN A GROUP*

ПРО РОЗТАШУВАННЯ ЕЛЕМЕНТІВ СКІНЧЕННОГО ПОРЯДКУ В ГРУПІ

We characterize a class of T_0-groups related to the infinite Burnside groups of odd period.

Охарактеризовано клас T_0-груп, пов'язаний із нескінченними бернсайдовськими групами непарного періоду.

When I was a student and attended lectures of my teacher Sergey Nikolaevich Chernikov on group theory, I learned that a free group is embedded as a subgroup in the Cartesian product of finite p-groups for any fixed prime number p. But any group is isomorphic to a factor group of the corresponding free group, and, therefore, keeping in mind what has been said above, it is isomorphic to a factor group of some subgroup of the Cartesian product of finite p-groups (p is a fixed prime number). These facts were discouraging for me: then I thought that if I know a finite group, I was able to say everything about an arbitrary group. Sure that it is not true, nevertheless my research on infinite groups was always done on a reliable base, i. e., in every important case they were based on properties of finite groups. In this connection one can understand my desire to select finite groups in the class of all groups, i. e., to find such conditions of finiteness when the group is still finite. On this way, rather recently a result was obtained giving a positive solution of the so-called restricted Burnside problem [1], i. e., it is proved (using the classification of finite simple groups) that there exists only a finite number of finite groups with a predetermined number of generators and with a predetermined finite period. At first the result was obtained for finite p-groups (A. I. Kostrikin [2] and E. I. Zelmanov [3, 4]), later using the famous Higman – Hall theorem [5] its justification in general was reduced to problem of classification of finite simple groups. The result obtained, and this is not difficult to see, may be formulated in terms of the characterization of finite groups in the class of all groups, i. e., the following theorem is valid:

A finitely generated group G is finite if and only if it satisfies the following conditions:

1) *the group G has finite period;*

2) *the group G is finitely approximable.*

Recently I succeeded in obtaining a result which is similar to this theorem, as noted above, giving a positive solution to the restricted Burnside problem [1] (see theorem in

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the end of the article). Using this result I succeeded in characterizing the class of T_0-groups which is closely related to free Burnside groups of odd period ≥ 665 (see the basic theorem and its corollaries).

In the 1960s P. S. Novikov and S. I. Adian published the results with full proofs concerning groups of finite period [6, 7], from which, in particular, follows the negative solution of the famous Burnside problem [8, 1, 2] for such groups. Let formulate these results as one theorem:

The free Burnside group $G = B(m, n)$ with a number of generators $m \geq 2$ and with odd period $n \geq 4381$ is infinite and the centralizer of any nontrivial element from G is finite and contained in a cyclic subgroup of order n in G.

The proof of this theorem [6, 7] is based on ideas briefly stated in [9].

Later S. I. Adian considerably reduced the period n, i. e. the above formulated theorem is correct for odd $n \geq 665$ and, moreover, in the group $B(m, n)$ with odd $n \geq 665$ any finite subgroup is cyclic [10]. In the same work [10] S. I. Adian constructs a torsion-free group $A(m, n)$, being a central extension of a cyclic group by means of groups of the type $B(m, n)$ with odd $n \geq 665$ (see also [11]). Using the groups of the types $A(m, n)$, $B(m, n)$ ($m > 1$, n is odd and $n \geq 665$), it is possible to substantiate the existence of a T_0-group (see Definition 1 and Example 1). Further, A. Yu. Olshanskii constructs a torsion-free group $O(p)$, being a central extension of a cyclic group by an infinite p-group $C(\infty)$, in which all proper subgroups have orders equal to a prime number p (for p large enough) [12, 13] (see Theorems 28.1, 31.4 in [13]). Proceeding from the properties of groups of the type $O(p)$, $C(\infty)$, the author finds another example of a T_0-group (see Example 2).

So, the groups $A(m, n)$, $B(m, n)$ ($m > 1$, n is odd and $n \geq 665$), $O(p)$, $C(\infty)$ (p is a sufficiently large prime number) (rather specific according to their properties, especially exotic are the properties of the groups $O(p)$, $C(\infty)$), turned out closely connected (through the concept of T_0-group) with general problems of embedding and disposition of the prime order elements in a group (see basic theorem and its corollaries).

Let us recall that a group possessing periodic part denotes a group, in which the set of all elements of finite orders constitutes a subgroup of this group.

Definition. *Let G be a group with involutions, i be one of its involutions, satisfying the following conditions*:

1) *all subgroups of the form* $\mathrm{gr}(i, i^g)$, $g \in G$ *are finite*;

2) *the Sylow 2-subgroups of G are cyclic or generalized quaternion groups*;

3) *the centralizer $C_G(i)$ is infinite and has finite periodic part*;

4) *the normalizer of any nontrivial (i)-invariant finite subgroup of G is either contained in $C_G(i)$ or has periodic part which is a Frobenius group with abelian kernel and finite complement factor of even order*;

5) $C_G(i) \neq G$ *and for any element c from $G \setminus C_G(i)$, strictly real with respect to i, i. e. $c^i = c^{-1}$, there exists an element s_c in $C_G(i)$ that the subgroup* $\mathrm{gr}(c, c^{s_c})$ *is infinite.*

Let call T_0-group a group G with involution i, satisfying with respect to the involution i, the conditions 1 – 5.

***Example* 1.** Let $A = A(m, n)$, $m > 1$, n is odd and $n \geq 665$. The group A has nontrivial center $Z(A) = (d)$, and $A/Z(A) = A/(d) \simeq B(m, n)$ [10]. Let consider the group $B = A(x) = (A \times A) \leftthreetimes (x)$, where x is an involution. Let us take from $A \times A$ the element $u = (d, d^{-1})$. Obviously, $u \in Z(A \times A)$ and $u^x = u^{-1}$. Further, the group $G = B/(u)$ and its involution $i = x(u)$ satisfy the conditions 1 – 5 from the definition of T_0-group. One can easily see this, proceeding from abstract properties of groups of the type $A(m, n)$, $B(m, n)$ [10] (see also the introduction to the present paper). Hence, $G = B/(u)$ is T_0-group (with respect to the involution $i = x(u)$). Besides, in $G = B/(u)$ any maximum periodic subgroup containing the involution i is a dihedral group of order $2n$.

Example 2. Let $V = O(p)$ (see the definition of groups of the type $O(p)$, $C(\infty)$ in the introduction). The group V has nontrivial center $Z(V) = (t)$ and $V/Z(V) = V/(t) \simeq C(\infty)$ [13].

Consider the group $T = V \wr (k) = (V \times V) \lambda (k)$, where k is an involution. Let us take from $V \times V$ the element $b = (t, t^{-1})$. Obviously, $b \in Z(V \times V)$ and $b^k = b^{-1}$. Let us take a factor group $M = T/(b)$, and in it an involution $j = k(b)$. Further, proceeding from abstract properties of the groups $V = O(p)$, $C(\infty)$ [13], it is easy to show that the group M and its involution j satisfy the conditions 1 – 5 of the definition. Hence, $M = T/(b)$ is T_0-group (with respect to the involution $j = k(b)$). Let us also note that in M any maximum periodic subgroup containing the involution j is a dihedral group of the order $2p$.

Basic theorem. *Let G be a group and a be an element of prime order p, satisfying the following conditions:*

1) *subgroups of the form* $\mathrm{gr}(a, a^g)$, $g \in G$, *are finite and almost all are solvable;*

2) *in the centralizer $C_G(a)$ the set of elements of finite order is finite;*

3) *in the group G the normalizer of any nontrivial (a)-invariant finite subgroup has periodic part;*

4) *for $p \neq 2$ and for $q \in \pi(G)$, $q \neq p$, any (a)-invariant elementary Abelian q-subgroup of G is finite.*

Then either G has almost nilpotent periodic part, or G is a T_0-group and $p = 2$.

Corollary 1. *Let G be a (periodic) group and a be an element of prime order $p \neq 2$, satisfying conditions 1 – 4 of the basic theorem.*

Then G has almost nilpotent periodic part.

The following statement, as it is easy to show, is equivalent to the basic theorem and gives an abstract characterization of T_0-groups in the class of all groups.

Corollary 2. *Let G be a group, a be an element of prime order p. The group G is a T_0-group and $p = 2$ if and only if for the pair (G, a) the conditions 1 – 4 of the basic theorems are satisfied and the subgroup* $\mathrm{gr}(a^g \mid g \in G)$ *is not periodic almost nilpotent.*

The particular case when $p = 2$ requires special consideration, since in this case condition 4 of the basic theorem is superfluous, i. e. the following statements are true.

Corollary 3. *Let G be a group with involutions and i be one of its involutions, satisfying the following conditions:*

1) *subgroups of the form* $\mathrm{gr}(i, i^g)$, $g \in G$ *are finite;*

2) *in the centralizer $C_G(i)$ the set of elements of finite order is finite;*

3) *in the group G the normalizer of any nontrivial (i)-invariant finite subgroup has periodic part.*

Then either G has almost nilpotent periodic part, or G is a T_0-group.

The conditions 1 – 3 of Corollary 3 are independent, i. e. each of them does not follow from the other two.

Corollary 4. *Let G be a group with involutions, i be one of its involutions. The group G is a T_0-group if and only if for the pair (G, i) conditions 1 – 3 of Corollary 3 are satisfied and the subgroup* $\mathrm{gr}(i^g \mid g \in G)$ *is not periodic almost nilpotent.*

As one can see from the basic theorem and its Corollaries 2, 4, the key for solution of some general problems of embedding and disposition of the prime order elements in a group is situated in the depth of the structure of T_0-groups. However, for the time being we possess information about the structure of such groups at the level of definition of T_0-groups and elementary properties, given above. In this connection, the formulation of problems directly concerning the study of T_0-groups becomes relevant. As an example, let us list the following questions.

1. Does a simple T_0-group exist?

2. Let U be a T_0-group, i be an involution and $U = \mathrm{gr}(i^g \mid g \in G)$. What can one say about the centralizer $C_U(i)$? In particular, can $G_U(i)$ be approximated by periodic groups?

Obviously, in Examples 1, 2, the centralizers $C_G(i)$, $C_M(j)$ are approximated by periodic groups, and the factor group of the centralizer $C_G(i)$ by its center is isomorphic to a group of type $B(m, n)$, $m > 1$, n is odd number and $n \geq 665$. In Example 2 the factor group of the centralizer $C_M(j)$ by its center is isomorphic to a group of type $C(\infty)$ for a prime number p large enough.

3. What is possible to say about the geometry appropriate to T_0-groups?

This problem deserves attention, taking in consideration the analogy with finite groups. In the theory of finite groups, if some class of groups is abstractly characterized, then to it, as a rule, corresponds a rather informative finite projective geometry. Besides, in Example 2 the properties of groups of the type $O(p)$, $C(\infty)$ are widely used for prime number p large enough, and the proof of their existence was obtained by A. Yu. Olshanskii by means of a geometric method developed in [13].

Theorem. *A nontrivial finitely generated group G is finite if and only if in it exists an element a of prime order p satisfying the following conditions:*

1) *the subgroups of the form $\mathrm{gr}(a, a^g)$, $g \in G$ are finite and almost all solvable;*

2) *the centralizer $C_G(a)$ is finite;*

3) *when $p \neq 2$ and for $q \in \pi(G)$, $q \neq p$, any (a)-invariant elementary abelian q-subgroup is finite.*

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