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W. Hołubowski (Inst. Math., Siles. Techn. Univ., Gliwice, Poland)

NOTE ON SYMMETRIC WORDS IN METABELIAN GROUPS

ЗАУВАЖЕННЯ ЩОДО СИМЕТРИЧНИХ СЛІВ У МЕТАБЕЛЕВИХ ГРУПАХ

We completely describe n-symmetric words in a free metabelian group.

Наведено повний опис n-симетричних слів для вільної метабелевої групи.

The problem of characterizing of n-symmetric words for a given group G was initiated by E. Plonka [1, 2]. The properties of n-symmetric words are fully described by the group $S^{(n)}(G)$, introduced in [1]. The n-symmetric words depends on identities in G and $S^{(n)}(G) \simeq S^{(n)}(F_n(\operatorname{var} G))$, where by $F_n(\operatorname{var} G)$ we denote n-generator relatively free group in variety defined by G. The group $S^{(n)}(\mathfrak{A}^2)$ ($\mathfrak{A}^2$ — the variety of all metabelian groups [3]) was described for $n = 2$ by Macedońska and Solitar [4] (another proofs one can find in [5, 6]). The case $n = 3$ was considered in [6] (see also [7, 8]). In our paper we extend these results describing $S^{(n)}(\mathfrak{A}^2)$ for arbitrary n.

We prove the following theorem.

Theorem. *The group $S^{(n)}(\mathfrak{A}^2)$ is an infinitely generated free abelian group.*

In order of the proof we give also full description of n-symmetric words in this case.

Let F_n be the free group on $x_1, \ldots, x_n$. A word $w \in F_n$ is called *n-symmetric word* for a group G if $w(g_1, \ldots, g_n) = w(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$ for all $g_1, \ldots, g_n$ in G and all permutations σ in the symmetric group S_n. We have a natural epimorphism $\varphi : F_n \to F_n(\operatorname{var} G)$. Let A be the group of automorphisms of $F_n(\operatorname{var} G)$ induced by the mappings $x_i \to x_{\sigma(i)}$, $1 \le i \le n$, where $\sigma \in S_n$ (symmetric group on n elements). The set

$$S^{(n)}(G) = \{ v \in F_n(\operatorname{var} G) : v = \alpha(v) \quad \text{for every} \quad \alpha \in A \} = \bigcap_{\alpha \in A} \operatorname{Fix}(\alpha)$$

is called a group of n-symmetric words for G because $\varphi^{-1}(S^{(n)}(G))$ consists of all n-symmetric words for G. So, in the case of free metabelian group we can carry out all our calculations in F while working modulo F''. We denote by $[x, y] = x^{-1}y^{-1}xy$ commutator of elements x, y.

Proof. Let $w \in F_n$ and let w be the n-symmetric word for free metabelian group F_n / F_n''. From Lemma 5 of [6] we have $w \in F_n'$. Since F_n' is generated by conjugates $[x_i, x_j]^u$ $(u \in F_n)$ and such conjugates commute modulo F'' we can assume that

$$w = \prod_{i<j} [x_i, x_j]^{w_{i,j}} \bmod F''$$

where $w_{i,j}$ are polynomials in commuting variables $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ with integral coefficients.

Any permutation σ does not change the number of variables of monomials in $w_{i,j}$ (the action of σ on w is given by $w^{\sigma}(x_1, \ldots, x_n) = w(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$) so we can assume that $w = w_1 w_2 w_3$, where for $k = 1, 2, 3$ we have $w_k = \prod_{i<j} [x_i, x_j]^{w_{k,ij}}$ and $w_{1,ij}$ are polynomials in variables x_i, x_j only, $w_{2,ij}$ are polynomials in which every monomial does not contains x_i, x_j and $w_{3,ij}$ contains all other polynomials.

We have $w^{\sigma} = w_1^{\sigma} w_2^{\sigma} w_3^{\sigma}$ and $w_i^{\sigma} = w_i$ for $i = 1, 2, 3$. If we put $\sigma = (ij)$ transposition interchanging i and j, then

$$\left([x_i, x_j]^{w_{2,ij}}\right)^{\sigma} = [x_j, x_i]^{w_{2,ij}} = [x_i, x_j]^{-w_{2,ij}} = [x_i, x_j]^{w_{2,ij}}.$$

It means that w_2 is a trivial word.

Since S_n is n-transitive, applying to w_1 permutation τ such that $\tau: i \to k,\ j \to m$, where $\{i, j\} \neq \{k, m\}$, $k < m$, we conclude that $w_1 = \prod_{i<j} [x_i, x_j]^{u_1(x_i, x_j)}$ for some fixed word u_1. Applying to w_1 a transposition $\sigma = (ij)$ we conclude that $[x_i, x_j]^{u_1}$ is 2-symmetric word for F_2/F_2'' and by Theorem 1 of [5] u_1 is a sum of polynomials of the form $x^r y^q - x^q y^r$ with integral coefficients.

Applying to w_3 a homomorphism which send $x_i \to x_i$, $x_j \to x_j$ and $x_k \to 1$ for $k \neq i, j$ we conclude that the image of w_3 is a 2-symmetric word and so the polynomial $w_{3,ij}(1, \ldots, 1, x_i, 1, \ldots, 1, x_j, 1, \ldots, 1)$ have the same form as u_1. It follows that $w_{3,ij}(x_1, \ldots, x_n) = q_{i,j}(x_i, x_j)\, s_{i,j}(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n)$ where polynomials $s_{i,j}$ have no constants and does not contain x_i and x_j.

For any k, l, $k < l$, there is a permutation which send $i \to k$ and $j \to l$. Applying such permutation to the word w_3 we conclude that $q_{i,j} = q_{k,l} = q$, $s_{i,j} = s_{k,l} = s$ for some fixed s and q. Applying to w_3 any permutation which fixed i and j we conclude that s is $n - 2$-symmetric polynomial.

So any n-symmetric word has a form

$$w(x_1, \ldots, x_n) = \prod_{i<j} v_1(x_i, x_j) \prod_{i<j} v_3(x_i, x_j),$$

where

$$v_1(x, y) = [x, y]^{u(x, y)}, \quad v_3(x, y) = [x, y]^{q(x_i, x_j) s(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n)}$$

and u, q are the sums of polynomials of the form $x^r y^q - x^q y^r$ with integral coefficients, s is $n - 2$-symmetric polynomial. Inverse statement is easy to check by direct computations.

The group $S^{(n)}(\mathfrak{A}^2)$ is infinitely generated because we have a natural epimorphism $S^{(n)}(\mathfrak{A}^2) \to S^{(2)}(\mathfrak{A}^2)$ defined by $w(x_1, \ldots, x_n) = w(x_1, x_2, 1, \ldots, 1)$ and $S^{(2)}(\mathfrak{A}^2)$ is infinitely generated by [4]. Theorem is proved.

Remark. From results of [9] it follows that in the free metabelian group any nontrivial relation occurring among the set $[x_i, x_j]^{w_{ij}}$ is a product of transforms of Jacobi identity $[x, y]^{1-z}[x, z]^{-1+y}[y, z]^{1-x} = 1$. So, the word $[x, y]^{(x-y)z}[x, z]^{(x-z)y} \times [y, z]^{(y-z)x}$ is nontrivial 3-symmetric word for $G = F_2/F_3''$.

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