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SPECTRUM AND STATES OF THE BCS HAMILTONIAN IN FINITE DOMAIN. III. THE BCS HAMILTONIAN WITH MEAN-FIELD INTERACTION*

СПЕКТР ТА СТАНИ ГАМІЛЬТОНІАНА БКШ В СКІНЧЕННІЙ ОБЛАСТІ. III. ГАМІЛЬТОНІАН БКШ З ВЗАЄМОДІЄЮ СЕРЕДНЬОГО ПОЛЯ

Spectra of model Hamiltonian with BCS and mean-field interaction in finite volume and periodic boundary conditions is investigated. Model Hamiltonian is considered on states of pairs and waves of density charges, and their excitations. It is represented as sum of three operators that describe noninteracting pairs, interaction between pairs, and interaction between pairs and waves of density charges. The last two operators tend to zero in the thermodynamic limit and spectra of model Hamiltonian coincide with spectra of noninteracting pairs with shifted by mean-field interaction chemical potential. It is shown that model and approximating Hamiltonians coincide in the thermodynamic limit on their ground and excited states and both have two branches of eigenvalues and eigenvectors.

Вивчено спектри модельного гамільтоніана з взаємодією БКШ та середнього поля в скінченному об'ємі та періодичних граничних умовах. Модельний гамільтоніан розглянуто на станах пар, хвиль густини заряду та їх збуджень. На цих станах модельний гамільтоніан представлено трьома операторами, що описують невзаємодіючі пари, взаємодію між парами та хвилями густини заряду. Останні два оператори прямують до нуля у термодинамічній границі, тому спектр модельного гамільтоніана асимптотично збігається зі спектром невзаємодіючих пар зі зсуненим взаємодією середнього поля хімічним потенціалом. Доведено, що модельний та апроксимуючий гамільтоніани збігаються у термодинамічній границі на їхніх основних та збуджених станах і обидва мають дві гілки власних значень та власних векторів.

Introduction. We investigate spectra of model Hamiltonian with BCS and mean-field interaction, proposed by Thirring and Ilieva [1, 2], in finite cube with periodic boundary condition. We used approach developed earlier in Petrina's papers [3, 4].

From general Fock space is extracted the subspace of pairs and waves of density charges invariant with respect of action of the model Hamiltonian. In this subspace the model Hamiltonian can be represented as sum of three operators that describe noninteracting pairs, interaction between pairs, and interaction between pairs and waves of density charges. The last two operators tend to zero in the thermodynamic limit and spectra of model Hamiltonian coincide with spectra of noninteracting pairs with shifted by mean-field interaction chemical potential. Spectra of ground and excited states are determined asymptotically exactly as $V \to \infty$. We show that the model Hamiltonian is thermodynamic equivalent to the approximating Hamiltonian on the ground and excited states. We also determine the ground and excited states of the approximating Hamiltonian and show that the model and approximating Hamiltonians are thermodynamic equivalent on them.

It follows from the obtained results that the model Hamiltonian has two branches of spectra: the first one connected with noninteracting pairs, with shifted chemical potential and their excitations, and the second one connected with ground and excited states of the approximating Hamiltonian.

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12. The model Hamiltonian and its action. *1. The Hamiltonian.* Consider a system of electrons enclosed in cube Λ in the three dimensional Euclidean space with periodic boundary conditions. Denote by L the length of the edge of the cube Λ centered at the origin. Denote by k the quasi discrete momenta which take values $k = \frac{2\pi}{L} n$, $n = (n_1, n_2, n_3)$, where the numbers $n_i, i = 1, 2, 3$, run through the entire set of integer numbers Z. Denote by σ the vector of spin of electron $\sigma = (1, -1)$ and by $\bar{k} = (k, \sigma)$. In that follows we will denote by k the vector $(k, 1)$ and by $-k$ the vector $(-k, -1)$.

Denote by $a^+_{\bar{k}}$ the operator of creation of electron with momenta k and spin σ, by $a_{\bar{k}}$ – the operator of annihilation of electron with momenta k and spin σ. We will also use the denotation

$$a^+_k = a^+_{k,1}, \quad a_k = a_{k,1}, \quad a^+_{-k} = a^+_{-k,-1}, \quad a_{-k} = a_{-k,-1}.$$

In what follows we will use the same denotation as in our previous papers [3, 4].

Consider the following model Hamiltonian [1, 2]

$$H_\Lambda = \sum_{\bar{k}} a^+_{\bar{k}} a_{\bar{k}} \left(\frac{k^2}{2m} - \mu \right) +$$
$$+ \frac{g}{V} \sum_{k,k'} v_k v_{k'} a^+_k a^+_{-k} a_{-k'} a_{k'} + \frac{g}{V} \sum_{k,k'} v_k v_{k'} a^+_k a_k a^+_{-k'} a_{-k'} =$$
$$= H_0 + H_B + H_M \tag{12.1}$$

where V is the volume of cube Λ, $V = L^3$, $v_k = v_{-k}$ is potential, μ — chemical potential , m — mass of electron, g — coupling constant, $g < 0$. We supose that potential has a support in layer D of the Fermi sphere $\left| \frac{k^2}{2m} - \mu \right| < \omega$, $\omega > 0$.

Hamiltonian (1.1) differs from usual BCS Hamiltonian by the last term H_M known as mean-field term.

2. Action of Hamiltonian. Consider the following states

$$f_{(p)_l,(p')_l} =$$
$$= a^+_{p_1} \ldots a^+_{p_l} a^+_{-p'_1} \ldots a^+_{-p'_l} \sum_{n=0}^{n_0} \sum_{k_1 \neq \ldots \neq k_n}{}' f_n(k_1, \ldots, k_n) a^+_{k_1} a^+_{-k_1} \ldots a^+_{k_n} a^+_{-k_n} |0\rangle =$$
$$= a^+_{p_1} \ldots a^+_{p_l} a^+_{-p'_1} \ldots a^+_{-p'_l} \sum_{n=0}^{n_0} \frac{1}{n!} \sum_{k_1, \ldots, k_n} f_n(k_1, \ldots, k_n) a^+_{k_1} a^+_{-k_1} \ldots a^+_{k_n} a^+_{-k_n} |0\rangle =$$
$$= \sum_{n=0}^{n_0} f^n_{(p)_l,(p')_l}, \quad f_0 = 1, \tag{12.2}$$

where all momenta $(p)_l$, $(p')_l$, $(k)_n$ belong to D, but $(p)_l$ and $(p')_l$ belong to the domain $D_I (p| -\omega < \frac{p^2}{2m} - \mu < 0)$, and $(k)_n$ belong to the domain $D_{II} (k| 0 < \frac{k^2}{2m} - \mu < \omega)$, n_0 will be fixed later. We suppose that $p_i \neq +p'_j$ for arbitrary $i, j \subset \subset (1, \ldots, l)$.

Note that all our results remain true if we use the domains $D_I \left(p| -\omega < \frac{p^2}{2m} - \mu < < \omega_0 \right)$, $D_{II} \left(k| \omega_0 < \frac{k^2}{2m} - \mu < \omega \right)$ with some ω_0, $-\omega < \omega_0 < \omega$. We will discuss later how to determine the parameter ω_0.

In state $f_{(p)_l,(p'_l)}$ the sequence of functions $f = (f_0, f_1(k_1), \ldots, f_n(k_1, \ldots, k_n), \ldots)$ has a support in D_{II}, functions are symmetric, and f belongs to the space $\mathcal{H}_V^P$ with norm

$$\|f\|'_V = \sum_{n=0}^{n_0} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n}' |f_n(k_1, \ldots, k_n)|^2 \tag{12.3}$$

and corresponding standart scalar product of two sequences $f \in \mathcal{H}_V^P$ and $g \in \mathcal{H}_V^P$.

We say that the state $f_{(p)_l,(p')_l}$ corresponds to the wave of density charge with the operators $a^+_{p_1} \ldots a^+_{p_l} a^+_{-p'_1} \ldots a^+_{-p'_l}$ and to the state of pairs with the sequence f. The reason why the state $f_{(p)_l,(p')_l}$ is constructed in such a way will be explained later.

Now consider action of H_Λ on the state $f_{(p)_l,(p')_l}$. By analogy with calculation fulfilled in our previous papers [3, 4] we obtain

$$H_\Lambda a^+_{p_1} \ldots a^+_{p_l} a^+_{-p'_1} \ldots a^+_{-p'_l} \sum_{n=0}^{n_0} \frac{1}{n!} \sum_{k_1, \ldots, k_n} f_n(k_1, \ldots, k_n) a^+_{k_1} a^+_{-k_1} \ldots a^+_{k_n} a^+_{-k_n} |0\rangle =$$

$$= \sum_{n=1}^{n_0} \frac{1}{n!} \sum_{k_1, \ldots, k_n} \left\{ \left[\sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) + \sum_{i=1}^{l} \left(\frac{p'^2_i}{2m} - \mu \right) + \right.\right.$$

$$\left. + \sum_{i=1}^{n} \left(\frac{2k_i^2}{2m} - 2\mu \right) \right] f_n(k_1, \ldots, k_n) +$$

$$+ \frac{g}{V} \sum_{i=1}^{n} \left[\sum_{p} v_{k_i} v_p f_n(k_1, \ldots, \overset{i}{\bar{p}}, \ldots, k_n) - \sum_{1=j\neq i}^{n} v_{k_i} v_{k_j} f_n(k_1, \ldots, \overset{i}{\bar{k_j}}, \ldots, k_n) \right] +$$

$$\left. + \frac{g}{V} \left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{l} v_{p_i} \right) \left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{l} v_{p'_i} \right) f_n(k_1, \ldots, k_n) \right\} \times$$

$$\times a^+_{p_1} \ldots a^+_{p_l} a^+_{-p'_1} \ldots a^+_{-p'_l} a^+_{k_1} a^+_{-k_1} \ldots a^+_{k_n} a^+_{-k_n} |0\rangle. \tag{12.4}$$

Formula (12.4) implies

$$H_\Lambda f_{(p)_l,(p')_l} = \left[\sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) I + \sum_{i=1}^{l} \left(\frac{p'^2_i}{2m} - \mu \right) I \right] f_{(p)_l,(p')_l} +$$

$$+ (A + B) f_{(p)_l,(p')_l} + Q_{k,p,p'} \cdot f_{(p)_l,(p')_l} \tag{12.5}$$

where I is unit operator, the operator A is defined by the third and fourth term, the operator B — by the fifth term, and the operator $Q_{k,p,p'}$ — by the last sixth term in (12.4).

Now we suppose that momenta $p_1, \ldots, p_l, -p'_1, \ldots, -p'_l$ *exhaust all domain* D_I *and* $p_i \neq +p'_j, (i, j) \subset (1, \ldots, l)$. *This means that after action of the operator* H_Λ *all the momenta* $k_1, \ldots, k_n$, $1 \leq n \leq n_0$, *again belong to* D_{II}.

3. Investigation of the operators A, B, $Q_{k;p;\bar{p}}$. a) The operator A acts on $f_n(k_1, \ldots, k_n)$ as follows

$$(Af)_n f_n(k_1, \ldots, k_n) =$$

$$= \left[\sum_{i=1}^{n} \left(\frac{2k_i^2}{2m} - 2\mu \right) f_n(k_1, \ldots, k_n) + \frac{g}{V} \sum_{p} v_{k_i} v_p f_n(k_1, \ldots, \overset{i}{\bar{p}}, \ldots, k_n) \right] \tag{12.6}$$

and it has alredy been investigated in our paper [3].

The only difference is that functions $f_n((k)_n)$ have suports in D_{II}, summation with respect to p in (12.6) is carried out over D_{II}. Momenta $(k)_n$ of the operators

$a^+_{k_1}a^+_{-k_1}\dots a^+_{k_n}a^+_{-k_n}$ in (12.4) belong also to D_{II} because momenta $(p)_l, (-p')_l$ of the operator $a^+_{p_1}\dots a^+_{p_l}a^+_{-p'_1}\dots a^+_{-p'_l}$ exhaust the domain D_I. This fact means that after the action of the Hamiltonian H_Λ momenta in $H_\Lambda f_{(p)_l,(p')_l}$ of the operators $a^+_{k_1}a^+_{-k_1}\dots$ $\dots a^+_{k_n}a^+_{-k_n}$ belong to D_{II}. Thus we can consider the operator A determined by (12.6) and with potentials v_k, v_p that have supports in D_{II}.

As known the operator A has eigenfunctions

$$f_n((k)_n) = \frac{1}{n!}\mathrm{sym}(f_1^1(k_1)\dots f_1^n(k_n)) \tag{12.7}$$

with eigenvalue $E = E_1 + \dots + E_n$ where functions $f_1^i(k)$ are defined as follows

$$f_1^i(k) = \frac{c^i v_k}{-\dfrac{2k^2}{2m} + 2\mu + E_i}, \qquad c^i = \frac{g}{V}\sum_p v_p f_1^i(p) \tag{12.8}$$

and numbers E_i are determined as solutions of equation

$$1 = \frac{g}{V}\sum_{p\subset D_{II}} \frac{v_p^2}{-\dfrac{2p^2}{2m} + 2\mu + E}. \tag{12.9}$$

Obviously E_i depend on L, $E_i = E_i(L)$.

Thus we can use all the results from our papers [3, 4] concerning the operator A. The operator A is selfadjoint in $\mathcal{H}^P_{V,II}$ i.e., in Hilbert space $\mathcal{H}^P_V$ consisting from sequences $f = (f_0, f_1(k_1), \dots, f_n((k)_n), \dots)$ of functions $f_n((k)_n)$ with supports in D_{II} with respect to all $k_i, i = 1, \dots, n$.

In what follows we consider only functions $f_n((k)_n)$ that satisfy conditions $\sup_{(k)_n}|f_n((k)_n)| \le f^n$, $f > 0$, $0 \le n \le n_0$, uniformly with respect to V.

b) Using this fact that summation in action of the operator B is restricted to D_{II} we obtain the following estimate

$$\left|(f_{(p)_l,(p')_l}, Bf_{(p)_l,(p')_l})'_V\right| \le \frac{gv^2}{V}\alpha^2 f^4 e^{\alpha f^2} \tag{12.10}$$

for sequences $f \in \mathcal{H}^P_{V,II}$ (see (5.8) from [3]).

We use denotation from [3, 4] where

$$v = \sup_{k\in D}|v_k|, \qquad \alpha = \frac{N_{II}}{V},$$

and N_{II} is number of quasimomenta in D_{II}. It follows from (12.10) that the averages $(f_{(p)_l,(p')_l}, Bf_{(p)_l,(p')_l})'_{V_{II}}$ tend to zero as $V \to \infty$ for arbitrary f such that $\sup_{(k)_n}|f_n((k)_n)| \le f^n$. It is easy to prove that $\|Bf_{(p)_l,(p')_l}\|'_V$ tends to zero as $V \to \infty$ on such f.

c) For the operator $Q_{k,p,p'}$ we have the following estimate

$$\left|(f_{(p)_l,(p')_l}, Q_{k,p,p'}f_{(p)_l,(p')_l})'_V\right| \le$$
$$\le \sum_{n=1}^{n_0}\frac{|g|}{V}\frac{1}{V^n}\sum'_{k_1\ne\dots\ne k_n}\left|\left(\sum_{i=1}^n v_{k_i} + \sum_{i=1}^l v_{p_i}\right)\times\right.$$
$$\left.\times\left(\sum_{i=1}^n v_{k_i} + \sum_{i=1}^l v_{p'_i}\right)\right||f_n(k_1,\dots,k_n)|^2 \le$$

$$\leq \frac{|g|v^2}{V}\sum_{n=1}^{n_0}\frac{N_{II}^n}{V^n}\frac{(n+l)^2}{n!}f^{2n}\leq$$

$$\leq \frac{|g|v^2(l+2)^2}{V}\left(\sum_{n=2}^{n_0}\frac{\alpha^n f^{2n}}{(n-2)!}+\alpha f^2\right)=\frac{|g|v^2(l+2)^2}{V}(\alpha f^2+\alpha^2 f^4 e^{\alpha f^2}). \quad (12.11)$$

It follows from estimate (12.11) that averages $(f_{(p)_l,(p')_l}, Q_{k,p,p'}f_{(p)_l,(p')_l})'_V$ exist for fixed V and l.

Now put $v_k = v$ in D and consider the following expression

$$C_M = \lim_{V\to\infty}\frac{g}{2V}\left(\sum_{i=1}^{n}v_{k_i}+\sum_{i=1}^{l}v_{p_i}\right)=\lim_{V\to\infty}\frac{g}{2V}\left(\sum_{i=1}^{n}v_{k_i}+\sum_{i=1}^{l}v_{p'_i}\right)=$$

$$=\lim_{V\to\infty}C_M^n(V)=\lim_{V\to\infty}\frac{gv}{2V}(n+l), \quad n\leq n_0. \quad (12.12)$$

Note that $l\to\infty$ as $V\to\infty$, $l=\alpha_I V$, $\alpha_I = 2\frac{V(D_I)}{(2\pi)^3}$, $V(D_I)$ is volume of the domain D_I.

In what follows we suppose that the number n_0 tends to ∞ as $V\to\infty$ but in such a way that $\lim_{V\to\infty}\frac{n_0}{l}=0$.

It follows from (12.12) that expressions C_M exists and is independent on n, $1\leq \leq n\leq n_0$,

$$\lim_{V\to\infty}C_M^n(V)=\lim_{V\to\infty}\frac{gv}{2V}(n+l)=\lim_{V\to\infty}\frac{gvl}{2V}=\frac{gv\alpha_I}{2}=C_M. \quad (12.13)$$

Obviously that

$$\frac{g}{V}\frac{1}{V}\sum_{k,k'}v_k v_{k'}a_k^+a_k^+a_{-k'}^+a_{-k'}=C_M^n(V)\sum_{\bar{k}}v_k a_{\bar{k}}^+a_{\bar{k}}$$

on $f^n_{(p)_l,(p')_l}$.

Now we are able to explain the choise of the operator $a^+_{p_1}\ldots a^+_{p_l}a^+_{-p'_1}\ldots a^+_{-p'_l}$, numbers l and n_0. With these operators and numbers the constant C_M is independent in the thermodynamic limit on numbers n, i.e. is the same for arbitrary $f^n_{(p)_l,(p')_l}$, $n\leq n_0$.

Denote by C_B the following constant

$$C_B=\frac{g}{V}\sum_k v_k f_1^0(k) \quad (12.14)$$

where $f_1^0(k)$ is the eigenfunction of the operator A (6.1) with the lowest eigenvalue E_0.

d) Consider the following operators

$$C_M\sum_{\bar{k}}v_k a_{\bar{k}}^+a_{\bar{k}} \quad (12.15)$$

and

$$\mathcal{E}_M=C_M\sum_{\bar{k}}v_k a_{\bar{k}}^+a_{\bar{k}}-\frac{g}{V}\sum_k v_k a_k^+a_k\sum_{k'}v_{k'}a_{-k'}^+a_{-k'}. \quad (12.16)$$

Now show that in some sense the operator $\mathcal{E}_M$ tends to zero as $V\to\infty$. We have according to the definition of the operator $Q_{k,p,p'}$

$$\mathcal{E}_M f_{(p)_l,(p')_l}=\sum_{n=1}^{n_0}\sum_{k_1\neq\ldots\neq k_n}{}' \left\{C_M\left[\sum_{i=1}^{n}(v_{k_i}+v_{k_i})+\sum_{i=1}^{l}(v_{p_i}+v_{p'_i})\right]-\right.$$

$$\left.-\frac{g}{V}\left(\sum_{i=1}^{n}v_{k_i}+\sum_{i=1}^{l}v_{p_i}\right)\left(\sum_{i=1}^{n}v_{k_i}+\sum_{i=1}^{l}v_{p'_i}\right)\right\}\times$$

$$\times f_n(k_1, \dots, k_n) a^+_{p_1} \dots a^+_{p_l} a^+_{-p'_1} \dots a^+_{-p'_l} a^+_{k_1} a^+_{-k_1} \dots a^+_{k_n} a^+_{-k_n} |0\rangle =$$

$$= \sum_{n=1}^{n_0} \sum_{k_1 \neq \dots \neq k_n}' \left\{ C_M(2vn + 2vl) - \frac{g}{V} v(n+l) v(n+l) \right\} f_n(k_1, \dots, k_n) \times$$

$$\times a^+_{p_1} \dots a^+_{p_l} a^+_{-p'_1} \dots a^+_{-p'_l} a^+_{k_1} a^+_{-k_1} \dots a^+_{k_n} a^+_{-k_n} |0\rangle. \tag{12.17}$$

According to definition (12.13) of the constant C_M, the expression $C_M - \frac{g}{2V} v(n+l)$ tends to zero as $V \to \infty$.

Further we have

$$\lim_{V \to \infty} \frac{1}{V} \left| \left(f_{(p)_l, (p')_l}, \varepsilon_M f_{(p)_l, (p')_l} \right)'_V \right| \leq$$

$$\leq \lim_{V \to \infty} \sum_{n=1}^{n_0} \frac{1}{V^n} \frac{N^n_{II}}{n!} f^{2n} \left| C_M - \frac{g}{2V} v(n+l) \right| \frac{2v(n+l)}{V} \leq$$

$$\leq 2v(\alpha + \alpha_I) \lim_{V \to \infty} \sum_{n=1}^{n_0} \frac{\alpha^n}{n!} f^{2n} \left| C_M - \frac{g}{2V} v(n+l) \right| = 0. \tag{12.18}$$

We used in (12.18) the fact that series are convergent uniformly with respect to V and the existence of limit (12.13).

e) Let estimate the operator

$$H_\Lambda - A - C_M \sum_{\bar{k}} v_{\bar{k}} a^+_{\bar{k}} a_{\bar{k}} - \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) I - \sum_{i=1}^{l} \left(\frac{p_i'^2}{2m} - \mu \right) I$$

on states $f_{(p)_l, (p')_l}$.

Consider the following expression

$$\lim_{V \to \infty} \frac{1}{V} \Bigg(f_{(p)_l, (p')_l}, \Bigg(H_\Lambda - A - C_M \sum_{\bar{k}} v_k a^+_{\bar{k}} a_{\bar{k}} -$$

$$- \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) I - \sum_{i=1}^{l} \left(\frac{p'^2_i}{2m} - \mu \right) I \Bigg) f_{(p)_l, (p')_l} \Bigg)'_V =$$

$$= \lim_{V \to \infty} \frac{1}{V} \left(f_{(p)_l, (p')_l}, (B - \mathcal{E}_M) f_{(p)_l, (p')_l} \right)'_V = 0. \tag{12.19}$$

In proving (12.19) we used representation (12.5), (12.16) and estimates (12.10), (12.18). Note that the average $\left(f_{(p)_l, (p')_l}, B f_{(p)_l, (p')_l} \right)'_V$ tends to zero as $V \to \infty$ even without the factor $\frac{1}{V}$. The factor $\frac{1}{V}$ is necessary to estimate the average $(f_{(p)_l, (p')_l}, \mathcal{E}_M f_{(p)_l, (p')_l})'_V$. Obtained above result will be used in Section 14.

Denote by $C_M(V)$ the following operator

$$C_M(V) f^n_{(p)_l, (p')_l} = C^n_M(V) f^n_{(p)_l, (p')_l}, \quad 0 \leq n \leq n_0. \tag{12.20}$$

Obviously that

$$\frac{g}{V} \sum_{k, k'} v_k v_{k'} a^+_k a_k a^+_{-k'} a_{-k'} = C_M(V) \sum_{\bar{k}} v_k a^+_{\bar{k}} a_{\bar{k}}$$

on $f_{(p)_l, (p')_l}$, i.e.

$$\frac{g}{V} \sum_{k, k'} v_k v_{k'} a^+_k a_k a^+_{-k'} a_{-k'} f_{(p)_l, (p')_l} = C_M(V) \sum_{\bar{k}} v_k a^+_{\bar{k}} a_{\bar{k}} f_{(p)_l, (p')_l}. \tag{12.21}$$

Now consider the following average

$$\lim_{V\to\infty}\left(f_{(p)_l,(p')_l},\left(H_\Lambda - A - C_M(V)\sum_{\bar k} v_k a^+_{\bar k} a_{\bar k} - \sum_{i=1}^{l}\left(\frac{p_i^2}{2m}-\mu\right)I -\right.\right.$$

$$\left.\left.-\sum_{i=1}^{l}\left(\frac{p'^2_i}{2m}-\mu\right)I\right)f_{(p)_l,(p')_l}\right)'_V = \lim_{V\to\infty}\left(f_{(p)_l,(p')_l}, Bf_{(p)_l,(p')_l}\right)'_V = 0. \quad (12.22)$$

It was used in (12.22) identity (12.21) and estimate (12.10).

Thus the operator H_Λ asymptotically coincides, in sense of (12.22), with the operator

$$A + C_M(V)\sum_{\bar k} v a^+_{\bar k} a_{\bar k} + \sum_{i=1}^{l}\left(\frac{p_i^2}{2m}-\mu\right)I + \sum_{i=1}^{l}\left(\frac{p'^2_i}{2m}-\mu\right)I =$$

$$= A_r + \sum_{i=1}^{l}\left(\frac{p_i^2}{2m}-\mu + vC_M(V)\right)I + \sum_{i=1}^{l}\left(\frac{p'^2_i}{2m}-\mu + vC_M(V)\right)I$$

where in the renormalized operator A_r the chemical potential is shifted by $-vC_M(V)$.

Note that eigenvalue of the operator A_r are the same as of the operatop A. The eigenfunctions of the operator A_r are given by formulae (12.8) with eigenvalue E_i of the operator A. As known the lowest eigenvalue E_0 is negative, there is the gap different from zero between E_0 and the next eigenvalue E_1, and the eigenfunction $f_1^0(k)$ is uniformly bounded with respect to V [3, 4].

Now prove the above formulated results. The eigenvalue problem for the operator $H_{2,r} = H_2 + 2vC^n_M(V)$ is defined by the following equation (for $f^n_{(p)_l,(p')_l}$)

$$H_{2,r}f_1(k) = \left(\frac{2k^2}{2m} - 2\mu + 2vC^n_M(V)\right)f_1(k) + \frac{g}{V}\sum_p v_k v_p f_1(p) = Ef_1(k).$$

The solutions of this equation is well known

$$f_1^i(k) = \frac{C_B^i v_k}{-\dfrac{2k^2}{2m} + 2\mu - 2vC^n_M(V) + E_{i,r}}, \qquad C_B^i = \frac{g}{V}\sum_{p\subset D_{II}} v_p f_1^i(p)$$

where eigenvalues $E_{i,r}$ are determined as solutions of the following algebraic equation

$$1 = \frac{g}{V}\sum_{p\subset D_{II}}\frac{v_p^2}{-\dfrac{2p^2}{2m} + 2\mu - 2vC^n_M(V) + E}. \quad (12.23)$$

It is obvions that solutions $E_{i,r}$ of equation (12.20) are given by formula $E_{i,r} = E_i + 2C^n_M(V)$ through solutions E_i of the equation

$$1 = \frac{g}{V}\sum_{p\subset D_{II}}\frac{v_p^2}{-\dfrac{2p^2}{2m} + 2\mu + E}.$$

The eigenfunctions $f_1^i(k)$ are equal to

$$f_1^i(k) = \frac{C_B^i v_k}{-\dfrac{2k^2}{2m} + 2\mu + E_i}$$

and does not depend on the shift of chemical potential.

Now we are able to investigate in detail the spectra of the Hamiltonian H_Λ. Namely, taking into account (12.12)–(12.20) we represent (12.5) as follows

$$H_\Lambda f^n_{(p)_l,(p')_l} = \left[\sum_{i=1}^{l}\left(\frac{p_i^2}{2m} - \mu + vC^n_M(V)\right)I + \sum_{i=1}^{l}\left(\frac{p_i'^2}{2m} - \mu + vC^n_M(V)\right)I\right]\times$$
$$\times f^n_{(p)_l,(p')_l} + (A_r + B)f^n_{(p)_l,(p')_l},$$
$$f^n_{(p)_l,(p')_l} = a^+_{p_1}\dots a^+_{p_l}a^+_{-p'_1}\dots a^+_{-p'_l}\sum_{k_1,\dots,k_n}\frac{1}{n!}f_n(k_1,\dots,k_n)a^+_{k_1}a^+_{-k_1}\dots a^+_{k_n}a^+_{-k_n}|0\rangle. \tag{12.24}$$

The operator A_r is defined as the operator A (12.6) but with shifted chemical potential $\mu - C^n_M(V)$. The operator A_r and B act only on the functions $f_n((k)_n)$ in $f_{(p)_l,(p')_l}$. It follows from (12.25) that to investigate the spectra of H_Λ it is sufficient to investigate the spectra of the operator $A_r + B$. The spectra of the operator A_r is defined according to (12.6), (12.9) and (12.23), (12.24) and is known. For the operator B we have the following estimate (see (3.9) from [3])

$$\|Bf_n\|'_V \le \frac{|g|v\|v\|}{V^{\frac12}}n(n-1)\|f_n\|'_V, \qquad \|v\|^2 = \frac{1}{V}\sum_{p\subset D_{II}} v_p^2 \tag{12.25}$$

from which one concludes that for fixed n the operator B can be considered as a perturbation of the operator A_r as $V\to\infty$. From the well known theorem of linear algebra (see, for example, [5–7]) one concludes that the eigenvalues of the operator $A_r + B$ differ from the eigenvalues of the operator A_r by values $\mathcal{E}\left(\frac{1}{V}\right)$ that are proportional to $\frac{1}{V^{\frac{1}{2m}}}$, where m is the multiplicity of eigenvalues of the operator A_r.

The obtained above result has two disadvantages:

1) for $n \sim V^{\frac14}$ the operator B cannot be considered as perturbation of A,

2) even for a fixed n the space $\mathcal{H}^{l,'}_V$ and operator A changes together with V, we have not standart problem of perturbation of spectra [7] with fixed space, fixed operator A_r, and perturbation $\mathcal{E}B$ with a smal parameter $\mathcal{E}$ and the fixed operator B.

Therefore even for fixed n the function $\mathcal{E}\left(\frac{1}{V}\right)$, in general case, can change with V, $\mathcal{E}\left(\frac{1}{V}\right) = \mathcal{E}_V\left(\frac{1}{V}\right)$ and it can be estimated effectively only if all eigenvalues are simple [5]. In our case $v_k = v$, all the eigenvalues are simple (see [3], Section VI), and we can obtain a desired estimate for $\mathcal{E}\left(\frac{1}{V}\right)$.

Namely, denote by $\tilde{E}(L)$ the point of the spectra of the operator $A_r + B$ that corresponds to the point $E(L)$ of spectra of the operator A_r. Then

$$|\tilde{E}(L) - E(L)| \le \frac{|g|}{V^{\frac12}}n(n-1)v\|v\|$$

and the function $\mathcal{E}\left(\frac{1}{V}\right)$ has the same majorant for all V

$$\mathcal{E}\left(\frac{1}{V}\right) \le \frac{|g|}{V^{\frac12}}n(n-1)v\|v\| \tag{12.26}$$

(see [5], Section 16, formula (16.50)).

Now obtain analogous estimate for the eigenvectors that corresponds to ground state. We use the fact that there is the gap between the eigenvalues of the ground state and the excited states. Namely $|E_0(L) - E_i(L)| \ge \Delta > 0$, $i \ge 1$, where Δ is the gap in the spectra of $H_{2,r}$ (see [3], Section VI).

For given fixed n $f_1^0(k_1)\dots f_1^0(k_n) = f_n^0(k_1,\dots,k_n)$ is the eigenvector of A_r with the lowest eigenvalue $E_0^n = nE_0$ and on distance greater then Δ from the rest of spectra E_i^n.

Denote by $\tilde{f}_n^0(k_1,\dots,k_n)$ the eigenvector of the operator $A_r + B$ that corresponds to the lowest eigenvalue $\tilde{E}_n^0, |\tilde{E}_n^0 - E_n^0| < \dfrac{gv\|v\|n(n-1)}{V^{\frac{1}{2}}}$.

Then according [5] (Section 16, formulae (16.50)) one has the following estimate

$$\|\tilde{f}_n^0 - f_n^0\|'_V \le \frac{gv\|v\|n(n-1)}{V^{\frac{1}{2}}} \sum_i \frac{1}{|E_0^n - E_i^n|} \le \frac{gv\|v\|n(n-1)}{V^{\frac{1}{2}}} \frac{1}{\Delta} N \qquad (12.27)$$

where summation is carried out over all eigenvalues E_i^n of the operator A_r in subspace of n pairs, and N is the number of all E_i^n. It is easy to show that

$$N \approx \left(\frac{V^{\frac{1}{3}}}{\pi} (2m(\mu+\omega))^{\frac{1}{2}} \right)^n$$

where $\dfrac{V^{\frac{1}{3}}}{\pi}(2m(\mu+\omega))^{\frac{1}{2}}$ is the estimate for the number of all eigenvalues of the operator $H_{2,r}$ (see [3], Section VI, (6.1)). This means that $\|\tilde{f}_n^0 - f_n^0\|'_V$ does not tends to zero as $V \to \infty$.

13. Approximating Hamiltonian and its coincidence with model Hamiltonian. ***1. Approximating Hamiltonian $H_{a;\Lambda}$ and its coincidence with H_Λ in the thermodynamic limit.*** Consider the following approximating Hamiltonian

$$H_{a,\Lambda} = \sum_k a_k^+ a_k \left(\frac{k^2}{2m} - \mu \right) + C_B \sum_k v_k a_k^+ a_{-k}^+ + C_B \sum_k v_k a_{-k} a_k +$$
$$+ C_M \sum_k v_k a_k^+ a_k + C_M \sum_k v_k a_{-k}^+ a_{-k} - g^{-1} C_B^2 V. \qquad (13.1)$$

Note that we use in $H_{a,\Lambda}$ the constant C_M (12.13) independent on n and V. Now we show that H_Λ tends in some sense to $H_{a,\Lambda}$ as $V \to \infty$ on certain states $\Phi_{(p)_l,(p')_l}$ defined as follows

$$\Phi_{(p)_l,(p')_l} = a_{p_1}^+ \dots a_{p_l}^+ a_{-p'_1}^+ \dots a_{-p'_l}^+ \Phi_0,$$
$$\Phi_0 = \sum_{n=0}^{n_0} \frac{1}{n!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \dots \sum_{k_n} f_1^0(k_n) a_{k_n}^+ a_{-k_n}^+ |0\rangle =$$
$$= \sum_{n=0}^{n_0} \sum'_{k_1 \ne \dots \ne k_n} f_1^0(k_1) \dots f_1^0(k_n) a_{k_1}^+ a_{-k_1}^+ \dots a_{k_n}^+ a_{-k_n}^+ |0\rangle \qquad (13.2)$$

where $f_1^0(k_1)$ is eigenfunction (12.8) with lowest eigenvalue $E_0(L)$. We say that $\Phi_{(p)_l,(p')_l}$ is the ground state of the model Hamiltonian H_Λ with given fixed $(p)_l, (p')_l$ with the same restrictions on $n_0, (p)_l, (p')_l$ as in Section I.

Note that the ground state $\Phi_{(p)_l,(p')_l}$ is the asymptotic, as $V \to \infty$, eigenvector of the operator

$$H_{\Lambda,r} = H_\Lambda - \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu + vC_M(V) \right) I -$$
$$- \sum_{i=1}^{l} \left(\frac{p_i'^2}{2m} - \mu + vC_M(V) \right) I - \frac{E_0(L)}{2} N, \qquad (13.3)$$

$$N = \sum_{\bar{p}} a_{\bar{p}}^{+} a_{\bar{p}}.$$

It follows from (12.22) that

$$\|H_{\Lambda,r}\Phi_{(p)_l,(p')_l}\|_V' = \|B\Phi_{(p)_l,(p')_l}\|_V'$$

because

$$\left(A_r - \frac{E_0(L)}{2}n\right) f_1^0(k)\dots f_1^0(k_n) = 0.$$

(Recall that $f_1^0(k)$ does not depend on $C_M^n(V)$, i.e. it is the same for different $C_M^n(V)$.)

According to estimate analogous to (12.10) one has $\lim_{V\to\infty}\|B\Phi_{(p)_l,(p')_l}\|_V' = 0$ and from (13.4) we obtain that

$$\lim_{V\to\infty}\|H_{\Lambda,r}\Phi_{(p)_l,(p')_l}\|_V' = 0$$

and conclude that $\Phi_{(p)_l,(p')_l}$ is the asymptotic, as $V\to\infty$, eigenvector of the renormalized Hamiltonian $H_{\Lambda,r}$ with eigenvalue zero.

Consider the following expression

$$(H_\Lambda - H_{a,\Lambda})\Phi_{(p)_l,(p')_l} =$$

$$= a_{p_1}^+\dots a_{p_l}^+ a_{-p_1'}^+\dots a_{-p_l'}^+\left\{\sum_{n=1}^{n_0}\frac{g}{V}\sum_{i=1}^{n}\left[\frac{1}{n!}\sum_{k_1} f_1^0(k_1)a_{k_1}^+a_{-k_1}^+\dots\right.\right.$$

$$\dots\sum_{k_i} v_{k_i}\sum_{p} v_p f_1^0(p)a_{k_i}^+a_{-k_i}^+\dots\sum_{k_n} f_1^0(k_n)a_{k_n}^+a_{-k_n}^+|0\rangle - \sum_{n=2}^{n_0}\frac{1}{n!}\sum_{k_1} f_1^0(k_1)a_{k_1}^+a_{-k_1}^+\dots$$

$$\left.\dots\sum_{1=j\neq i}^{n} v_{k_i}v_{k_j} f_1^0(k_j)a_{k_i}^+a_{-k_i}^+\dots\sum_{k_n} f_1^0(k_n)a_{k_n}^+a_{-k_n}^+|0\rangle\right] +$$

$$+\frac{g}{V}\sum_{n=1}^{n_0}\left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{l} v_{p_i}\right)\left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{l} v_{p_i'}\right)\times$$

$$\times\frac{1}{n!}\sum_{k_1} f_1^0(k_1)a_{k_1}^+a_{-k_1}^+\dots\sum_{k_n} f_1^0(k_n)a_{k_n}^+a_{-k_n}^+|0\rangle -$$

$$\left.-C_B\sum_{n=1}^{n_0+1}\sum_{i=1}^{n}\frac{1}{n!}\sum_{k_1} f_1^0(k_1)a_{k_1}^+a_{-k_1}^+\dots\sum_{k_i} v_{k_i}a_{k_i}^+a_{-k_i}^+\dots\sum_{k_n} f_1^0(k_n)a_{k_n}^+a_{-k_n}^+|0\rangle\right] -$$

$$-\sum_{n=0}^{n_0-1} g^{-1}C_B^2V\frac{1}{n!}\sum_{k_1} f_1^0(k_1)a_{k_1}^+a_{-k_1}^+\dots\sum_{k_n} f_1^0(k_n)a_{k_n}^+a_{-k_n}^+|0\rangle +$$

$$+\sum_{n=0}^{n_0-1} g^{-1}C_B^2V\frac{1}{n!}\sum_{k_1} f_1^0(k_1)a_{k_1}^+a_{-k_1}^+\dots\sum_{k_n} f_1^0(k_n)a_{k_n}^+a_{-k_n}^+|0\rangle +$$

$$+\sum_{n=1}^{n_0-1} C_B\sum_{k=k_1,\dots,k=k_n} v_k f_1^0(k)\frac{1}{n!}\sum_{k_1} f_1^0(k_1)a_{k_1}^+a_{-k_1}^+\dots\sum_{k_n} f_1^0(k_n)a_{k_n}^+a_{-k_n}^+|0\rangle -$$

$$-\sum_{n=1}^{n_0} C_M\left[\left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{n} v_{p_i}\right) + \left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{n} v_{p_i'}\right)\right]\frac{1}{n!}\sum_{k_1} f_1^0(k_1)a_{k_1}^+a_{-k_1}^+\dots$$

$$\dots\sum_{k_n} f_1^0(k_n)a_{k_n}^+a_{-k_n}^+|0\rangle + g^{-1}C_B^2V\times$$

$$\times \frac{1}{n_0!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_{n_0}} f_1^0(k_{n_0}) a_{k_{n_0}}^+ a_{-k_{n_0}}^+ |0\rangle \Bigg\} =$$

$$= a_{p_1}^+ \ldots a_{p_l}^+ a_{-p'_1}^+ \ldots a_{-p'_l}^+ \Bigg\{ \sum_{n=2}^{n_0} \sum_{i=1}^{n} \Bigg(-\frac{1}{n!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots$$

$$\ldots \frac{g}{V} \sum_{1=j\neq i} v_{k_i} v_{k_j} f_1^0(k_j) a_{k_i}^+ a_{-k_i}^+ \ldots \sum_{k_n} f_1^0(k_n) a_{k_n}^+ a_{-k_n}^+ |0\rangle \Bigg) +$$

$$+ \sum_{n=1}^{n_0} \Bigg\{ \frac{g}{V} \left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{l} v_{p_i} \right) \left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{l} v_{p'_i} \right) -$$

$$-C_M \left[\left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{l} v_{p_i} \right) + \left(\sum_{i=1}^{n} v_{k_i} + \sum_{i=1}^{l} v_{p'_i} \right) \right] \Bigg\} \times$$

$$\times \frac{1}{n!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_n} f_1^0(k_n) a_{k_n}^+ a_{-k_n}^+ |0\rangle +$$

$$+ \sum_{n=1}^{n_0-1} C_B \sum_{k=k_1,\ldots,k=k_n} v_k f_1^0(k) \frac{1}{n!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots$$

$$\ldots \sum_{k_n} f_1^0(k_n) a_{k_n}^+ a_{-k_n}^+ |0\rangle + g^{-1} C_B^2 V \frac{1}{n_0!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots$$

$$\ldots \sum_{k_{n_0}} f_1^0(k_{n_0}) a_{k_{n_0}}^+ a_{-k_{n_0}}^+ |0\rangle -$$

$$-C_B \sum_{i=1}^{n_0+1} \frac{1}{(n_0+1)!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots$$

$$\ldots \sum_{k_i} v_{k_i} a_{k_i}^+ a_{-k_i}^+ \ldots \sum_{k_{n_0+1}} f_1^0(k_{n_0+1}) a_{k_{n_0+1}}^+ a_{-k_{n_0+1}}^+ |0\rangle \Bigg\} =$$

$$= B\Phi_{(p)_l,(p')_l} - \mathcal{E}_M \Phi_{(p)_l,(p')_l} + C_B B_1 \Phi_{(p)_l,(p')_l} + a_{p_1}^+ \ldots a_{p_l}^+ a_{-p'_1}^+ \ldots a_{-p'_l}^+ \times$$

$$\times \Bigg\{ \frac{g^{-1} C_B^2 V}{n_0!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_{n_0}} f_1^0(k_{n_0}) a_{k_{n_0}}^+ a_{-k_{n_0}}^+ |0\rangle -$$

$$-C_B \sum_{i=1}^{n_0+1} \frac{1}{(n_0+1)!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_i} v_{k_i} a_{k_i}^+ a_{-k_i}^+ \ldots$$

$$\ldots \sum_{k_{n_0+1}} f_1^0(k_{n_0+1}) a_{k_{n_0+1}}^+ a_{-k_{n_0+1}}^+ |0\rangle \Bigg\}. \tag{13.4}$$

The operators B and $\mathcal{E}_M$ were defined according to (12.4), (12.5) and (12.16), (12.17) respectively and they are equal to the first and second terms in (13.4), the operator B_1 is defined by the third term in (13.4).

Now estimate the following average

$$\lim_{V\to\infty} \frac{1}{V} (\Phi_{(p)_l,(p')_l}, (H_\Lambda - H_{a,\Lambda}) \Phi_{(p)_l,(p')_l})'_V =$$

$$= \lim_{V\to\infty} \frac{1}{V} (\Phi_{(p)_l,(p')_l}, (B - \mathcal{E}_M + C_B B_1)\Phi_{(p)_l,(p')_l})'_V +$$

$$+ \lim_{V\to\infty} \frac{1}{V} \Bigg(\Phi_{(p)_l,(p')_l}, a^+_{p_1} \ldots a^+_{p_l} a^+_{-p'_1} \ldots a^+_{-p'_l} \times$$

$$\times \Bigg(\frac{g^{-1} C_B^2 V}{n_0!} \sum_{k_1} f_1^0(k_1) a^+_{k_1} a^+_{-k_1} \ldots \sum_{k_{n_0}} f_1^0(k_{n_0}) a^+_{k_{n_0}} a^+_{-k_{n_0}} |0\rangle -$$

$$-C_B \sum_{i=1}^{n_0+1} \frac{1}{(n_0+1)!} \sum_{k_1} f_1^0(k_1) a^+_{k_1} a^+_{-k_1} \ldots \sum_{k_i} v_{k_i} a^+_{k_i} a^+_{-k_i} \ldots$$

$$\ldots \sum_{k_{n_0+1}} f_1^0(k_{n_0+1}) a^+_{k_{n_0+1}} a^+_{-k_{n_0+1}} |0\rangle \Bigg) \Bigg)'_V . \tag{13.5}$$

The average (13.5) for the operator B and $\mathcal{E}_M$ were estimated in the Section 12 and according to (12.10) and (12.18) they are equal to zero. The average (13.4) for the operator B_1 is the same as in paper [3], namely

$$\frac{1}{V} \left| (\Phi_{(p)_l,(p')_l}, C_B B_1 \Phi_{(p)_l,(p')_l})'_V \right| \le |C_B| \frac{1}{V} v\alpha f^2 e^\alpha f^2. \tag{13.6}$$

Note that in estimates (12.10), (12.18) and (13.5) we put $f = \sup_k |f_1^0(k)|$.

The modulo of the last terms in (13.4) can be estimated as follows

$$|C_B| \frac{1}{V^{n_0}} \sum_{(k)_{n_0}}{}' |f_1^0(k_1)| \ldots |f_1^0(k_{n_0})| \le |C_B| \frac{1}{V^{n_0}} \frac{N^{n_0}}{n_0!} f^{2n_0} \le |C_B| \frac{\alpha^{n_0}}{n_0!} f^{2n_0} \tag{13.6'}$$

and analogous estimate for the second last term.

According to the definition of the number n_0 it tends to ∞ as $V \to \infty$ and therefore the last expression tends to zero as $V \to \infty$.

Taking into account all the above desribed facts we conclude that

$$\lim_{V\to\infty} \frac{1}{V} (\Phi_{(p)_l,(p')_l}, (H_\Lambda - H_{a,\Lambda})\Phi_{(p)_l,(p')_l})'_V = 0. \tag{13.7}$$

2. *Hamiltonians H_Λ and $H_{a;\Lambda}$ on excited states.* Consider the following excited states of the ground state

$$\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} = a^+_{q_1} \ldots a^+_{q_{m_1}} a_{q'_1} a^+_{-q'_1} \ldots a^+_{q'_{m_2}} a^+_{-q'_{m_2}} \Phi_{(p)_l,(p')_l} \tag{13.8}$$

where $(q)_{m_1} = (q_1, \ldots, q_{m_1})$, $(q')_{m_2} = (q'_1, -q'_1, \ldots, q'_{m_2}, -q'_{m_2})$ belong to D_{II}. In the excited states (13.8) there are m_1 electrons with momenta $(q)_{m_1}$ and m_2 pairs with opposite momenta $(q')_{m_2}$ (all from D_{II}). We do not especially fix spin of electrons with momenta $(q)_{m_1}$, let all be $+1$.

Consider H_Λ on $\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}$. By analogy with (12.4) we obtain

$$H_\Lambda \Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} = \sum_{n=1}^{n_0} \frac{1}{n!} \sum_{k_1,\ldots,k_n} \Bigg\{ \Bigg[\sum_{i=1}^{m_1} \left(\frac{q_i^2}{2m} - \mu \right) + \sum_{i=1}^{m_2} \left(\frac{2{q'}_i^2}{2m} - 2\mu \right) +$$

$$+ \sum_{i=1}^{l} \left(\frac{{p'}_i^2}{2m} - \mu \right) + \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) + \sum_{i=1}^{n} \left(\frac{2k_i^2}{2m} - 2\mu \right) f_n^0(k_1, \ldots, k_n) +$$

$$+ \frac{g}{V} \sum_{i=1}^{n} v_{k_i} v_p f_n^0(k_1, \ldots, \overset{i}{\overline{p}}, \ldots, k_n) - \frac{g}{V} \sum_{1=j\ne i}^{n} v_{k_i} v_{k_j} f_n^0(k_1, \ldots, \overset{i}{\overline{k_j}}, \ldots, k_n) -$$

$$-\frac{g}{V}\sum_{1=j\neq i}^{m_1} v_{k_i}v_{q_j} f_n^0(k_1,\ldots,\overset{i}{q_j},\ldots,k_n) - \frac{g}{V}\sum_{1=j\neq i}^{m_2} v_{k_i}v_{q'_j} f_n^0(k_1,\ldots,\overset{i}{q'_j},\ldots,k_n) +$$

$$+\frac{g}{V}\left(\sum_{i=1}^{m_1} v_{q_i} + \sum_{i=1}^{m_2} v_{q'_i} + \sum_{i=1}^{l} v_{p_i} + \sum_{i=1}^{n} v_{k_i}\right)\times$$

$$\times\left(\sum_{i=1}^{m_2} v_{q'_i} + \sum_{i=1}^{l} v_{p'_i} + \sum_{i=1}^{n} v_{k_i}\right) f_n^0(k_1,\ldots,k_n)\times$$

$$\left.\times a^+_{q_1}\ldots a^+_{q_{m_1}} a^+_{q'_1}a^+_{-q'_1}\ldots a^+_{q'_{m_2}}a^+_{-q'_{m_2}} a^+_{p_1}\ldots a^+_{p_l}a^+_{-p'_1}\ldots a^+_{-p'_l}a^+_{k_1}a^+_{-k_1}\ldots a^+_{k_n}a^+_{-k_n}|0\rangle\right] +$$

$$+\frac{g}{V}\sum_{j=1}^{m_2}\sum_{k_n+1} v_{k_{n+1}}v_{q'_j} f_n^0(k_1,\ldots,k_n) a^+_{q_1}\ldots a^+_{q_{m_1}} a_{q'_1}a^+_{-q'_1} \overset{j}{\vee} a^+_{q'_{m_2}}a^+_{-q_{m_2}}\times$$

$$\left.\times a^+_{p_1}\ldots a^+_{p_l}a^+_{-p'_1}\ldots a^+_{-p'_l}a^+_{k_1}a^+_{-k_1}\ldots a^+_{k_n}a^+_{-k_n}a^+_{k_{n+1}}a^+_{-k_{n+1}}|0\rangle\right\} =$$

$$=\left[\sum_{i=1}^{l}\left(\frac{p_i^2}{2m}-\mu\right)+\sum_{i=1}^{l}\left(\frac{p'^2_i}{2m}-\mu\right)+\sum_{i=1}^{m_1}\left(\frac{q_i^2}{2m}-\mu\right)+\sum_{i=1}^{m_2}\left(\frac{2q'^2_i}{2m}-2\mu\right)+\right.$$

$$\left.+A+B_{(q)_{m_1},(q')_{m_2}}+Q_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}\right]\times$$

$$\times\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} + C_{(q')_{m_2}}\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}, \tag{13.9}$$

$$f_n^0(k_1,\ldots,k_n) = f_n^0(k_1)\ldots f_n^0(k_n) = f_n^0((k)_n).$$

Representation (13.9) is also true for general $f_n(k_1,\ldots,k_n)$ with restrictions imposed in Section 12.

In (13.8) the operator A is defined as before for the state $\Phi_{(p)_l,(p')_l}$, the operator $B_{(q)_{m_1},(q)_{m_2}}$ differs from the operator B by additional terms connected with momenta $(q)_{m_1},(q')_{m_2}$ and it includes all the terms with sign "minus"; the operator $Q_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}$ differs from the operator $Q_{(p)_l,(p')_l}$ by the two additional terms connected with momenta $(q)_{m_1},(q')_{m_2}$; the operator $C_{(q)_{m_2}}$ is defined by the last term in (13.9).

Now proceed to estimate the above defined operators. The operator A acts only on the functions $f_n^0((k)_n)$ exactly as for the state $\Phi_{(p)_l,(p')_l}$.

For the operator $B_{(q)_{m_1},(q)_{m_2}}$ we have the following estimate (see estimate (8.6) in [4])

$$\left(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}, B_{(q)_{m_1},(q)_{m_2}}\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}\right)'_V \le$$

$$\le |g|v^2\frac{m_1+m_2+1}{V}\alpha^2 f^4 e^{\alpha f^2}, \qquad m_1\ge 0, \qquad m_2\ge 0. \tag{13.10}$$

By little modification of estimate (12.11) we obtain the following estimate for the operator $Q_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}$

$$\frac{1}{V}\left|\left(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}, Q_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}\right)\right| \le$$

$$\le\sum_{n=1}^{n_0}\frac{|g|}{V^2}\frac{1}{V^n}\sum_{k_1\neq\ldots\neq k_n}' |f_n^0(k_1,\ldots,k_n)|^2\left|\left(\sum_{i=1}^{m_1} v_{q_i}+\sum_{i=1}^{m_2} v_{q'_i}+\sum_{i=1}^{l} v_{p_i}+\sum_{i=1}^{n} v_{k_i}\right)\times\right.$$

$$\times \left(\sum_{i=1}^{m_2} v_{q'_i} + \sum_{i=1}^{l} v_{p'_i} + \sum_{i=1}^{n} v_{k_i} \right) \Bigg| \leq$$

$$\leq \sum_{n=1}^{n_0} \frac{|g|}{V^2} \frac{1}{V^n} \frac{N_{Il}^n}{n!} v^2 (m_1 + m_2 + l + n)(m_2 + l + n) f^{2n} \leq$$

$$\leq \sum_{n=1}^{n_0} v^2 \frac{|g|}{V^2} \frac{\alpha^n}{n!} (m_1 + m_2 + l + n)(m_2 + l + n) f^{2n} \leq \sum_{n=1}^{n_0} |g| v^2 (\alpha + \alpha_I)^2 \frac{\alpha^n}{n!} f^{2n} \leq$$

$$\leq |g| v^2 (\alpha + \alpha_I)^2 e^{\alpha f^2}. \tag{13.11}$$

Now estimate the operator $C_{(q)_{m_2}}$. We have

$$\left| \left(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}, C_{(q)_{m_2}} \Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} \right)'_V \right| \leq$$

$$\leq \sum_{n=0}^{n_0} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n}' |f_n^0(k_1, \ldots, k_n)|^2 \frac{g}{V} \sum_{i=1}^{m_2} |v_{k_i} v_{k_i}| \leq$$

$$\leq \sum_{n=0}^{n_0} \frac{N^n}{V^n} \frac{1}{n!} f^{2n} \frac{v^2 |g|}{V} m_2 \leq \frac{|g|}{V} m_2 v^2 e^{\alpha f^2}. \tag{13.12}$$

By using representation (13.9) we obtain the following analog of (13.4)

$$\lim_{V \to \infty} \frac{1}{V} \left(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}, (H_\Lambda - H_{a,\Lambda}) \Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} \right)'_V =$$

$$= \lim_{V \to \infty} \frac{1}{V} \Big(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l},$$

$$\left[B_{(q)_{m_1},(q')_{m_2}} + C_{(q)_{m_2}} - \mathcal{E}_M + C_B B_{1(q)_{m_1},(q')_{m_2}} \right] \Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} \Big)'_V +$$

$$+ \lim_{V \to \infty} \frac{1}{V} \left(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}, C_B \sum_{i=1}^{m_2} v_{q'_i} \Phi_{(q)_{m_1},\overset{i}{\check{(q')}}_{m_2},(p)_l,(p')_l} \right) +$$

$$+ \lim_{V \to \infty} \frac{1}{V} \Bigg(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}, a^+_{q_1} \ldots a^+_{q_m} a^+_{q'_1} a^+_{-q'_1} \ldots a^+_{q'_{m_2}} a^+_{-q'_{m_2}} \times$$

$$\times a^+_{p_1} \ldots a^+_{p_l} a^+_{-p'_1} \ldots a^+_{-p'_l} \Bigg(\frac{g^{-1} C_B^2 V}{n_0!} \sum_{k_1} f_1^0(k_1) a^+_{k_1} a^+_{-k_1} \ldots \sum_{k_{n_0}} f_1^0(k_{n_0}) a^+_{k_{n_0}} a^+_{-k_{n_0}} |0\rangle -$$

$$- C_B \sum_{i=1}^{n_0+1} \frac{1}{(n_0+1)!} \sum_{k_1} f_1^0(k_1) a^+_{k_1} a^+_{-k_1} \ldots \sum_{k_i} v_{k_i} a^+_{k_i} a^+_{-k_i} \ldots$$

$$\ldots \sum_{k_{n_0+1}} f_1^0(k_{n_0+1}) a^+_{k_{n_0+1}} a^+_{-k_{n_0+1}} |0\rangle \Bigg) \Bigg)'_V. \tag{13.13}$$

Note that in the operator $\mathcal{E}_M$ and $B_{1(q)_{m_1},(q')_{m_2}}$ the terms connected with the momenta $(q)_{m_1}, (q')_{m_2}$ are added. For example

$$C_B B_{1(q)_{m_1},(q')_{m_2}} \Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} =$$
$$= a^+_{q_1} \ldots a^+_{q_m} a^+_{q'_1} a^+_{-q'_1} \ldots a^+_{q'_{m_2}} a^+_{-q'_{m_2}} a^+_{p_1} \ldots a^+_{p_l} a^+_{-p'_1} \ldots a^+_{-p'_l} \times$$

$$\times \sum_{n=0}^{n_0-1} \sum_{\substack{k=(k)_n \\ k=(q)_{m_1} \\ k=(q')_{m_2}}} v_k f_1^0(k) \frac{1}{n!} \sum_{k_1} f_1^0(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_n} f_1^0(k_n) a_{k_n}^+ a_{-k_n}^+ |0\rangle. \quad (13.14)$$

The following estimate is true

$$\frac{1}{V} \left| \left(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}, C_B B_{1(q)_{m_1},(q')_{m_2}} \Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} \right)_V' \right| \leq$$

$$\leq C_B \frac{1}{V} v(m_1 + m_2 + 1) \alpha f^2 e^{\alpha f^2}. \quad (13.15)$$

The operator $\mathcal{E}_M$ acts on $\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}$ as follows

$$\mathcal{E}_M \Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} =$$

$$= \sum_{n=1}^{n_0} \sum_{k_1,\ldots,k_n} \frac{1}{n!} \left[C_M \left(\sum_{i=1}^{m_1} v_{q_i} + 2 \sum_{i=1}^{m_2} v_{q'_i} + \sum_{i=1}^{l} v_{p_i} + \sum_{i=1}^{l} v_{p'_i} + 2 \sum_{i=1}^{n} v_{k_i} \right) - \right.$$

$$\left. - \frac{g}{V} \left(\sum_{i=1}^{m_1} v_{q_i} + \sum_{i=1}^{m_2} v_{q'_i} + \sum_{i=1}^{l} v_{p_i} + \sum_{i=1}^{n} v_{k_i} \right) \left(\sum_{i=1}^{m_2} v_{q'_i} + \sum_{i=1}^{l} v_{p'_i} + \sum_{i=1}^{n} v_{k_i} \right) \right] \times$$

$$\times f_n^0(k_1, \ldots, k_n) a_{q_1}^+ \ldots a_{q_{m_1}}^+ a_{q'_1}^+ a_{-q'_1}^+ \ldots a_{q'_{m_2}}^+ a_{-q'_{m_2}}^+ a_{p_1}^+ \ldots a_{p_l}^+ a_{-p'_1}^+ \ldots$$

$$\ldots a_{-p'_l}^+ a_{k_1}^+ a_{-k_1}^+ \ldots a_{k_n}^+ a_{-k_n}^+ |0\rangle. \quad (13.16)$$

Use the following identity for the expression in square bracket

$$[\ldots] = C_M v(2m_1 + 2m_2 + 2l + 2n - m_1) - \frac{gv}{2V}(m_2 + l + n) v(2m_1 + 2m_2 + 2l + 2n).$$

Obviously

$$\lim_{V \to \infty} \frac{gv}{2V}(m_2 + l + n) = \lim_{V \to \infty} \frac{gvl}{2V} = C_M.$$

We obtain the following estimate

$$\lim_{V \to \infty} \frac{1}{V} \left| \left(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}, \mathcal{E}_M \Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} \right) \right| \leq$$

$$\leq \lim_{V \to \infty} \sum_{n=1}^{n_0} \left[\frac{\alpha^n}{n!} f^{2n} \left| C_M - \frac{gv}{2V}(m_2 + n + l) \right| \times \right.$$

$$\left. \times \frac{2v(m_1 + m_2 + l + n)}{V} + \frac{|C_M| v}{V} m_1 \frac{\alpha^n}{n!} f^{2n} \right] = 0 \quad (13.17)$$

because the series converges uniformly with respect to V and each term tends to zero as $V \to \infty$.

Note that $\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}$ is orthogonal to $\sum_{i=1}^{m_2} v_{q'_i} \Phi_{(q)_{m_1},(\overset{\downarrow}{q'})_{m_2},(p)_l,(p')_l}$ and their scalar product is equal to zero.

We have the following estimate

$$\frac{1}{V} \left| \left(\sum_{i=1}^{m_2} v_{q'_i} \Phi_{(q)_{m_1},(\overset{\downarrow}{q'})_{m_2},(p)_l,(p')_l}, \sum_{i=1}^{m_2} v_{q'_i} \Phi_{(q)_{m_1},(\overset{\downarrow}{q'})_{m_2},(p)_l,(p')_l} \right)_V' \right| \leq \frac{1}{V} v^2 m_2 e^{\alpha f^2}. \quad (13.18)$$

From representation (13.3), estimates (13.6′), (13.10), (13.12), (13.15)–(13.18) and estimate (13.17) for $\mathcal{E}_M$ it follows that

$$\lim_{V\to\infty}\frac{1}{V}\left(\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l},(H_\Lambda - H_{a,\Lambda})\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}\right)'_V = 0. \quad (13.19)$$

Summarize the above obtained results in the following theorem.

Theorem 13. *The Hamiltonians H_Λ and $H_{a,\Lambda}$ are thermodynamically equivalent on ground and excited states $\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}$, $m_1 \geq 0$, $m_2 \geq 0$ in sense of* (13.7) *and* (13.19).

Now show that the excited states $\Phi_{(q')_{m_2},(p)_l,(p')_l}$ is the asymptotic eigenvector of the renormalized Hamiltonian

$$H_{\Lambda,r} = H_\Lambda - \sum_{i=1}^{l}\left(\frac{p_i^2}{2m} - \mu + vC_M(V)\right)I - \sum_{i=1}^{l}\left(\frac{p_i'^2}{2m} - \mu + vC_M(V)\right)I -$$
$$-\sum_{i=1}^{m_2}\left(\frac{2q_i'^2}{2m} - \mu + vC_M(V)\right)I - \frac{E_0(L)}{2}N \quad (13.20)$$

with eigenvalue equal to zero.

There the operator $C_M(V)$ is defined as in (12.20) but with $C_M^n(V) = \frac{g}{2V}(m_2 + + n + l)$ instead of $\frac{g}{2V}(n+l)$.

It follows from (13.9) that

$$H_{\Lambda,r}\Phi_{(q')_{m_2},(p)_l,(p')_l} = (B_{(q')_{m_2}} + C_{(q')_{m_2}})\Phi_{(q')_{m_2},(p)_l,(p')_l}$$

because $\left(A_r - \frac{E_0(L)}{2}n\right) f_1^0(k_1)\ldots f_1^0(k_n) = 0.$

According to estimates analogous to (13.10) and (13.12) ones

$$\lim_{V\to\infty}\|B_{(q')_{m_2}}\Phi_{(q')_{m_2},(p)_l,(p')_l}\|'_V = 0, \qquad \lim_{V\to\infty}\|C_{(q')_{m_2}}\Phi_{(q')_{m_2},(p)_l,(p')_l}\| = 0.$$

This implies that

$$\lim_{V\to\infty}\|H_{\Lambda,r}\Phi_{(q')_{m_2},(p)_l,(p')_l}\| = 0 \quad (13.21)$$

and, thus, $\Phi_{(q')_{m_2},(p)_l,(p')_l}$ is the asymptotic eigenvector of $H_{\Lambda,r}$ with eigenvalue zero.

We have proved the following theorem.

Theorem 14. *The ground and excited states $\Phi_{(q')_{m_2},(p)_l,(p')_l}$, $m_2 \geq 0$, are the asymptotic, as $V\to\infty$, eigenvectors of the renormalized Hamiltonians $H_{\Lambda,r}$* (13.20).

Note that $H_{\Lambda,r}$ changes together with $\Phi_{(q')_{m_2},(p)_l,(p')_l}$. If one considers the Hamiltonian $H_{\Lambda,r}$ with the following two operators

$$H_{\Lambda,r} = H_\Lambda - \sum_{i=1}^{l}\left(\frac{p_i^2}{2m} - \mu + vC_M(V)\right)I - \sum_{i=1}^{l}\left(\frac{p_i'^2}{2m} - \mu + vC_M(V)\right)I \quad (13.22)$$

then $\Phi^n_{(q)_{m_2},(p)_l,(p')_l}$ is the asymptotic, as $V\to\infty$, eigenvector of the Hamiltonian (13.22) with eigenvalue

$$nE_0 + 2\mathcal{E}q_1' + \ldots + 2\mathcal{E}q_{m_2}', \qquad \mathcal{E}_q = \frac{q^2}{2m} - \mu + vC_M.$$

We are not able to prove the theorem for excited states $\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}$ with $m_1 > 0$ electrons with momenta $(q)_{m_1}$ because in this case (see (13.16))

$$\left[C_M^n(V)\sum_k v_k a_k^+ a_k - \frac{g}{V}\sum_{k,k'} v_k v_k' a_k^+ a_k a_{-k'}^+ a_{-k'}\right]\Phi^n_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l} =$$
$$= C_M^n(V) m_1 \Phi^n_{(q)_{m_1},(q)_{m_2},(p)_l,(p')_l}$$

and the last term does not tend to zero as $V \to \infty$. Note that the theorem is still true if one considers the states with m_1 electrons with spin 1 and m_1 electrons with spin -1 and impose on their momenta in D_{II} the same conditions as for $(p)_l$, $(p')_l$, but $2m_1$ electrons do not exhaust the domain D_{II}.

14. Hilbert space of excited states. ***1. Ground state of the model Hamiltonian for quasiparticles.*** Consider the following operators

$$\alpha_k = u_k a_k + w_k a_{-k}^+, \qquad \alpha_k^+ = u_k a_k^+ + w_k a_{-k},$$
$$\alpha_{-k} = u_k a_{-k} - w_k a_k^+, \qquad \alpha_{-k}^+ = u_k a_{-k}^+ - w_k a_k, \tag{14.1}$$
$$u_k = \frac{1}{\sqrt{1+(f_1^0(k))^2}}, \qquad w_k = \frac{-f_1^0(k)}{\sqrt{1+(f_1^0(k))^2}}, \quad k \in D_{II}.$$

The ground state $\Phi_{(p)_l,(p')_l}$ (13.2), but with summation with respect to n from 0 to ∞ (in this section we will consider only the such ground state), is the vacuum for the operators α_k, α_{-k}. One can check that

$$\alpha_k \Phi_{(p)_l,(p')_l} = 0, \qquad \alpha_{-k}\Phi_{(p)_l,(p')_l} = 0, \quad k \in D_{II}. \tag{14.2}$$

As known the function $f_1^0(k)$ has support in D_{II} and $u_k = 1, w_k = 0, k \in D_I$, and the operators (1.3) reduce to the operators $a_k, a_k^+, a_{-k}, a_{-k}^+$ for $k \in D_I$.

Consider the states

$$\varphi_{(q)_{m_1},(q')_{m_2}} = \alpha_{q_1}^+ \ldots \alpha_{q_{m_1}}^+ \alpha_{q_1'}^+ \alpha_{-q_1'}^+ \ldots \alpha_{q_{m_2}'}^+ \alpha_{-q_{m_2}'}^+ \Phi_{(p)_l,(p')_l} \tag{14.3}$$

that is excited states of the ground state $\Phi_{(p)_l,(p')_l}$ with m_1 quasiparticles and m_2 pairs of quasiparticles. By direct calculation one can show that the states $\varphi_{(q)_{m_1},(q')_{m_2}}$ are linear combination of the states $\Phi_{(q)_{m_i},(q')_{m_j},(p)_l,(p')_l}$, $0 \le m_i \le m_1, 0 \le m_j \le m_2$ (see (10.8), (10.9) from [4]). Therefore, by using (13.19) we obtain

$$\lim_{V\to\infty}\frac{1}{V}\left(\varphi_{(q)_{m_1},(q')_{m_2}}, (H_\Lambda - H_{a,\Lambda})\varphi_{(q)_{m_1},(q')_{m_2}}\right)_V' = 0. \tag{14.4}$$

States $\varphi_{(q)_{m_1},(q')_{m_2}}$ are orthogonal for different q, q', m_1, m_2 and they constitute a base of Hilbert spase $\mathcal{H}_V^P \times \mathcal{H}_F$.

It follows from (14.4) *that*

$$\lim_{V\to\infty}\frac{1}{V}(f, (H_\Lambda - H_{a,\Lambda})f)_V' = 0 \tag{14.5}$$

for arbitrary f *that are finite linear combination of* $\varphi_{(q)_{m_1},(q')_{m_2}}$.

2. Ground state of the approximating Hamiltonian. Consider the approximating Hamiltonian

$$H_{a,\Lambda} = \sum_k a_k^+ a_k\left(\frac{k^2}{2m} - \mu + vC_M\right) +$$
$$+ C_B\sum_k v_k a_k^+ a_{-k}^+ + C_B\sum_k v_k a_{-k} a_k - g^{-1}C_B^2 V.$$

It can be diagonalized by using linear canonical transformation

$$\begin{gathered}\alpha_k = u_k a_k + w_k a^+_{-k}, \qquad \alpha^+_k = u_k a^+_k + w_k a_{-k}, \\ \alpha_{-k} = u_k a_k - w_k a^+_k, \qquad \alpha^+_{-k} = u_k a^+_{-k} - w_k a_k, \\ u_k = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{\mathcal{E}_k}{\sqrt{\mathcal{E}_k^2 + C_B^2 v_k^2}}}, \qquad w_k = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\mathcal{E}_k}{\sqrt{\mathcal{E}_k^2 + C_B^2 v_k^2}}}, \\ \mathcal{E}_k = \left(\frac{k^2}{2m} - \mu + vC_M\right), \qquad k \in D_{II}.\end{gathered} \tag{14.6}$$

If $k \in D_I, u_k = 1, w_k = 0$.

The spectra of the diagonalized approximating Hamiltonian

$$\begin{gathered}H_{a,\Lambda} = \sum_{\bar{k}} E_k \alpha^+_{\bar{k}} \alpha_{\bar{k}} + V\left\{\frac{1}{V}\sum_k \left[\mathcal{E}_k - \sqrt{\mathcal{E}_k^2 + C_B^2 v_k^2}\right] - g^{-1}C_B^2\right\} = \\ = \sum_{\bar{k}} E_k \alpha^+_{\bar{k}} \alpha_{\bar{k}} + VC_V, \\ E_k = \sqrt{\mathcal{E}_k^2 + C_B^2 v_k^2}, \qquad E_k = \mathcal{E}_k, \quad k \subset D_{II},\end{gathered} \tag{14.7}$$

can be determined exactly. Namely the state

$$\begin{gathered}\Phi^a_{(p)_l,(p')_l} = a^+_{p_1} \ldots a^+_{p_l} a^+_{-p'_1} \ldots a^+_{-p'_l} \times \\ \times \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1} f_1^a(k_1) a^+_{k_1} a^+_{-k_1} \ldots \sum_{k_n} f_1^a(k_n) a^+_{k_n} a^+_{-k_n} |0\rangle, \\ f_1^a(k) = \frac{-w_k}{u_k}\end{gathered} \tag{14.8}$$

is the vacuum for operators (6.3) because

$$\alpha_k \Phi^a_{(p)_l,(p')_l} = 0, \qquad \alpha_{-k} \Phi^a_{(p)_l,(p')_l} = 0.$$

Note that the summation with respect to n in $\Phi^a_{(p)_l,(p')_l}$ (14.8) as well as in $\Phi_{(p)_l,(p')_l}$ (14.2) is carried out from 0 to ∞. It does not change the previons estimates (13.5)–(13.7) and (13.10)–(13.19) because for $\Phi_{(p)_l,(p')_l}$ and $\Phi_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}$

$$\begin{gathered}\lim_{V\to\infty} \frac{1}{V}\frac{|g|}{V} \sum_{n=n_0+1}^{\infty} \alpha^n f^{2n} \frac{v^2(n+l)^2}{n!} \le \lim_{V\to\infty} |g| v^2 \sum_{n=n_0+1}^{\infty} \frac{\alpha^n f^{2n}}{n!}(\alpha_0 + \alpha_I)^2 = 0, \\ \lim_{V\to\infty} \frac{1}{V}\frac{|g|}{V} \sum_{n=n_0+1}^{\infty} \frac{\alpha^n f^{2n} v^2 (m_1 + m_2 + n + l)^2}{n!} \le \\ \le \lim_{V\to\infty} |g| v^2 \sum_{n=n_0+1}^{\infty} \frac{\alpha^n f^{2n}}{n!}(\alpha + \alpha_I)^2 = 0.\end{gathered} \tag{14.9}$$

Recall that $\lim_{V\to\infty} n_0 = \infty$.

(The contributions from $\Phi^n_{(p)_l,(p')_l}$, $\Phi^n_{(q)_{m_1},(q')_{m_2},(p)_l,(p')_l}$, $n > n_0$ in estimates (13.5)–(13.7) and correspondingly (13.10)–(13.19) tend to zero as $V \to \infty$. We could also use $\Phi_{(p)_l,(p')_l}$ with $0 \le n < \infty$ in the previous sections. We used $0 \le n \le n_0$ only for the save of simplicity.)

The function $f_1^a(k) = \frac{-w_k}{u_k}$ is bounded uniformly with respect to V and estimate (14.9) also holds for $\Phi^a_{(p)_l,(p')_l}$ with corresponding $f = \sup_k |f_1^a(k)|$. Consider the states

$$\varphi^a_{(q)_{m_1},(q')_{m_2}} = \alpha^+_{q_1} \ldots \alpha^+_{q_{m_1}} \alpha^+_{q'_1} \alpha^+_{-q'_1} \ldots \alpha^+_{q'_{m_2}} \alpha^+_{-q'_{m_2}} \Phi^a_{(p)_l,(p')_l}. \tag{14.10}$$

They are excited states of the ground state $\Phi^a_{(p)_l,(p')_l}$ with m_1 quasiparticles and m_2 pairs of quasiparticles. Repeating calculation fulfilled for $\varphi_{(q)_{m_1},(q')_{m_2}}$ one can proof that

$$\lim_{V\to\infty} \frac{1}{V} \left(\varphi^a_{(q)_{m_1},(q')_{m_2}}, (H_\Lambda - H_{a,\Lambda})\varphi^a_{(q)_{m_1},(q')_{m_2}}\right)'_V = 0,$$
$$\lim_{V\to\infty} \frac{1}{V} (f, (H_\Lambda - H_{a,\Lambda})f)'_V = 0 \tag{14.11}$$

where f are arbitrary finite linear combination of $\varphi^a_{(q)_{m_1},(q')_{m_2}}$.

The states $\varphi^a_{(q)_{m_1},(q')_{m_2}}$ are eigenvectors of $H_{a,\Lambda}$ with eigenvalues $VC_V + E_{q_1} + + E_{q_{m_1}} + 2E_{q'_1} + \ldots + 2E_{q'_{m_2}} + \mathcal{E}_{p_1} + \ldots + \mathcal{E}_{p_l} + \mathcal{E}_{p'_1} + \ldots + \mathcal{E}_{p'_l}$, i.e.,

$$H_{a,\Lambda}\varphi^a_{(q)_{m_1},(q')_{m_2}} = (VC_V + E_{q_1} + E_{q_{m_1}} +$$
$$+2E_{q'_1} + \ldots + 2E_{q_{m_2}} + \mathcal{E}_{p_1} + \ldots + \mathcal{E}_{p_l} + \mathcal{E}_{p'_1} + \ldots + \mathcal{E}_{p'_l})\varphi^a_{(q)_{m_1},(q')_{m_2}}. \tag{14.12}$$

According to (14.11) the states $\varphi^a_{(q')_{m_2}}$ are also eigenvectors of the model operator H_Λ in the thermodynamic limit (see detail in [4], Section 11).

Thus the model Hamiltonian H_Λ *has two asymptotic as* $V \to \infty$ *systems of eigenvectors* 1) $\Phi^n_{(q')_{m_2},(p)_l,(p')_l}$ *with the eigenvalues* $nE_0 + 2\mathcal{E}_{q'_1} + \ldots + 2\mathcal{E}_{q_{m_2}} + \mathcal{E}_{p_1} + \ldots \ldots + \mathcal{E}_{p_l} + \mathcal{E}_{p'_1} + \ldots + \mathcal{E}_{p'_l}$ *and* 2) $\varphi^a_{(q')_{m_2}}$ *with the eigenvalues* $VC_V + + 2E_{q'_1} + \ldots + 2E_{q'_{m_2}} + \mathcal{E}_{p_1} + \ldots + \mathcal{E}_{p_l} + \mathcal{E}_{p'_1} + \ldots + \mathcal{E}_{p'_l}$ *(in sense of* (14.11)).

Note that parameter ω_0 from definition of the domains D_I, D_{II} should be defined from the condition of minimum of energy of ground states of the operators H_Λ or $H_{a,\Lambda}$.

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