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ON THE CONVERGENCE OF FOURIER SERIES WITH ORTHOGONAL POLYNOMIALS INSIDE AND ON THE CLOSURE OF A REGION

ПРО ЗБІЖНІСТЬ РЯДІВ ФУР'Є З ОРТОГОНАЛЬНИМИ ПОЛІНОМАМИ ВСЕРЕДИНІ ТА НА ЗАМИКАННІ ОБЛАСТІ

We study the rate of convergence of Fourier series of orthogonal polynomials over an area inside and on the closure of regions of the complex plane.

Досліджено швидкість збіжності рядів Фур'є ортогональних по площі поліномів всередині та на замиканні областей комплексної площини.

1. Introduction and main results. Let $G \subset \mathbb{C}$ be a finite region bounded by a Jordan curve $L := \partial G$, let $h(z)$ be a weight function on G, and let $\{K_n(z)\}_{n=0}^{\infty}$ be a unique system of orthonormal polynomials $K_n(z) := \alpha_n z^n + \ldots,\ \alpha_n > 0$, over the region G with respect to the weight function $h(z)$, i.e.,

$$\iint_G h(z) K_n(z) \overline{K_m(z)} d\sigma_z = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases}$$

where σ is a two-dimensional Lebesgue measure on G.

Let $A_2(h, G)$ be the set of analytic functions f on G satisfying the condition

$$\|f\|_{A_2}^2 := \|f\|_{A_2(h,G)}^2 = \iint_G h(z) |f(z)|^2 d\sigma_z < \infty. \tag{1}$$

For every $f \in A_2(h, G)$ and each $n = 0, 1, 2, \ldots$ the following quantity a_n and series corresponding to f are defined:

$$a_n := a_n(f) := \iint_G h(z) f(z) \overline{K_n(z)} d\sigma_z < \infty \tag{2}$$

and

$$f(z) \sim \sum_{n=0}^{\infty} a_n K_n(z). \tag{3}$$

The solution of the problem of the convergence of (3) to f in G depends on the completeness of the system of polynomials in $A_2(h, G)$ with respect to norm (1). It is well known that (see, e.g., [1]) if the weight function is bounded above and below by

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positive constants, then the system $\{K_n(z)\}_{n=0}^{\infty}$ of orthogonal polynomials is complete in G.

We set

$$S_n(f;z) := \sum_{k=0}^{n} a_k K_k(z),$$

and

$$\omega_n(z) := |f(z) - S_n(f;z)|, \quad z \in G.$$

Then we have

$$\lim_{n\to\infty} \omega_n(z) = 0, \quad z \in G,$$

and, consequently,

$$f(z) = \sum_{n=0}^{\infty} a_n K_n(z), \quad z \in G. \tag{4}$$

In this paper, we investigate the rate of convergence to zero $\{\omega_n(z)\} \to 0$, $n \to \infty$, first, inside the region G and, second, on the closure of the region G for different weight functions. Our aim is to determine the rate of convergence, depending on the properties of the region G and weight function $h(z)$.

Now let us give several definitions.

Throughout this paper $c, c_1, c_2, \ldots$ are positive constants and $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants, in general, dependent on G.

Definition 1. *We say that*

a) $G \in C(p,\alpha)$, $p = 1, 2, \ldots$, $0 < \alpha < 1$, *if* $z = z(s)$, $0 \le s \le \operatorname{mes} L$ *is a natural representation of* L, $z = z(s)$ *is* p*-times continuously differentiable, and* $z^{(p)}(s) \in \operatorname{Lip}\alpha$;

b) $G \in C_\theta$ *if* $L = \partial G$ *has a continuous tangent* $\theta(s) := \theta(z(s))$ *for every point* $z(s)$;

c) $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, *if* $L = \partial G$ *consists of the union of finite* C_θ *arcs such that they have exterior (with respect to* G*) angles* $\lambda_j\pi$ *at the corners where two arcs meet,* $0 < \lambda_j < 2$, $\lambda = \min_j \{\lambda_j\}$.

Let

$$h(z) = |D(z)|^2, \quad D(z) \in H^\alpha(\overline{G}), \quad 0 < \alpha \le 1, \quad \text{and} \quad D(z) \ne 0, \quad z \in \overline{G}, \tag{5}$$

where $H^\alpha(\overline{G})$ is the class of analytic functions f in G, and $f \in \operatorname{Lip}\alpha$ on $\overline{G}$.

P. K. Suetin [2] proved that if $L \in C(p+1, \alpha)$ and $h(z)$ defined by (5) is such that $D(z) \in W^{(p)}H^\alpha(\overline{G})$ (i.e., $D^{(p)} \in H^\alpha(\overline{G})$), $p \ge 0$, $0 < \alpha < 1$, then, for all $z \in F \Subset G$, we have

$$\omega_n(z) \le c\,[d(F,L)]^{-(p+3)} \frac{1}{n^{p+\alpha}} E_n(f, A_2), \tag{6}$$

where c is a constant independent of z and n and

$$E_n(f, A_2) := \min_{P_n} \left(\iint_G h(z)\,|f(z) - P_n(z)|^2 \, d\sigma_z \right)^{\frac{1}{2}}, \quad \deg P_n \le n.$$

is the best approximation in the class $A_2(h, G)$ by algebraic polynomials $P_n(z)$.

One can see that the rate of the convergence $\omega_n(z) \to 0$, $n \to \infty$ depends on the distance from the compact subset $F \Subset G$ to the boundary countor L.

Let $\delta(z) := d(z, L) = \inf\{|z - \zeta| : \zeta \in L\}$ and, for $\lambda > 0$,

$$\vartheta(\lambda) := \begin{cases} \dfrac{1}{2-\lambda}, & \lambda \le \dfrac{2}{3}, \\ \dfrac{1}{2\lambda}, & \dfrac{2}{3} < \lambda < 1, \\ \dfrac{1}{2}, & \lambda \ge 1. \end{cases} \tag{7}$$

Theorem 1. *Let $L \in C_\theta(\lambda)$ for some λ, $0 < \lambda < 2$, and let $h(z)$ be defined by (5). Then*

$$\omega_n(z) \le c\delta^{\frac{4\lambda-5}{2(2-\lambda)}}(z) E_n(f, A_2) n^{-\mu} \tag{8}$$

for all μ such that $0 < \mu < \min\left\{\frac{1}{2}; \frac{\lambda}{2-\lambda}\right\}$ if $\alpha > \vartheta(\lambda)$, and $0 < \mu < \alpha \min\{1; \lambda\}$ if $\alpha \le \vartheta(\lambda)$. Here, c is independent of z and n, and $\vartheta(\lambda)$ is defined by (7).

Corollary 1. *Under the conditions of Theorem 1, if $\lambda = 1$, then*

$$\omega_n(z) \le c\delta(z)^{-\frac{1}{2}} E_n(f, A_2) n^{-\mu} \tag{9}$$

for all $\mu \in (0, \frac{1}{2})$ if $\frac{1}{2} < \alpha \le 1$ and $\mu \in (0, \alpha)$ if $0 < \alpha \le \frac{1}{2}$.

This shows that if $0 < \alpha \le \frac{1}{2}$, then (9) is better than (6) in the case $p = 0$.

Definition 2 [3]. *We say that L is a K-quasiconformal ($K \ge 1$) arc or curve if there is a K-quasiconformal mapping f of a region D containing L such that $f(L)$ is a line segment or a circle, respectively.*

Let $F(L)$ denote the set of all sense-preserving plane homeomorphisms f of regions D containing L such that $f(L)$ is a line segment or a circle and

$$K_L = \inf_{f \in F(L)} K(f),$$

where $K(f)$ is the maximal dilatation of the mapping f. Then L is a K-quasiconformal curve if and only if $K_L \le K < \infty$.

In case where L is a K-quasiconformal curve or a piecewise K-quasiconformal curve, estimates of the type (6) were investigated in [4] and [5], respectively.

We now choose a weight function as follows:

$$h \in C(\overline{G}) \quad \text{and} \quad h(z) \ge c > 0. \tag{10}$$

Our aim is to investigate the same problem on the closure of G.

In the case where $L \in C(1, \alpha)$, $0 < \alpha < 1$, P. K. Suetin [2] proved that, for every $f \in A(\overline{G})$, i.e., for f analytic in G and continuous on $\overline{G}$, one has

$$\omega_n := \max_{z \in \overline{G}} \omega_n(z) \le cnE_n(f, \overline{G}), \tag{11}$$

where

$$E_n(f, \overline{G}) := \inf_{P_n} \max_{z \in \overline{G}} |f(z) - P_n(z)|$$

is the best approximation of $f \in A(\overline{G})$ by polynomials $P_n(z)$, $\deg P_n \leq n$.

In the case where L is a K-quasiconformal curve, a similar result of the type (11) was established by F. G. Abdullaev in [6].

We now give several results similar to (11) for different regions and obtain, by using some known results, the subclass of analytic functions that can be expanded into uniformly convergent series on $\overline{G}$ as in (4).

Theorem 2. *Let $G \in C_\theta(\lambda)$ for some λ, $0 < \lambda < 2$, and let $f \in A(\overline{G})$. Then, for arbitrary $\varepsilon > 0$ and $\lambda^* = \max\{1; \lambda\}$, we have*

$$\omega_n \leq c n^{\lambda^* + \varepsilon} E_n(f, \overline{G}).$$

Corollary 2. *Suppose that the conditions of Theorem 2 are satisfied and $f \in W^{(r)} H^\alpha(\overline{G})$, $r + \alpha > \frac{\lambda^*}{\lambda_*}$, $\lambda_* = \min\{1; \lambda\}$. Then*

$$\omega_n \leq c n^{-\eta} \tag{12}$$

for all $\eta < (r + \alpha)\lambda_* - \lambda^*$.

Thus, every $f \in W^{(r)} H^\alpha(\overline{G})$, $r + \alpha > \frac{\lambda^*}{\lambda_*}$, can be expanded into series (4) uniformly convergent on $\overline{G}$.

Let ψ be a conformal mapping of $\{w: |w| < 1\}$ to G with $\psi(0) = z_0$ and $\psi'(0) > 0$ for arbitrary fixed $z_0 \in G$.

Definition 3. *We say that G is a k-quasidisk, $0 \leq k < 1$, if any conformal mapping ψ can be extended to a Q-quasiconformal self-homeomorphism of the plane $\overline{\mathbb{C}}$, where $Q = \frac{1+k}{1-k}$. In this case, the curve $L = \partial G$ is called a k-quasicircle. The region G (curve L) is called a quasidisk (quasicircle) if it is a k-quasidisk (k-quasicircle) for some k, $0 \leq k < 1$.*

For example, let G be the region bounded by two circle arcs symmetric with respect to the x-axis and y-axis, each crossing x-axis at $\pm\delta$, where $\delta > 0$, and the angle between the arcs is $\pi(1 - k)$, where $0 \leq k < 1$. This region is a k-quasidisk.

Theorem 3. *Let G be a k-quasidisk for some k, $0 \leq k < 1$, and let $f \in A(\overline{G})$. Then*

$$\omega_n \leq c n^{1+k} E_n(f, \overline{G}).$$

Corollary 3. *If G is a k-quasidisk, $0 \leq k < \frac{1}{3}$, and $f \in W^{(r)} H^\alpha(\overline{G})$, $r + \alpha > \frac{1+k}{1-k}$, then*

$$\omega_n \leq c n^{-\eta}, \tag{13}$$

where $\eta := (r + \alpha)(1 - k) - (1 + k)$.

Thus, f can be expanded into series (4) uniformly convergent in $\overline{G}$.

Remark 1. If $L = \partial G$ is a K-quasiconformal curve, then $k = \frac{K^2-1}{K^2+1}$. Therefore, (13) is satisfied for every $\alpha > K^2 - r$. This is better than [6] (Theorem 3) for $K < \sqrt{2}$.

In some cases it is difficult to find the quasiconformality coefficient k for a given region. Nevertheless, the rate of convergence must be found for such regions, too. Therefore, we give this rate depending on other parameters.

Let $w = \Phi(z)$ be a conformal mapping of $\Omega = C\overline{G}$ onto $\Delta := \{w : |w| > 1\}$ normalized by $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$, and $\Psi = \Phi^{-1}$.

Definition 4. *We say that $G \in Q_\beta^\gamma$, $0 < \beta, \gamma \leq 1$, if the following conditions are satisfied:*

i) $L = \partial G$ is a quasicircle;

ii) $\Phi \in \mathrm{Lip}\,\beta$, $z \in \overline{\Omega}$, and $\Psi \in \mathrm{Lip}\,\gamma$, $|w| \geq 1$.

Theorem 4. *Let $G \in Q_\beta^\gamma$ for some β and γ, $0 < \beta, \gamma \leq 1$. Then, for every $f \in A(\overline{G})$, we have*

$$\omega_n \leq c n^{\beta_*} E_n(f, \overline{G}), \quad z \in \overline{G},$$

where $\beta_* := \min\left\{2; \frac{1}{\beta}\right\}$.

Corollary 4. *Suppose that the conditions of Theorem 4 are satisfied and $f \in W^{(r)} H^\alpha(\overline{G})$, $r + \alpha \geq \frac{\beta_*}{\gamma}$. Then*

$$\omega_n \leq c n^{-\eta}, \tag{14}$$

where $\eta := (r + \alpha)\gamma - \beta_*$.

Thus, f can be expanded into the uniformly convergent series (4) *on $\overline{G}$.*

Remark 2. a) If G is a convex region, then $\Psi \in \mathrm{Lip}\,1$ [7] and $\Phi \in \mathrm{Lip}\,1$ [8, p. 582]. Hence, for every $f \in W^{(r)} H^\alpha(\overline{G})$, $r \geq 1$, $0 < \alpha \leq 1$, we have

$$\omega_n \leq c n^{-\eta},$$

where $\eta := r + \alpha - 1$.

b) If $G \in C_\theta$, then $G \in Q_\beta^\gamma$ for all $0 < \beta, \gamma < 1$. Hence, for every $f \in W^{(r)} H^\alpha(\overline{G})$, $r + \alpha > 1$, we have

$$\omega_n \leq c n^{-\eta},$$

where $\eta := (r + \alpha) - 1 - \varepsilon$ for arbitrary $\varepsilon > 0$.

c) If G is an L-shaped region, then $\Phi \in \mathrm{Lip}\,\frac{2}{3}$ and $\Psi \in \mathrm{Lip}\,\frac{1}{2}$ [9]. Hence, for every $f \in W^{(r)} H^\alpha(\overline{G})$, $r + \alpha > 3$, we have

$$\omega_n \leq c n^{-\eta},$$

where $\eta := (r + \alpha)\frac{1}{2} - \frac{3}{2}$.

d) Let L be quasismooth, i.e., for every $z_1, z_2 \in L$, if $s(z_1, z_2)$ is the smaller of the lengths of the arcs joining z_1 and z_2 on L, then there exists a constant $c > 1$ such that

$$s(z_1, z_2) \leq c\,|z_1 - z_2|,$$

let $\Phi \in \mathrm{Lip}\,\beta$, $\beta = \frac{1}{2}(1 - \frac{1}{\pi}\arcsin\frac{1}{c})^{-1}$, and let $\Psi \in \mathrm{Lip}\,\gamma$, $\gamma = \frac{2}{(1+c)^2}$ [10, 11]. Then, for every $f \in W^{(r)}H^{\alpha}(\overline{G})$, $r + \alpha > \frac{\beta_*}{\gamma}$, we have

$$\omega_n \leq cn^{-\eta},$$

where $\eta := (r+\alpha)\gamma - \beta_*$.

e) If L is c-quasiconformal (see, e.g., [9]), then $\Phi \in \mathrm{Lip}\,\beta$ for $\beta = \frac{\pi}{2\pi - 2\arcsin\frac{1}{c}}$ and $\Psi \in \mathrm{Lip}\,\gamma$ for $\gamma = \frac{2(\arcsin\frac{1}{c})^2}{\pi^2 - \pi\arcsin\frac{1}{c}}$. Also, if L is an asymptotic conformal curve, then $\Phi \in \mathrm{Lip}\,\beta$ and $\Psi \in \mathrm{Lip}\,\beta$ for $\beta < 1$ [9].

Hence, for every $f \in W^{(r)}H^{\alpha}(\overline{G})$, $r \geq 1$, $0 < \alpha \leq 1$, one can also calculate ω_n.

Definition 5. *We say that*

a) $G \in Q(\lambda)$ *if* $L = \partial G$ *is a quasicircle and, for every* $z \in L$, *there exist* $r > 0$ *and* $0 < \lambda < 1$ *such that the closed circular sector of radius* r *and opening* $\lambda\pi$ *lies in* $\overline{\Omega}$ *with vertex at* z.

b) $G \in Q_{\beta}(\lambda)$ *for some* λ, $0 < \lambda < 1$, *and* β, $0 < \beta \leq 1$, *if*

i) $G \in Q(\lambda)$;

ii) $\Phi \in \mathrm{Lip}\,\beta$, $|w| \geq 1$.

***Remark* 3.** If $G \in Q_{\beta}(\lambda)$ for some λ, $0 < \lambda < 1$, and β, $0 < \beta \leq 1$, then, for every $f \in W^{(r)}H^{\alpha}(\overline{G})$, $r + \alpha > \frac{\beta_*}{\lambda}$, we have

$$\omega_n \leq cn^{-\eta},$$

where $\eta = (r+\alpha)\lambda - \beta_*$.

2. Some auxiliary results. The relations $a \prec b$ and $a \asymp b$ are equivalent to $a \leq cb$ and $c_1 a \leq b \leq c_2 b$ for some constants c, c_1, and c_2. Let G be a finite Jordan region, let $z_0 \in G$, let $w = \varphi(z, z_0)$ be the conformal mapping of G onto the unit disc $B = \{w : |w| < 1\}$ normalized by $\varphi(z_0, z_0) = 0$, $\varphi'(z_0, z_0) > 0$, and let $\psi = \varphi^{-1}$.

Let G be a quasidisk. Then there exists a quasiconformal reflection $y(\cdot)$ with respect to L such that $y(G) = \Omega$, $y(\Omega) = G$, and $y(\cdot)$ fixes a point of L. The quasiconformal reflection $y(\cdot)$ satisfies the following conditions [12, p. 26]:

$$|y(\zeta) - z| \asymp |\zeta - z|, \quad z \in L, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon},$$

$$\left|y_{\overline{\zeta}}\right| \asymp |y_{\zeta}| \asymp 1, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \tag{15}$$

$$\left|y_{\overline{\zeta}}\right| \asymp |y(\zeta)|^2, \quad |\zeta| < \varepsilon,$$

$$\left|y_{\overline{\zeta}}\right| \asymp |\zeta|^{-2}, \quad |\zeta| > \frac{1}{\varepsilon}.$$

For $t > 0$, let

$$L_t := \{z : |\varphi(z)| = t \ \text{ if } \ t < 1, |\Phi(z)| = t \ \text{ if } \ t > 1\}, \quad L_1 := L.$$

$G_t := \mathrm{int}\, L_t$, and $\Omega_t := \mathrm{ext}\, L_t$.

Lemma 1 [5]. *Suppose that G is a k-quasidisk for some $0 \leq k < 1$, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z\colon |z - z_1| \prec d(z_1, L_{r_0})\}$, and $w_j = \Phi(z_j)$, $j = 1, 2, 3$.*

Then the following assertions are true:

(i) the relations $|z_1 - z_2| \prec |z_1 - z_3|$ and $|w_1 - w_2| \prec |w_1 - w_3|$ are equivalent. Furthermore, $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$;

(ii) if $|z_1 - z_2| \prec |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $0 < r_0 < 1$ is a certain constant dependent on G and k.

Lemma 2 [13]. *Let G be a k-quasidisk for some $0 \leq k < 1$. Then*

$$|\Psi(w_1) - \Psi(w_2)| \succ |w_1 - w_2|^{1+k} \tag{16}$$

for all $w_1, w_2 \in \overline{\Omega'}$.

For every $R > 1$, let $L^* := y(L_R)$, $G^* := \operatorname{int} L^*$, and $\Omega^* := \operatorname{ext} L^*$. Let $\Phi_*\colon \Omega^* \to \Delta$ be a conformal mapping such that $\Phi_*(\infty) = \infty$ and $\Phi'_*(\infty) > 0$. According to [14], for all $z \in L^*$ and $t \in L$ such that $|z - t| = d(z, L)$, we have

$$d(z, L) \asymp d(t, L_R) \asymp d(z, L_R), \tag{17}$$

$$|\Phi_*(t)| \leq |\Phi_*(z)| \leq 1 + c(R - 1).$$

Lemma 3. *Let G be a quasidisk, let $P_n(z)$ be an arbitrary polynomial of at most nth degree, and let $R = 1 + \frac{c}{n}$. Then*

$$\|P_n\|_{C(\overline{G})} \leq c_1 \|P_n\|_{C(\overline{G}^*)},$$

where $c_1 = c_1(G, c) > 0$ is a constant.

Proof. Let

$$F(z) := \frac{P_n(z)}{[\Phi_*(z)]^{n+1}}, \quad z \in \Omega^*. \tag{18}$$

It is clear that $F \in A(\overline{\Omega^*})$, $F(\infty) = 0$, and, for every $z \in L^*$,

$$|F(z)| = |P_n(z)|.$$

Then, according to the maximum modulus principle, we obtain

$$|F(\xi)| \leq \max_{z \in L^*} |F(z)| = \max_{z \in L^*} |P_n(z)|, \quad \xi \in \overline{\Omega^*}. \tag{19}$$

From (18) and (19), we get

$$|P_n(\xi)| \leq |\Phi_*(\xi)|^{n+1} \|P_n\|_{C(\overline{G}^*)}, \quad \xi \in \overline{\Omega^*}. \tag{20}$$

Taking $\xi \in L$, in view of (20)–(17) we obtain the proof of the lemma.

Lemma 4 [3]. *Let L be a K-quasicomformal curve. Then, for all u, $0 < u < R_0 - 1$, and $z_0 \in G$, we have*

$$\operatorname{mes}((\varphi \circ y)(G_{1+u} \setminus G, z_0)) \prec \delta^{-1}(z_0)\delta^{K^{-2}}(\xi), \qquad (21)$$

where $\delta(z) := d(z, L)$, $\xi = \varphi^{-1}(\tau)$, and

$$|\tau| = \inf\{|w| : w \in (\varphi \circ y)(L_{1+u}, z_0)\}.$$

Lemma 5 [3]. *Let L be a K-quasicomformal curve. Then, for every $n \geq 1$, there exists a polynomial $P_n(z, z_0)$ such that $P_n(z_0, z_0) = 0$, $P_n'(z_0, z_0) = \varphi'(z_0, z_0)$, and*

$$\|\varphi(\cdot, z_0) - P_n(\cdot, z_0)\|_{A_2} \prec n^{-1} + \delta^{-1}(z_0)\left[\operatorname{mes}\varphi(y(G_R \setminus G), z_0)\right]^{\frac{1}{2}}. \qquad (22)$$

Lemma 6. *Let $G \in C_0(\lambda)$ for some λ, $0 < \lambda < 2$. Then, for every $n \geq 1$ and $0 < \mu < \min\left\{\frac{1}{2}; \frac{\lambda}{2-\lambda}\right\}$, there exists a polynomial $P_n(z, z_0)$ such that $P_n(z_0, z_0) = 0$, $P_n'(z_0, z_0) = \varphi'(z_0, z_0)$, and*

$$\|\varphi(\cdot, z_0) - P_n(\cdot, z_0)\|_{A_2} \prec \delta^{-\frac{5-2\lambda}{2(2-\lambda)}}(z_0)n^{-\mu}. \qquad (23)$$

Proof. Since $G \in C_0(\lambda)$, it is easy to see that L satisfies the Ahlfors three-point condition [15, p. 81], i.e., L is a quasiconformal curve with some quasiconformality coefficient $c(K)$. Thus, by using Lemma 5, we get

$$\|\varphi(\cdot, z_0) - P_n(\cdot, z_0)\|_{H_2'} \prec n^{-1} + \delta^{-1}(z_0)\left[\operatorname{mes}\varphi(y(G_R \setminus G), z_0)\right]^{\frac{1}{2}}$$

for the polynomial $P_n(z, z_0)$ such that $P_n(z_0, z_0) = 0$ and $P_n'(z_0, z_0) = \varphi'(z_0, z_0)$.

In the case where z_0 is fixed, the estimate for $\operatorname{mes}\{\varphi(y(G_R \setminus G), z_0)\}$ is given in [16]. In the case where z_0 is arbitrary, we add more facts to the process of the proof presented in [16] in order to obtain an estimate for $\operatorname{mes}\{\varphi(y(G_R \setminus G), z_0)\}$ when necessary.

Let $z_\lambda \in L$ be a corner with exterior angle $\lambda\pi$ and let $b_i(z_\lambda)$, $i = 1, 2$, be two sub arcs of L that meet at z_λ and belong to a sufficiently small neighborhood of z_λ. Without loss of generality, we can assume that $\Phi(z_\lambda) = 1$. Let b_t denote the smaller length of the arcs $\Phi(b_i(z_\lambda))$. We take

$$E_t := \left\{re^{i\theta} : |\theta| < \operatorname{mes} b_t,\ 1 < r < R\right\},$$

$$E_z = \Psi(E_t), \quad E_{z^*} = y(E_z), \quad E_w = \varphi(E_{z^*}).$$

Further, consider the mappings $\xi = \xi(z^*, z_0) = (z^* - z_\lambda)^{\frac{1}{2-\lambda}}$, $w = w(\xi, z_0)$, and $w = w(\xi(z_0, z_0)) = 0$ and let $\xi_0 := \xi(z_0, z_0) = (z_0 - z_\lambda)^{\frac{1}{2-\lambda}}$, $E_\xi = \xi(E_{z^*})$, and $E_w = w(E_\xi)$. Then $G_\xi := \xi(G)$ is a region with smooth boundary and L_ξ is a $(1+\varepsilon)$-quasiconformal curve. Hence, by using Lemma 4, we get

$$\operatorname{mes} E_w \prec \delta^{-1}(\xi_0)\delta^{1-\varepsilon} \qquad (24)$$

for all $\varepsilon > 0$, and the mapping $w(\xi)$ and its inverse can be extended to the entire plane $(1+\varepsilon)^2$-quasiconformally [15, p. 75]. Therefore, by virtue of the Goldstein theorem, we get

$$\operatorname{mes} w(E_\xi) \prec (\operatorname{mes} E_\xi)^\eta \tag{25}$$

for all $0 < \eta < (1+\varepsilon)^{-2}$. On the other hand, since ∂E_w is quasiconformal, one can show that there exists ε_1 such that

$$\operatorname{mes} E_w \geq \varepsilon_1(\rho - |w'|). \tag{26}$$

Since $(\rho - |w'|) \geq c_3 \delta^{(1+\varepsilon)^2}(\xi')$ by virtue of [3] (Lemma 2.1), from (24)–(26) we get

$$\operatorname{mes} E_w \prec \delta^{-1}(\xi_0)(\operatorname{mes} E_\xi)^{1-\varepsilon} \tag{27}$$

for all $\varepsilon > 0$. The estimate for $\operatorname{mes} E_\xi$ is obtained by the procedure used in [16] to estimate $\operatorname{mes} F_w$. Using [16] and the mapping $\xi = \xi(z)$, we get

$$\operatorname{mes} E_w \prec \delta^{-\frac{1}{2-\lambda}}(z_0)(R-1)^{\mu'} \tag{28}$$

for all $0 < \mu' < \min\left\{\frac{2\lambda}{2-\lambda}; 1\right\}$.

Lemma 7. *Let $G \in C_\theta(\lambda)$ for some λ, $0 < \lambda < 2$, and let $h(z)$ be defined by* (5). *Then, for $z_0 \in G$ and $n \geq 1$, there exists a polynomial $T_n(z, z_0)$ such that $T_n(z_0, z_0) = \frac{\varphi'(z_0, z_0)}{D(z_0)}$ and*

$$\left\| \frac{\varphi'(\cdot, z_0)}{D(\cdot)} - T_n(\cdot, z_0) \right\|_{A_2} \prec \delta^{-\frac{\lambda-2\lambda}{2(2-\lambda)}}(z_0) n^{-\mu} \tag{29}$$

for all μ such that $0 < \mu < \min\left\{\frac{\lambda}{2-\lambda}; \frac{1}{2}\right\}$ if $\alpha > \vartheta(\lambda)$ and $0 < \mu < \alpha \min\{1; \lambda\}$ if $\alpha \leq \vartheta(\lambda)$. Here, c is independent of z and n.

Proof. By assumption, we have $\frac{1}{D(z)} \in \operatorname{Lip} \alpha$, $z \in \overline{G}$. Hence, since L is quasiconformal (with a quasiconformality coefficient $c(K) > 1$), by virtue of [17] (Theorem 3) there exist polynomials $\tilde{Q}_m(z)$ such that

$$\max_{z \in \overline{G}} \left| \frac{1}{D(z)} - \tilde{Q}_m(z) \right| \prec d^{\alpha}(z, L_{1+\frac{1}{m}}). \tag{30}$$

Let $Q_m(z, z_0) := \tilde{Q}_m(z) - \tilde{Q}_m(z_0) + \frac{1}{D(z_0)}$. From (30) we get

$$\max_{z \in \overline{G}} \left| \frac{1}{D(z)} - Q_m(z, z_0) \right| \prec m^{-\mu'} \tag{31}$$

for all $0 < \mu' < \alpha \min\{1; \lambda\}$.

Then

$$\left\| \frac{\varphi'(\cdot, z_0)}{D(\cdot)} - P_l'(\cdot, z_0) Q_m(\cdot, z_0) \right\|_{A_2} \le$$

$$\le \left\| \frac{1}{D(\cdot)} \right\|_{C(\overline{G})} \|\varphi'(\cdot, z_0) - P_l'(\cdot, z_0)\|_{A_2} +$$

$$+ \left\| \frac{1}{D(\cdot)} - Q_m(\cdot, z_0) \right\|_{C(\overline{G})} \|\varphi'(\cdot, z_0) - P_l'(\cdot, z_0)\|_{A_2} +$$

$$+ \|\varphi'(\cdot, z_0)\|_{A_2} \left\| \frac{1}{D(\cdot)} - Q_m(\cdot, z_0) \right\|_{C(\overline{G})} \prec$$

$$\prec \delta^{-\frac{5-2\lambda}{2(2-\lambda)}}(z_0) \left(l^{-\mu} + l^{-\mu} m^{-\mu'} \right) + m^{-\mu'}$$

for all $0 < \mu < \min\left\{\frac{\lambda}{2-\lambda}; \frac{1}{2}\right\}$ and $0 < \mu' < \alpha \min\{1; \lambda\}$. Defining $m := n$ if $\mu' \le \mu$, $m := \left[n^{\frac{\mu}{\mu'}}\right] + 1$ if $\mu' > \mu$, and $T_n := P_l' Q_m$, we complete the proof.

Lemma 8. *Let $G \in C_\theta(\lambda)$ for some λ, $0 < \lambda < 2$, and let $h(z)$ be defined by (5). Then, for every $z_0 \in G$, we have*

$$\sum_{k=n}^{\infty} |K_k(z_0)|^2 = O(|\lambda_0|^2 \delta^{-\mu_1}(z_0) n^{-\mu_2}) \tag{32}$$

for all $\mu_1 = \frac{5-2\lambda}{2(2-\lambda)}$ and $\mu_2 = 2\min\{\mu; \mu'\}$.

Proof. It is well known that the function that minimizes the integral

$$J(f) := \iint_G h(z) |f(z)|^2 d\sigma_z \tag{33}$$

in the class of functions f analytic in G and square integrable over G and takes the value $f_0(z_0, z_0) = \frac{\varphi'(z_0, z_0)}{D(z_0)} =: \lambda_0$ for $z_0 \in G$ is $f_0(z, z_0) := \frac{\varphi'(z, z_0)}{D(z)}$ [2].

On the other hand, the polynomial that minimizes (33) in the class of polynomials of at most $(n-1)$th degree and takes the value λ_0 for $z_0 \in G$ is

$$\widetilde{Q}_{n-1}(z) := \lambda_0 \frac{\sum_{k=0}^{n-1} \overline{K_k(z_0)} K_k(z)}{\sum_{k=0}^{n-1} |K_k(z)|^2}, \quad \text{and} \quad J(\widetilde{Q}_{n-1}) = \frac{|\lambda_0|^2}{\sum_{k=0}^{n-1} |K_k(z_0)|^2}. \tag{34}$$

We also have

$$\pi \leq \iint_G h(z)\left|\widetilde{Q}_{n-1}(z,z_0)\right|^2 d\sigma_z =$$

$$= \pi + \iint_G h(z)\left|f_0(z,z_0) - \widetilde{Q}_{n-1}(z,z_0)\right|^2 d\sigma_z \leq$$

$$\leq \pi + \iint_G h(z)\left|f_0(z,z_0) - T_{n-1}(z,z_0)\right|^2 d\sigma_z, \tag{35}$$

where T_{n-1} is an arbitrary polynomial for which $T_{n-1}(z_0,z_0) = \lambda_0$. Taking T_{n-1} as in Lemma 7, we get

$$\frac{|\lambda_0|^2}{\sum\limits_{k=0}^{n-1}|K_k(z_0)|^2} = \pi + O(\delta^{-\mu_1}(z_0)n^{-\mu_2})$$

and, consequently,

$$\sum_{k=0}^{n-1}|K_k(z_0)|^2 = \frac{|\lambda_0|^2}{\pi} - O(|\lambda_0|^2\delta^{-\mu_1}(z_0)n^{-\mu_2}). \tag{36}$$

Let $m > n$. Relation (36) yields

$$\sum_{k=n}^{m}|K_k(z_0)|^2 = O(|\lambda_0|^2\delta^{-\mu_1}(z_0)n^{-\mu_2}) - O(|\lambda_0|^2\delta^{-\mu_1}(z_0)m^{-\mu_2}).$$

Taking the limit as $m \to \infty$, we get

$$\sum_{k=n}^{\infty}|K_k(z_0)|^2 = O(|\lambda_0|^2\delta^{-\mu_1}(z_0)n^{-\mu_2}). \tag{37}$$

3. Proof of theorems. ***Proof of Theorem* 1.** By using the Minkowski inequality, we obtain

$$\omega_n(z) = |f(z) - S_n(f,z)| =$$

$$= \left|\sum_{k=0}^{\infty}a_kK_k(z) - \sum_{k=0}^{n}a_kK_k(z)\right| = \left|\sum_{k=n+1}^{\infty}a_kK_k(z)\right| \leq$$

$$\leq \left(\sum_{k=n+1}^{\infty}|a_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=n+1}^{\infty}|K_k(z)|^2\right)^{\frac{1}{2}}. \tag{38}$$

It is well known that

$$E_n(f,A_2) = \left(\sum_{k=n+1}^{\infty}|a_k|^2\right)^{\frac{1}{2}}$$

in $A_2(h, G)$. Hence, by using Lemma 8, we complete the proof of Theorem 1.

Now let G be a finite region bounded by a Jordan curve and let $T_n(z)$ be the polynomial of the best approximation of $f(z)$ in $\overline{G}$. We have

$$T_n(z) = \sum_{k=0}^{n} a_k^{(n)} K_k(z), \tag{39}$$

where

$$a_k^{(n)} := \iint_G h(\zeta) T_n(\zeta) \overline{K_k(\zeta)} d\sigma_\zeta.$$

Then

$$\omega_n(z) = \left| f(z) - \sum_{k=0}^{n} a_k K_k(z) \right| \leq$$

$$\leq |f(z) - T_n(z)| + \left| T_n(z) - \sum_{k=0}^{n} a_k K_k(z) \right| \leq$$

$$\leq E_n(f, \overline{G}) + \left| T_n(z) - \sum_{k=0}^{n} a_k K_k(z) \right|. \tag{40}$$

$Q_n(z) := T_n(z) - \sum_{k=0}^{n} a_k K_k(z)$ is a polynomial of at most nth degree. Then, by using (2) and (39), we get

$$|Q_n(z)| = \left| \sum_{k=0}^{n} (a_k^{(n)} - a_k) K_k(z) \right| =$$

$$= \left| \sum_{k=0}^{n} \iint_G \sqrt{h(\zeta)} \left[T_n(\zeta) - f(\zeta) \right] \overline{K_k(\zeta)} K_k(z) d\sigma_\zeta \right| \leq$$

$$\leq \iint_G \sqrt{h(\zeta)} \left| T_n(\zeta) - f(\zeta) \right| \left| \sum_{k=0}^{n} \overline{K_k(\zeta)} K_k(z) \right| d\sigma_\zeta \leq$$

$$\leq \max_{z \in \overline{G}} |T_n(z) - f(z)| \iint_G \sqrt{h(\zeta)} \left| \sum_{k=0}^{n} \overline{K_k(\zeta)} K_k(z) \right| d\sigma_\zeta.$$

By using Hölder inequality, we obtain

$$\leq c E_n(f, \overline{G}) \left(\sum_{k=0}^{n} |K_k(z)|^2 \right)^{\frac{1}{2}}. \tag{41}$$

Let

$$F(z, \zeta) = \sum_{k=0}^{n} \overline{K_k(\zeta)} K_k(z)$$

be a bilinear series. We apply the mean-value theorem to $F(z, \zeta)$ in the disk $|z - \zeta| < < d(z, L)$, $z \in G$. Then

$$\iint_{|z-\zeta|<d(z,L)} |F(z, \zeta)|^2 d\sigma_\zeta \leq \frac{1}{\pi d^2(z, L)} \iint_G |F(z, \zeta)|^2 d\sigma_\zeta,$$

$$\left(\sum_{k=0}^{n}|K_k(z)|^2\right)^2 \leq \frac{1}{c\pi d^2(z,L)} \iint_G h(\zeta)\left|\sum_{k=0}^{n}\overline{K_k(\zeta)}K_k(z)\right|^2 d\sigma_\zeta =$$

$$= cd^{-2}(z,L)\sum_{k=0}^{n}|K_k(z)|^2,$$

$$\sum_{k=0}^{n}|K_k(z)|^2 \leq cd^{-2}(z,L).$$

Then, by using relations (40), (41), we get

$$\omega_n \leq cE_n(f,\overline{G})d^{-1}(z,L),\ z \in G. \tag{42}$$

Now let $z \in L^*$. Then, according to Lemma 3 and (17), we can write

$$\omega_n \leq cE_n(f,\overline{G})d^{-1}(t,L_R), \qquad z \in \overline{G}, \tag{43}$$

where $t \in L$ is such that $d(z,L) = |z-t|$.

***Proof of Theorem* 2.** If $G \in C_\theta(\lambda)$ for some λ, $0 < \lambda < 2$, then

$$c_1 n^{-\lambda^*} \leq d(z, L_{1+\frac{1}{n}}) \leq c_2 n^{-\lambda_*}, \tag{44}$$

where $\lambda^* := \max\{1;\lambda\}$ and $\lambda_* := \min\{1;\lambda\}$. The required result now follows from (43) and (44).

***Proof of Corollary* 2.** If $f \in W^{(r)}H^\alpha(\overline{G})$, then, by using [12] (Theorem 4.12), we obtain

$$|f(z) - P_n(z)| \leq d^{r+\alpha}(z, L_{1+\frac{1}{n}}), \quad z \in \overline{G}. \tag{45}$$

Taking into account Theorem 2, (44), and (45), we get (12).

***Proof of Theorem* 3.** If G is a k-quasidisk, $0 \leq k < 1$, then, according to Lemma 16, we get

$$d(z, L_{1+\frac{1}{n}}) \geq cn^{-(1+k)}. \tag{46}$$

The required result now follows from (43) and (46).

***Proof of Corollary* 3.** It follows from [18] (Theorem 2) that

$$|f(z) - P_n(z)| \leq n^{-(r+\alpha)(1-k)}. \tag{47}$$

Relation (13) now follows from Theorem 3 and (47).

***Proof of Theorem* 4.** If $G \in Q_\beta^\gamma$, then, by using [12, p. 58], we obtain

$$cn^{-\beta_*} \leq d(z, L_{1+\frac{1}{n}}) \leq cn^{-\gamma}, \tag{48}$$

where $\beta_* := \min\left\{2, \frac{1}{\beta}\right\}$. Relations (43) and (48) now yield Theorem 4.

***Proof of Corollary* 4.** By virtue of Theorem 4, relations (48) and (44) yield (14).

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