UDC 517.9 + 531.19

D. Ya. Petrina (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

SPECTRUM AND STATES OF THE BCS HAMILTONIAN IN FINITE DOMAIN. II. SPECTRA OF EXCITATIONS*

СПЕКТР ТА СТАНИ ГАМІЛЬТОНІАНА БКШ В СКІНЧЕННІЙ ОБЛАСТІ. II. СПЕКТРИ ЗБУДЖЕНЬ

Coincidence of averages per volume of the BCS and approximating Hamiltonians over all the excited states is established in the thermodynamic limit. Earlier it has been established only for the ground state.

Встановлено, що у термодинамічній границі середні по усіх збуджених станах на одиницю об'єму від модельного гамільтоніана БКШ та відповідного апроксимуючого гамільтоніана збігаються. Раніше це було встановлено тільки для основного стану.

Introduction. We conceived a series of papers devoted to investigation of spectral properties of BCS Hamiltonian in finite domain and corresponding states. In the first paper [1] we have investigated spectra of the BCS Hamiltonian in a finite cube Λ with periodic boundary conditions. It has been proved that in certain subspace of pairs the BCS Hamiltonian can be represented as a sum of two operators A and B. The operator A describes the spectra of noninteracting pairs, and the operator B describes the interaction between pairs and tends to zero as volume $V(\Lambda) = V$ of the cube Λ tends to infinity. The pairs may be in the ground or excited states with corresponding eigenvalues. The complete description of spectra of the BCS Hamiltonian in the subspace of pairs has been established.

It has also been proved that the average energies per volume V of the BCS Hamiltonian H_Λ and the approximating Hamiltonian $H_{a,\Lambda}$ over the ground state Φ_0 of H_Λ coincide in the thermodynamic limit

$$\lim_{V\to\infty} \frac{1}{V} (\Phi_0, (H_\Lambda - H_{a,\Lambda})\Phi_0)'_V = 0. \tag{1}$$

Given paper is a direct continuation of the previous one and is devoted to the proof of the thermodynamic equivalence of the BCS Hamiltonian H_Λ and the approximating Hamiltonian $H_{a,\Lambda}$. Namely, we prove that

$$\lim_{V\to\infty} \frac{1}{V} \left(\Phi_{(p)_l,(q)_m}, (H_\Lambda - H_{a,\Lambda})\, \Phi_{(p)_l,(q)_m}\right)'_V = 0, \tag{2}$$

where

$$\Phi_{(p)_l,(q)_m} = a^+_{p_1} \dots a^+_{p_l} a^+_{q_1} a^+_{-q_1} \dots a^+_{q_m} a^+_{-q_m} \Phi_0, \qquad l \geq 0, \quad m \geq 0, \quad m + l \geq 1, \tag{3}$$

is the excited state of the ground state Φ_0 of H_Λ. (We use the same denotation as in the previous paper [1].)

It follows from (1), (2) that

$$\lim_{V\to\infty} \frac{1}{V} (f, (H_\Lambda - H_{a,\Lambda})f)'_V \tag{4}$$

for arbitrary f from certain Hilbert space which elements are linear combinations of vectors $\Phi_{(p)_l,(q)_m}$ (2).

* Financial support of the DFG through SFB 256, Bonn University and project No. 004 UKR 99/L2, No. 436 UKR 113/43 is gratefully acknowledged.

We also proved that

$$\lim_{V\to\infty} \frac{1}{V}\left(\Phi_0^a, (H_\Lambda - H_{a,\Lambda})\Phi_0^a\right)'_V = 0,$$
$$\lim_{V\to\infty} \frac{1}{V}\left(\varphi^a_{(p)_l,(q)_m}, (H_\Lambda - H_{a,\Lambda})\varphi^a_{(p)_l,(q)_m}\right)'_V, \tag{5}$$
$$\varphi^a_{(p)_l,(q)_m}) = \alpha^+_{\overline{p}_1}\ldots\alpha^+_{\overline{p}_l}\alpha^+_{q_1}\alpha^+_{-q_1}\ldots\alpha^+_{q_m}\alpha^+_{-q_m}\Phi_0^a,$$

where Φ_0^a is the ground state of the approximating Hamiltonian $H_{a,\Lambda}$. The operators $\alpha^+_{\overline{k}}$, $\alpha_{\overline{k}}$ are obtained by the canonical transformation of the operator $a^+_{\overline{k}}$, $a_{\overline{k}}$ and Φ_0^a is their vacuum, $\alpha_{\overline{k}}\Phi_0^a = 0$. The first equality (5) had been established by Bogolyubov [2]. Equality (4) holds also for arbitrary f from certain Hilbert space which elements are linear combination of vectors $\varphi^a_{(p)_l,(q)_m}$.

Equalities (1) – (5) mean that the Hamiltonians H_Λ and $H_{a,\Lambda}$ are thermodynamic equivalent.

We introduced a new conception of the thermodynamic equivalence on ground and excited states (1) – (5).

By using the canonical Bogolyubov's transformation, the approximating Hamiltonian $H_{a,\Lambda}$ can be diagonalized and its spectrum can be determined exactly. Namely, the states $\varphi^a_{(p)_l,(q)_m}$ are the eigenvectors of $H_{a,\Lambda}$ with eigenvalues

$$\sum_{i=1}^{l} E_{p_i} + 2\sum_{j=1}^{m} E_{q_j} + E_0^a, \tag{6}$$

where $E_k = \sqrt{\varepsilon_k^2 + c^2 v_k^2}$ is the energy of the one-particle excitation ($\varepsilon_p = p^2/2m - \mu$), E_0^a is the energy of the ground state Φ_0^a.

The spectrum of the BCS Hamiltonian H_Λ can be determined asymptotically exactly. Namely, consider the renormalized Hamiltonian $H_\Lambda^{ren} = H_\Lambda - E_0 N/2$ where N is the operator of number of particles and E_0 is the lowest eigenvalue of pair with eigenfunction $f_1^0(k)$. Then

$$H_\Lambda^{ren}\Phi_{(p)_l,(q)_m} = \left(\sum_{i=1}^{n}\left(\varepsilon_{p_i} - \frac{E_0}{2}\right) + 2\sum_{j=1}^{m}\left(\varepsilon_{q_j} - \frac{E_0}{2}\right)\right)\Phi_{(p)_l,(q)_m} +$$
$$+ \left(B_{(p)_l,(q)_m} + C_{(p)_l,(q)_m}\right)\Phi_{(p)_l,(q)_m}, \tag{7}$$

where

$$\lim_{V\to\infty}\left\|\left(B_{(p)_l,(q)_m} + C_{(p)_l,(q)_m}\right)\Phi_{(p)_l,(q)_m}\right\|'_V = 0.$$

It follows from (7) that

$$H_\Lambda\Phi^n_{(p)_l,(q)_m} = \left(nE_0 + \sum_{i=1}^{l}\varepsilon_{p_i} + 2\sum_{j=1}^{m}\varepsilon_{q_i}\right)\Phi^n_{(p)_l,(q)_m} +$$
$$+ \left(B_{(p)_l,(q)_m} + C_{(p)_l,(q)_m}\right)\Phi^n_{(p)_l,(q)_m}, \tag{8}$$
$$\Phi^n_{(p)_l,(q)_m} = a^+_{p_1}\ldots a^+_{p_l}a^+_{q_1}a^+_{-q_1}\ldots a^+_{q_m}a^+_{-q_m}\Phi_0^n,$$

where

$$\lim_{V\to\infty}\left\|\left(B_{(p)_l,(q)_m} + C_{(p)_l,(q)_m}\right)\Phi^n_{(p)_l,(q)_m}\right\|'_V = 0.$$

Formulae (6), (7) determine asymptotically exactly (as $V \to \infty$) spectra of the BCS Hamiltonian that correspond to ground state of fixed number n of noninteracting pairs and their excitations.

It will be shown in the third paper of the given series that the spectra (6) of the approximating Hamiltonian $H_{a,\Lambda}$ describe excitations of the single entire system while the spectra (8) describe excitations of $(l+2m+2n)$-particle subsystem. It will be especially clear for nonzero temperature because in this case the energy of excitation E_k depends on temperature.

Thus there are two ground states Φ_0 and Φ_0^a and their excitations $\Phi_{(p)_l,(q)_m}$ (2) and $\Phi^a_{(p)_l,(q)_m}$ (5) and two kind of the spectra: those of the BCS Hamiltonian H_Λ (7), (8) and those of the approximating Hamiltonian $H_{a,\Lambda}$ (6). The Hilbert spaces that are linear combinations of two different systems of vectors $\Phi_{(p)_l,(q)_m}$ and $\Phi^a_{(p)_l,(q)_m}$, respectively, are unitary nonequivalent in the thermodynamic limit. These facts had been unknown before.

VIII. Hamiltonian BCS and its spectra on excited states. *1. H_Λ on exited states.* Denote by $f_{(p)_l,(q)_m} = f_{p_1,\ldots,p_l;\, q_1,\ldots,q_m}$ the following state*:

$$f_{(p)_l,(q)_m} = a^+_{p_1}\ldots a^+_{p_l} a^+_{q_1} a^+_{-q_1}\ldots a^+_{q_m} a^+_{-q_m} f =$$
$$= a^+_{p_1}\ldots a^+_{p_l} a^+_{q_1} a^+_{-q_1}\ldots a^+_{q_m} a^+_{-q_m} \times$$
$$\times \sum_{n=0}^{\infty} \sum_{k_1\neq\ldots\neq k_n} f^s_n(k_1,\ldots,k_n) a^+_{k_1} a^+_{-k_1}\ldots a^+_{k_n} a^+_{-k_n} \mid 0\rangle, \tag{8.1}$$

where f was defined above by (1.10), $f \in \mathcal{H}^P_V$. We will say that $f_{(p)_l}$ is excited state with momenta $p_1,\ldots,p_l$ of l particles and momenta $(q_1,-q_1),\ldots,(q_m,-q_m)$ of m pairs. We suppose that any two momenta (p_i,p_j) do not coincide with some pairs of momenta from the sets $(k_1,-k_1,\ldots,k_n,-k_n) = (k)_n$, $n = 1,2,\ldots$, $k_i \subset D$ or from the set $(q_1,-q_1,\ldots,q_m,-q_m) = (q)_m$, $q_j \subset D$, but some p_i can coincide with some momenta from sets $(k)_n$. If some momenta from $(p)_l$ coincide with some momenta from $(q)_m$ then $f_{(p)_l,(q)_l} = 0$.

We consider $f_{(p)_l,(q)_m}$ as an element of $\mathcal{H}^F$ with respect to momenta $(p)_l$, $(q)_m$ and as an element of $\mathcal{H}^P_V$ with respect to momenta $(k)_n$, $n = 1,2,\ldots$, i.e. $f_{(p)_l,(q)_m} \in \in \mathcal{H}^F \otimes \mathcal{H}^P_V$.

We will use the denotation $(f_{(p)_l,(q)_m}, g_{(p)_l,(q)_m})'_V$ for scalar product of two elements $f_{(p)_l,(q)_m}$ and $g_{(p)_l,(q)_m}$ from $\mathcal{H}^F \otimes \mathcal{H}^P_V$.

The scalar product of two sequences $f_{(p)_l,(q)_m}$ and $g_{(p)_l,(q)_m}$ are equal to (1.11), but these momenta from the sets $(k)_n$ that coincide with some momenta from the set $(p)_l$ should be omitted in it

$$(f_{(p)_l,(q)_m}, g_{(p)_l,(q)_m})'_V = \sum_{n=0}^{\infty} \frac{1}{V^n} \sum'_{(k)_n\neq(p)_l\neq(q)_m} \overline{f^s_n(k_1,\ldots,k_n)} g^s_n(k_1,\ldots,k_n) =$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{V^n} \sum_{(k)_n\neq(p)_l\neq(q)_l} \overline{f^s_n(k_1,\ldots,k_n)} g^s_n(k_1,\ldots,k_n). \tag{8.2}$$

The states $f_{(p)_l,(q)_m}$, $g_{(p)_l,(q)_m}$ belong to $\mathcal{H}^F \otimes \mathcal{H}^P_V$.

Consider the action of H_Λ on $f_{(p)_l,(q)_m}$. By analogy with (2.5) we obtain

* For the sake of simplicity we consider only operators a^+_p with spins $+1$.

$$H_\Lambda a^+_{p_1}\dots a^+_{p_l}\, a^+_{q_1}\, a^+_{-q_1}\dots a^+_{q_m}\, a^+_{-q_m} \sum_{k_1,\dots,k_n} f_n(k_1,\dots,k_n)\, a^+_{k_1} a^+_{-k_1}\dots a^+_{k_n} a^+_{-k_n}|0\rangle =$$

$$= \sum_{k_1,\dots,k_n}\Bigg\{\Bigg[\sum_{i=1}^{l}\left(\frac{p_i^2}{2m}-\mu\right)+\sum_{i=1}^{m}\left(\frac{2q_i^2}{2m}-2\mu\right)+\sum_{i=1}^{m}\left(\frac{k_i^2}{2m}-2\mu\right)\Bigg]f_n(k_1,\dots,k_n)+$$

$$+\frac{g}{V}\sum_{i=1}^{n}\Bigg[\sum_{p}Vk_i,p f_n\Big(k_1,\dots,\overset{i}{\overline{p}},\dots,k_n\Big)-\sum_{1=j\neq i}^{n}Vk_i,k_j f_n\Big(k_1,\dots,\overset{i}{\overline{k_j}},\dots,k_n\Big)-$$

$$-\sum_{j=1}^{l}Vk_i,p_j f_n\left(k_1,\dots,\overset{i}{\overline{p_j}},\dots,k_n\right)-\sum_{j=1}^{m}Vk_i,q_j f_n\left(k_1,\dots,\overset{i}{\overline{q_j}},\dots,k_n\right)\Bigg]\times$$

$$\times a^+_{p_1}\dots a^+_{p_l}\, a^+_{q_1}\, a^+_{-q_1}\dots a^+_{q_m}\, a^+_{-q_m} a^+_{k_1} a^+_{-k_1}\dots a^+_{k_n} a^+_{-k_n}\,|\,0\rangle+$$

$$+\frac{g}{V}\sum_{j=1}^{m}\sum_{k_{n+1}}Vk_{n+1},q_j f_n(k_1,\dots,k_n)\, a^+_{p_1}\dots a^+_{p_l}\, a^+_{q_1}\, a^+_{-q_1}\overset{j}{\overset{\vee}{\dots}}\, a^+_{q_m}\, a^+_{-q_m} a^+_{k_1} a^+_{-k_1}\dots$$

$$\dots a^+_{k_n} a^+_{-k_n} a^+_{k_{n+1}} a^+_{-k_{n+1}}\,|\,0\rangle\Bigg\}. \tag{8.3}$$

This yields

$$H_\Lambda f_{(p)_l,(q)_m}=\left[\sum_{i=1}^{l}\left(\frac{p_i^2}{2m}-\mu\right)+\sum_{i=1}^{m}\left(\frac{2q_j^2}{2m}-2\mu\right)\right]f_{(p)_l,(q)_m}+$$
$$+\left(A+B_{(p)_l,(q)_m}\right)f_{(p)_l,(q)_m}+C_{(p)_l,(q)_m}\,f_{(p)_l,(q)_m}. \tag{8.4}$$

The operator A is defined by the third and fourth terms, the operator $B_{(p)_l,(q)_m}$ is defined by the fifth–seventh terms and the operator $C_{(p)_l,(q)_m}$ is defined by the last eighth term in (8.3). The operator $B_{(p)_l,(q)_m}$ in (8.3), (8.4) differs from the operator B (2.7) by the additional terms in (8.3) connected with the momenta $(p)_l$

$$-\frac{g}{V}\sum_{i=1}^{n}\left[\sum_{j=1}^{l}Vk_i,p_j f_n\Big(k_1,\dots,\overset{i}{\overline{p_j}},\dots,k_n\Big)+\sum_{j=1}^{m}Vk_i,q_j f_n\Big(k_1,\dots,\overset{i}{\overline{q_j}},\dots,k_n\Big)\right].$$

If some $p_j\not\subset D$ then $Vk_i,p_j=0$ and this term is absent in (8.3), (8.4). Note that the terms with some $k_i=k_j$, $k_i=p_j$, or $k_i=q_j$ are equal to zero.

2. ***Estimate of the operator*** $\boldsymbol{H_\Lambda - A-\sum_{i=1}^{l}\left(\frac{p_i^2}{2m}-2\mu\right)I-\sum_{i=1}^{m}\left(\frac{2q_j^2}{2m}-2\mu\right)I}$ ***on the excited states*** $\boldsymbol{f_{(p)_l,(q)_m}}$. Consider the exited states $f_{(p)_l,(q)_m}$ where the sequences f satisfy the same conditions as in Theorem 4. The action of Hamiltonian H_Λ on the excited states $f_{(p)_l,(q)_m}$ was defined by formulae (8.3), (8.4) and we have

$$\left[H_\Lambda-A-\sum_{i=1}^{l}\left(\frac{p_i^2}{2m}-2\mu\right)I-\sum_{i=1}^{m}\left(\frac{2q_j^2}{2m}-2\mu\right)I\right]f_{(p)_l,(q)_m}=$$
$$=\left[B_{(p)_l,(q)}+C_{(p)_l,(q)_m}\right]f_{(p)_l,(q)_m}, \tag{8.5}$$

where I is the unit operator.

For the operator $B_{(p)_l,(q)_m}$ we have

$$\left|\left(f_{(p)_l,(q)_m}, B_{(p)_l,(q)_m}, f_{(p)_l,(q)_m}\right)_V'\right| \leq \sum_{n=2}^{N} \frac{|g|}{V} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n}' |f_n(k_1, \ldots, k_n)| \times$$

$$\times \left\{ \sum_{i=1}^{n} \left[\sum_{1=j\neq i}^{n} |Vk_i, k_j| \left| f_n \left(k_1, \ldots, \overset{i}{\overline{k_j}}, \ldots, k_n \right) \right| + \right.\right.$$

$$+ \sum_{j=1}^{l} |Vk_i, p_j| \left| f_n \left(k_1, \ldots, \overset{i}{\overline{p_j}}, \ldots, k_n \right) \right| +$$

$$\left.\left. + \sum_{j=1}^{m} |Vk_i, q_j| \left| f_n \left(k_1, \ldots, \overset{i}{\overline{q_j}}, \ldots, k_n \right) \right| \right] \right\} \leq$$

$$\leq \frac{|g|\, v^2}{V} \sum_{n=2}^{N} \frac{N^n}{V^n} \frac{n\,(n+l+m-1)}{n!} f^{2n} \leq \frac{|g|\, v^2\,(l+m+1)}{V} \sum_{n=2}^{N} \frac{\alpha^n f^{2n}}{(n-2)!} \leq$$

$$\leq \frac{|g|\, v^2\,(l+m+1)}{V} \alpha^2 f^4 e^{\alpha f^2}, \qquad l \geq 0, \quad m \geq 0. \tag{8.6}$$

Now estimate the operator $C_{(p)_l,(q)_m}$. We have

$$\left|\left(f_{(p)_l,(q)_m},\ C_{(p)_l,(q)_m}\ f_{(p)_l,(q)_m}\right)_V'\right| \leq$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n}' |f_n(k_1, \ldots, k_n)|^2 \frac{|g|}{V} \sum_{j=1}^{m} |Vq_j, q_j| \leq$$

$$\leq \sum_{n=0}^{\infty} \frac{N^n}{V^n} \frac{1}{n!} f^{2n} \frac{v^2 |g| m}{V} = \frac{g}{V} m v^2 e^{\alpha f^2}. \tag{8.7}$$

Further we have

$$\left(\left\|C_{(p)_l,(q)_m}\ f_{(p)_l,(q)_m}\right\|_V'\right)^2 = \left(C_{(p)_l,(q)_m}\ f_{(p)_l,(q)_m}, C_{(p)_l,(q)_m}\ f_{(p)_l,(q)_m}\right)_V' \leq$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n}' \left(\frac{g^2}{V^2} \sum_{i=1}^{m} \sum_{k_{n+1}} |V_{k_{n+1}}, q_i|^2 + \right.$$

$$\left. + \frac{g^2}{V^2} \sum_{i \neq j=1}^{m} |Vq_i, q_i||Vq_j, q_j| \right) |f_n(k_1, \ldots, k_n)|^2 \leq$$

$$\leq \sum_{n=0}^{\infty} \frac{N^n}{V^n n!} f^{2n} \left(\frac{g^2}{V} \alpha v^4 m + \frac{g^2}{V} m^2 v^4 \right) \leq \frac{g^2}{V} e^{\alpha f^2} \left(\alpha v^4 m + \frac{m^2 v^4}{V} \right). \tag{8.8}$$

Obtaining estimates (8.7), (8.8) one subscribes factor $\left(V^{n/2}\right)^{-1}$ to $f_n(k_1, \ldots, k_n)$ and performs a standard calculation with the operators of creation and annihilation. (It is easy to prove that $\left\|B_{(p)_l} f_{(p)_l}\right\|_V'$ tends to zero as $V \to \infty$.)

Thus, averages $\left(f_{(p)_l,(q)_m}, B_{(p)_l,(q)_m} f_{(p)_l,(q)_m}\right)_V'$ and $\left(f_{(p)_l,(q)_m}, C_{(p)_l,(q)} f_{(p)_l,(q)_m}\right)_V'$ tend to zero as $V \to \infty$.

It implies that the following analog of Theorem 4 is true.

Theorem 7. *If the states of pairs* $f = (1, 0, f_1(k_1), \ldots, f_n(k_1, \ldots, k_n), \ldots)$ *satisfy the conditions*

$$|f_n(k_1, \ldots, k_n)| \leq f^n, \qquad f < \infty, \quad n \geq 1,$$

uniformly with respect to V *and have supports in* D^n, *then*

$$\left(f_{(p)_l,(q)_m}, \left[H_\Lambda - A - \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) I - \sum_{i=1}^{m} \left(\frac{2q_i^2}{2m} - 2\mu \right) I \right] f_{(p)_l,(q)_m} \right)'_V \tag{8.9}$$

tends to zero when $V \to \infty$ for arbitrary fixed l, m and the estimates (8.6) – (8.8) *hold.*

Note that the numbers l and m can tend to infinity together with V but in such away that $(l+m)/V$ tends to zero. Remark that $m + l \le N \approx \alpha V$.

***Corollary* 2.** *It is obvious that theorem also holds for the excitations of the ground state Φ_0*

$$\Phi_{(p)_l,(q)_m} = a_{p_1}^+ \dots a_{p_l}^+ \, a_{q_1}^+ \, a_{-q_1}^+ \dots a_{q_m}^+ \, a_{-q_m}^+ \Phi_0$$

because the eigenfunction $f_1^0(k)$ of ground state of one pair is uniformly bounded with respect to V (see Section 5).

3. *Hamiltonian H_Λ on normalized excited states.* Consider a normalized analog of states $f(p)_l$

$$f_l = \sum_{n=0}^{\infty} \sum_{(p)_l,(k)_n} \psi_l((p)_l) \, f_n((k)_n) \, a_{p_1}^+ \dots a_{p_l}^+ a_{k_1}^+ a_{-k_1}^+ \dots a_{k_n}^+ a_{-k_n}^+ |0\rangle =$$

$$= \sum_{(p)_l} \psi_l((p)_l) \, f_{(p)_l}, \quad (p)_l = (p_1, \dots, p_l), \quad (k)_n = (k_1, \dots, k_l), \tag{8.10}$$

where the functions $\psi_l((p)_l)$ is antisymmetric with respect to $(p)_l$ and equal to zero if $p_i = -p_j$ for some $(i,j) \subset (1, \dots, l)$ and the function $f_n((k)_n)$ satisfies the condition of Theorem 7. Linear operations with (8.10) are obvious.

Now introduce the following scalar product of two elements f_l and

$$g_l = \sum_{(p)_l} h_l((p)_l) \, g_{(p)_l}, g_{(p)_l} =$$

$$= \sum_{(p)_l,(k)_n} h_l((p)_l) \, g_n((k)_n) a_{p_1}^+ \dots a_{p_l}^+ a_{k_1}^+ a_{-k_1}^+ \dots a_{k_n}^+ a_{-k_n}^+ |0\rangle, \tag{8.11}$$

$$(f_l, g_l) = \sum_{n=0}^{\infty} \frac{1}{V^{l+n}} \sum_{(p)_l \ne (k)_n}{}' \overline{\psi_l}((p)_l) \, h_l((p)_l) \overline{f_n}((k)_n) \, g_n((k)_n). \tag{8.12}$$

Note that summation in (8.12) is carried out over momenta $p_1 \ne \dots \ne p_l \ne k_1 \ne \dots \dots \ne k_n$ and sets that differ only by permutation are identified. Scalar product (8.12) will be useful for performing the thermodynamic limit.

We introduce the following norm $\|f_l\|'_V = \left((f_l, f_l)'_V \right)^{1/2}$. The Hilbert space with elements f_l, g_l and scalar product (8.12) will be denoted by $\mathcal{H}_V^F \otimes \mathcal{H}_V^P$. The normalized states f_l belong to $\mathcal{H}_V^F \otimes \mathcal{H}_V^P$.

Now consider the Hamiltonian H_Λ on f_l. H follows from (8.4) and (8.10) that

$$H_\Lambda f_l = \sum_{(p)_l} \psi_l((p)_l) \left[\sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) I + A + B_{(p)_l} \right] f_{(p)_l}, \tag{8.13}$$

where the operators A and $B_{(p)_l}$ have been defined by (8.3).

We have the following estimates that are an analog of estimates (8.6)

$$\sum_{(p)_l}' \frac{1}{V^l} \left| \left(\psi_l((p)_l) f_{(p)_l}, \psi_l((p)_l) B_{(p)_l} f_{(p)_l}\right)_V' \right| \leq$$

$$\leq \frac{|g|}{V} \sum_{n=1}^{\infty} \frac{1}{V^{l+n}} \sum_{(p)_l \neq (k)_n}' |\psi_l((p)_l)|^2 \, |f_n((k)_N)| \times$$

$$\times \left\{ \sum_{i=1}^{n} \left[\sum_{1=j\neq i}^{n} |V_{k_i,k_j}| |f_n((k)_n)|_{k_i=k_j} + \sum_{j=1}^{l} |V_{k_i,p_j}| |f_n((k)_n)|_{k_i=p_j} \right] \right\} \leq$$

$$\leq \frac{g}{|V|} \sum_{n=0}^{\infty} \frac{N^{l+n}}{V^{n+l}} \frac{v^2 n(n+l-1)}{(l+n)!} \psi^{2l} f^{2n} \leq$$

$$\leq \frac{|g|v^2}{V} \sum_{n=0}^{\infty} \frac{\alpha^{l+n}}{(l+n-2)!} \chi^{2(l+n)} \leq \frac{|g|v^2}{V} \alpha^2 \chi^4 e^{\alpha \chi^2}, \tag{8.14}$$

$$\psi^l = \sup_{(p)_l} |\psi_l((p)_l)|, \qquad f^n = \sup_{(k)_n} |f_n((k)_n)|, \qquad \chi = \max\{\psi, f\}.$$

The series in (8.14) are convergent and tend to zero as $V \to \infty$ uniformly with respect to numbers l.

Now we are able to prove the following analog of Theorem 7.

Theorem 8. *If states of pairs $f = (1, 0, f_1(k_1), \ldots, f_n((k)_n), \ldots)$ and functions $\psi_l((p)_l)$ satisfy the condltions*

$$|f_n((k)_n)| \leq f^n, \quad f < \infty, \quad n \geq 1, \qquad |\psi_l((p)_l)| < \psi^l, \quad \psi < \infty,$$

uniformly with respect to V and have supports in D with respect to each variables, then

$$\left(f_l, \left[H_\Lambda - A - \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) I \right] f_l \right)_V' =$$

$$= \frac{1}{V^l} \sum_{(p)_l} \psi_l((p)_l) \left(f_{(p)_l}, \psi_l((p)_l) \left[H_\Lambda - A - \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) I \right] f_{(p)_l} \right)_V' \tag{8.15}$$

tends to zero as $V \to \infty$ uniformly with respect to numbers l and estimates (8.12) hold.

Proof follows directly from (8.13), (8.14). Note that, according to (8.14), expressions (8.15) tend to zero as $V \to \infty$ uniformly with respect to l while in Theorem 7 analogical expressions (8.9) for $f_{(p)_l}$ depend on l accordingly to (8.6), (8.7).

***Corollary* 3.** *It is obvious that Theorem* 8 *also holds for normalized excitations of the ground state Φ_0*

$$\Phi_l = \sum_{(p)_l} \psi_l((p)_l) \Phi_{(p)_l} \tag{8.16}$$

because the eigenfunctions $f_1^0(k)$ of the pair with lowest eigenvalue E_0 is uniformly bounded with respect to V (see Section 5*) and we suppose that functions $\psi_l(p)_l$ satisfy the conditions of Theorem* 8.

Also consider the following states

$$\Phi_{lm} = \sum_{(p)_l, (q)_m} \psi_l((p)_l) \chi_m((q)_m) a_{p_1}^+ \ldots a_{p_l}^+ a_{q_1}^+ a_{-q_1}^+ \ldots a_{q_m}^+ a_{-q_m}^+ \Phi_0 \tag{8.17}$$

where functions $\psi_l((p)_l)$ are antisymmetric, $|\psi_l((p)_l)| \leq \psi^l$, and functions $\chi_m((q)_m)$ are symmetric, $|\chi_m((q)_m)| \leq \chi^m$.

Φ_{lm} is a state where l electrons are excited with wave function $\psi_l((p)_l)$, m pairs are excited with wave function $\chi_m((q)_m)$ and the rest of pairs are in ground state with wave function $f_1^0(k)$. Φ_{lm} can be represented as follows

$$\Phi_{lm} = \sum_{(p)_l} \psi_l((p)_l) a^+_{p_1} \ldots a^+_{p_l} \sum_{n=m}^{\infty} \sum_{(k)_n} f_n((k)_n) a^+_{k_1} a^+_{-k_1} \ldots a^+_{k_n} a^+_{-k_n} |0\rangle =$$
$$= \sum_{(p)_l} \psi_l((p)_l) \Phi_{(p)_l} = f_l, \tag{8.18}$$

where

$$f_n((k)_n) = \frac{1}{n(n-1)\ldots(n-m-1)} sym\, [\chi_m((k)_m) f_1^0(k_{m+1}) \ldots f_1(k_n)], \quad n \geq m,$$

and

$$|f_n((k)_n)| \leq \chi^m f^{n-m} \leq (\max\{\chi, f\})^n, \qquad |f_1^0(k)| < f, \quad n \geq m.$$

It is obvious that Theorem 8 holds for the state Φ_{lm} with l excited electrons and m excited pairs (uniformly with respect to l and m) because functions $f_n((k)_n)$ in (8.18) satisfy the conditions of Theorem 8.

Namely, the following theorem is true.

Theorem 8′.

$$\left(\Phi_{lm}, \left[H_\Lambda - A - \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu\right) I\right] \Phi_{lm}\right) =$$
$$= \left(f_l, \left[H_\Lambda - A - \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu\right) I\right] f_l\right) \tag{8.15′}$$

tends to zero as $V \to \infty$ *uniformly with respect to numbers* l *and* m *and estimates* (8.12) *hold* (*with* $f_l, f_n((k)_n)$), $n \geq m$, *determined according to* (8.18).

4. Spectra of renormalized Hamiltonian. Consider the renormalized Hamiltonian

$$H_{\Lambda,r} = H_\Lambda - \frac{E_0}{2} N, \qquad N = \sum_{\overline{p}} a^+_{\overline{p}} a_{\overline{p}}, \tag{8.19}$$

where $E_0 = E_0(L)$ is the lowest eigenvalue of one pair. It is obvious that the ground state Φ_0 is the asymptotical (in the limit $\Lambda \nearrow R^3$) eigenstate of $H_{\Lambda,r}$ with eigenvalue equal to zero. It follows directly from formula

$$H_{\Lambda,r}\Phi_0 = \left(H_\Lambda - \frac{E_0}{2} N\right) \Phi_0 = B\Phi_0$$

and the fact that $\|B\Phi_0\|'_V \to 0, V \to \infty$ (see (5.8), (5.11).

Consider, for example, excited state $\Phi_{(q)_1} = a^+_{q_1} a^+_{-q_1} \Phi_0$.

We have

$$H_{\Lambda,r}\Phi_{(q)_1} = \left(H_\Lambda - \frac{E_0}{2} N\right) \Phi_{(q)_1} =$$
$$= \left(-E_0 + 2\left(\frac{q_1^2}{2m} - \mu\right)\right) \Phi_{(q)_1} + \left(B_{(q)_1} + C_{(q)_1}\right) \Phi_{(q)_1}.$$

Taking into account that $\|B_{(q)_1}\Phi_{(q)_1}\|'_V \to 0, V \to \infty$, $\|C_{(q)_1}\Phi_{(q)_1}\|'_V \to 0, V \to \to \infty$ (see (8.6), (8.8)) we conclude that the excited state $\Phi_{(q)_1}$ is the asymptotical (in the limit $\Lambda \nearrow R^3$) eigenstate of $H_{\Lambda,r}$ with eigenvalue $-E_0 + 2\left(\frac{q_1^2}{2m} - \mu\right) \geq |\Delta|$ (see Section VI from [1]).

For the general excited states $\Phi_{(p)_l,(q)_m} = a_{p_1}^+ \ldots a_{p_l}^+ \, a_{q_1}^+ \, a_{-q_1}^+ \ldots a_{q_m}^+ \, a_{-q_m}^+ \Phi_0$, $l \geq 0, m \geq 0, m + l > 0$, one obtains

$$H_{\Lambda,r}\Phi_{(p)_l,(q)_m} = \left(H_\Lambda - \frac{E_0}{2}N\right)\Phi_{(p)_l,(q)_m} =$$

$$= \left[\sum_{i=1}^{l}\left(\frac{p_i^2}{2m} - \mu\right) + \sum_{i=1}^{m}\left(\frac{2q_i^2}{2m} - 2\mu\right) - \frac{E_0}{2}(l + 2m)\right]\Phi_{(p)_l,(q)_m} +$$

$$+ \left(B_{(p)_l,(q)_m} + C_{(p)_l,(q)_m}\right)\Phi_{(p)_l,(q)_m}. \tag{8.20}$$

and

$$\left\|B_{(p)_l,(q)_m}\Phi_{(p)_l,(q)_m}\right\|_V' \to 0, \quad V \to \infty,$$

$$\left\|C_{(p)_l,(q)_m}\Phi_{(p)_l,(q)_m}\right\|_V' \to 0, \quad V \to \infty.$$

It follows from the above obtained formulae that the eigenvalue of eigenstate $\Phi_{(p)_l,(q)_m}$ are asymptotically equal to

$$\sum_{i=1}^{l}\left(-\frac{E_0}{2} + \frac{p_i^2}{2m} - \mu\right) + \sum_{i=1}^{m}\left(-E_0 + \frac{2q_i^2}{2m} - 2\mu\right). \tag{8.21}$$

Thus there exists the gap equal to $|\Delta|/2$ in eigenvalues of the excitations of the ground states $\Phi_{(p)_1}$, the gap equal to $|\Delta|$ for $\Phi_{(q)_1}$, and $|\Delta|l/2 + |\Delta|m$ for $\Phi_{(p)_l,(q)_m}$.

There also exists the gap equal to Δ in spectra of H_2 that corresponds to the eigenvectors $f_i^1(k)$ of exited pairs (see Section VI). We are not able to control behavior of $f_i^1(k)$ as $V \to \infty$. It seems to us that vectors $\Phi_{(p)_l,(q)_m}$ also describe correctly excited pairs in the thermodynamic limit.

IX. Hamiltonian BCS and approximating Hamiltonian on excited states. ***1. Operators A^I, A^+, and A^- on excited states.*** Consider the following excitations of the ground state Φ_0

$$\Phi_{(p)_l,(q)_m} = a_{p_1}^+ \ldots a_{p_l}^+ \, a_{q_1}^+ \, a_{-q_1}^+ \ldots a_{q_m}^+ \, a_{-q_m}^+ \Phi_0. \tag{9.1}$$

Momenta $(p)_1$, $(q)_m$ satisfy the same conditions as for $f_{(p)_l,(q)_m}$ (see Section VIII.1). Consider the approximating Hamiltonian

$$H_{a,\Lambda} = \sum_{\overline{p}} a_{\overline{p}}^+ a_{\overline{p}}\left(\frac{p^2}{2m} - \mu\right) + c\sum_p v_p a_p^+ a_{-p}^+ + c\sum_p v_p a_{-p} a_p - g^{-1}c^2 V,$$

$$c = \frac{g}{V}\sum_p v_p f_1^0(p).$$

By analogy with the ground state (see Section VII) and according to (8.3), (8.4) we have

$$A^I\Phi_{(p)_l,(q)_m} = A^+\Phi_{(p)_l,(q)_m}, \quad A^+ = c\sum_p v_p a_p^+ a_{-p}^+, \quad A^- = c\sum_p v_p a_{-p} a_p,$$

$$A^-\Phi_{(p)_l,(q)_m} = g^{-1}c^2 V\Phi_{(p)_l,(q)_m} + cB^1_{(p)_l,(q)_m}\Phi_{(p)_l,(q)_m} + c\sum_{i=1}^{m} v_{q_i}\Phi_{(p)_l,(\overset{i}{\vee}{q})_m}, \tag{9.2}$$

where the operator $B^1_{(p)_l,(q)_m}$ is defined as in (7.7), but one has $\sum_{k=(k)_n, k=(p)_l, k=(q)_m} v_k f_1^0(k)$ instead of $\sum_{k=(k)_n} v_k f_1^0(k)$, and A^I is the part of the operator A that describes interaction — the fourth term on the right-hand side of (8.3). If some momenta from the set $(p)_l$, $(q)_m$ do not belong to the domain D then they should be omitted in the operator $B^1_{(p)_l,(q)_m}$.

By analogy with (7.8) the following estimates are obvious

$$\frac{1}{V}\left(\Phi_{(p)_l,(q)_m}, B^1_{(p)_l,(q)_m}\Phi_{(p)_l,(q)_m}\right)'_V \leq \frac{1}{V} v\,(l+m+1)\,\alpha f^2 e^{\alpha f^2},$$
$$\frac{1}{V}\left|\left(\sum_{i=1}^m v_{q_i}\Phi_{(p)_l,(\overset{i}{\vee}{q})_m}, \sum_{i=1}^m v_{q_i}\Phi_{(p)_l,(\overset{i}{\vee}{q})_m}\right)'_V\right| \leq \frac{1}{V} v^2 m^2 e^{\alpha f^2}. \tag{9.3}$$

It follows from (9.3) that the averages

$$\frac{1}{V}\left|\left(\Phi_{(p)_l,(q)_m}, B^1_{(p)_l,(q)_m}\Phi_{(p)_l,(q)_m}\right)'_V\right|,$$
$$\frac{1}{V}\left|\left(\sum_{i=1}^m v_{q_i}\Phi_{(p)_l,(\overset{i}{\vee}{q})_m}, \sum_{i=1}^m v_{q_i}\Phi_{(p)_l,(\overset{i}{\vee}{q})_m}\right)'_V\right|, \tag{9.4}$$
$$\frac{1}{V}\left|\left(\Phi_{(p)_l,(q)_m}, \sum_{i=1}^m v_{q_i}\Phi_{(p)_l,(\overset{i}{\vee}{q})_m}\right)'_V\right| \leq \left\|\Phi_{(p)_l,(q)_m}\right\|'_V \left\|\sum_{i=1}^m v_{q_i}\Phi_{(p)_l,(\overset{i}{\vee}{q})_m}\right\|'_V$$

tend to zero as $V \to \infty$ for arbitrary fixed l and m and even for l and m that tend to infinity together with V but in such a way that $\lim_{V\to\infty}((l+m)/V) = 0$. Note that $\Phi_{(p)_l,(q)_m}$ is orthogonal to $\sum_{i=1}^m v_{q_i}\Phi_{(p)_l,(\overset{i}{\vee}{q})_m}$ and their scalar product is equal to zero.

2. *Asymptotic coincidence of H_Λ and $H_{a,\Lambda}$.* Now we are able to prove the following analog of Theorem 6 about the thermodynamic equivalence of the Hamiltonians H_Λ and $H_{a,\Lambda}$ on the excitations of the ground states $\Phi_{(p)_l,(q)_m}$.

Theorem 9. *The averages*

$$\frac{1}{V}\left(\Phi_{(p)_l,(q)_m}, (H_\Lambda - H_{a,\Lambda})\Phi_{(p)_l,(q)_m}\right)'_V \tag{9.5}$$

tend to zero in the thermodynamic limit as $V \to \infty$ for arbitrary fixed l, m.

Proof. Consider the identity

$$\frac{1}{V}\left(\Phi_{(p)_l,(q)_m}, (H_\Lambda - H_{a,\Lambda})\Phi_{(p)_l,(q)_m}\right)'_V =$$
$$= \frac{1}{V}\left(\Phi_{(p)_l,(q)_m}, \left(H_\Lambda - A - \sum_{i=1}^l\left(\frac{p_i^2}{2m} - \mu\right)I - \right.\right.$$
$$\left.\left. - \sum_{i=1}^m\left(\frac{2q_i^2}{2m} - 2\mu\right)I\right)\Phi_{(p)_l,(q)_m}\right)'_V +$$
$$+ \frac{1}{V}\left(\Phi_{(p)_l,(q)_m}, \left(A + \sum_{i=1}^l\left(\frac{p_i^2}{2m} - \mu\right)I + \right.\right.$$
$$\left.\left. + \sum_{i=1}^m\left(\frac{2q_i^2}{2m} - 2\mu\right)I - H_{a,\Lambda}\right)\Phi_{(p)_l,(q)_m}\right)'_V. \tag{9.6}$$

According to (8.6), (8.7) the first term in (9.6) tends to zero as $V \to \infty$ (even without term $1/V$).

To estimate the second term in (9.6) we use the following identities

$$H_{a,\Lambda} = \sum_{\bar{p}} \left(\frac{p^2}{2m} - \mu \right) a^+_{\bar{p}} a_{\bar{p}} + A^+ + A^- - g^{-1} c^2 V,$$

$$A = \sum_{\bar{p}} \left(\frac{p^2}{2m} - \mu \right) a^+_{\bar{p}} a_{\bar{p}} + A^I - \sum_{i=1}^{l} \left(\frac{p_i^2}{2m} - \mu \right) I - \sum_{i=1}^{m} \left(\frac{2q_i^2}{2m} - 2\mu \right) I, \tag{9.7}$$

take into account identities (9.2), and represent the second term in (9.5) as follows

$$\frac{1}{V} \left(\Phi_{(p)_l,(q)_m}, \left[cB^1_{(p)_l,(q)_m} \Phi_{(p)_l,(q)_m} + c \sum_{i=1}^{m} v_{q_i} \Phi_{(p)_l,(\overset{i}{\vee}{q})_m} \right] \right)'_V. \tag{9.8}$$

It follows from (9.3), (9.4) that average (9.8) tends to zero as $V \to \infty$. Thus, both terms on the right-hand side of (9.6) tend to zero as $V \to \infty$ for arbitrary fixed l, m, and l, m can even tend to infinity together with V but in such a way that $\lim_{V\to\infty} ((l+m)/V) = 0$. The theorem is proved.

Note that averages

$$\frac{1}{V} \left\| (H_\Lambda - H_{a,\Lambda}) \Phi_{(p)_l,(q)_m} \right\|'_V \tag{9.9}$$

also tend to zero as $V \to \infty$. The proof is almost the same as in Theorem 9.

Remark. Theorem 8 also holds for arbitrary states

$$\Phi = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1} f_1(k_1) a^+_{k_1} a^+_{-k_1} \ldots \sum_{k_n} f_1(k_n) a^+_{k_n} a^+_{-kn} \mid 0\rangle$$

if functions $f_1(k)$ satisfy the following condition $f = \sup_k |f_1(k)| < \infty$ uniformly with respect to V. To every such states there exists corresponding approximating Hamiltonian $H_{a,\Lambda}$ with $c = (q/V) \sum_p v_p f_1(p)$.

Now consider the operators A^I, A^+, A^- on normalized excitations of the ground states Φ_{lm} (8.17). By analogy with (8.13) we have the following modification of formulae (9.2)

$$A^I \Phi_{lm} = \sum_{(p)_l,(q)_m} \psi_l((p)_l) \chi_m((q)_m) A^I \Phi_{(p)_l,(q)_m} = \sum_{(p)_l,(q)_m} \psi_l((p)_l) \chi_m((q)_m) A^+ \Phi_{(p)_l,(q)_m},$$

$$A^- \Phi_{lm} = g^{-1} c^2 V \Phi_{lm} + \sum_{(p)_l,(q)_m} \psi_l((p)_l) \chi_m((q)_m) cB^1_{(p)_l,(q)_m} \Phi_{(p)_l,(q)_m} + \sum_{(p)_l,(q)_m} \psi_l((p)_l) \chi_m((q)_m) c \sum_{i=1}^{m} v_{q_i} \Phi_{(p)_l,(\overset{i}{\vee}{q})_m}. \tag{9.10}$$

The following estimates hold

$$\frac{1}{V}\frac{1}{V^{l+m}}\left|\sum_{(p)_l,(q)_m}{}'\left(\psi_l((p)_l)\chi_m((q)_m)\Phi_{(p)_l,(q)_m},\psi_l((p)_l)\chi_m((q)_m)B^1_{(p)_l,(q)_m}\Phi_{(p)_l,(q)_m}\right)'_V\right| \leq$$

$$\leq \frac{1}{V}\sum_{n=1}^{\infty}\frac{1}{V^{l+m+n}}\sum_{(p)_l\neq(q)_m\neq(k)_n}{}' |\psi_l((p)_l)|^2|\chi_m((q)_m)|^2|f_n((k)_n)|^2 \sum_{\substack{k=(k)_n\\k=(p)_l\\k=(q)_m}} |v_k f_1^0(k)| \leq$$

$$\leq \frac{1}{V}\sum_{n=0}^{\infty}\frac{N^{l+m+n}}{V^{l+m+n}}\frac{l+m+n}{(l+m+n)!}\beta^{2(l+m+n)+1}v \leq$$

$$\leq \frac{1}{V}v\beta\sum_{n=0}^{\infty}\alpha^{l+m+n}\frac{1}{(l+m+n-1)!}\beta^{2(l+m+n)} \leq \frac{1}{V}v\beta^3\alpha e^{\alpha\beta^2}, \tag{9.11}$$

where

$$f_n((k)_n) = f_1^0(k_1)\ldots f_1^0(k_n),$$

$$\sup\left|\psi_l((p)_l)\right| \leq \psi^l, \qquad \sup\left|\chi_m((q)_m)\right| \leq \chi^m,$$

$$f = \sup\left|f_1^0(k)\right|, \qquad \beta = \max\{\psi,\chi,f\}.$$

Series (9.11) is convergent and tends to zero as $V\to\infty$ uniformly with respect to l, m.

We have also the following estimates by using (9.3)

$$\frac{1}{V}\frac{1}{V^{l+m}}\left|\sum_{(p)_l,(q)_m}{}'\left(\psi_l((p)_l)\chi_m((q)_m)\sum_{i=1}^{m}v_{q_i}\Phi_{(p)_l,(\overset{i}{\check{q}})_m},\right.\right.$$

$$\left.\left.\psi_l((p)_l)\chi_m((q)_m)\sum_{j=1}^{m}v_{q_j}\Phi_{(p)_l,(\overset{j}{\check{q}})_m}\right)'_V\right| \leq$$

$$\leq \frac{1}{V}\sum_{n=0}^{\infty}\frac{1}{V^{l+m+n}}\sum_{(p)_l\neq(q)_m\neq(k)_n}{}' |\psi_l((p)_l)|^2\left|\chi_m((q)_m)\right|^2\left|f_n(k)\right|^2 mv^2 \leq$$

$$\leq \frac{1}{V}v^2\sum_{n=0}^{\infty}\frac{N^{l+m+n}}{V^{l+m+n}}\frac{\beta^{2(l+m+n)}m}{(l+m+n)!} \leq$$

$$\leq \frac{1}{V}v^2\sum_{n=0}^{\infty}\alpha^{l+m+n}\beta^{2(l+m+n)}\frac{1}{(l+m+n-1)!} \leq \frac{1}{V}v^2\alpha\beta^2e^{\alpha\beta^2}. \tag{9.12}$$

Theorem 10. *The averages $V^{-1}(\Phi_{lm},(H_\Lambda - H_{a,\Lambda})\Phi_{lm})$ tend to zero in the thermodynamic limit as $V\to\infty$ uniformly with respect to l and m.*

Proof is some modification of the proof of Theorem 9. Indeed, consider the identity

$$\frac{1}{V}(\Phi_{lm}, (H_\Lambda - H_{a,\Lambda})\Phi_{lm})'_V =$$

$$= \frac{1}{V}\left(\frac{1}{V^{l+m}}\sum_{(p)_l,(q)_m}{}' \psi_l((p)_l)\chi_m((q)_m)\Phi_{(p)_l,(q)_m}, \psi_l((p)_l)\chi_m((q)_m) \times \right.$$

$$\left. \times\left[H_\Lambda - A - \sum_{i+1}^{l}\left(\frac{p_i^2}{2m} - \mu\right)I - \sum_{j-1}^{m}\left(\frac{2q_j^2}{2m} - 2\mu\right)I\right]\Phi_{(p)_l,(q)_m}\right)'_V +$$

$$+\frac{1}{V}\left(\frac{1}{V^{l+m}}\sum_{(p)_l,(q)_m}{}' \psi_l((p)_l)\chi_m((q)_m)\Phi_{(p)_l,(q)_m}, \psi_l((p)_l)\chi_m((q)_m) \times \right.$$

$$\left. \times\left[A + \sum_{i+1}^{l}\left(\frac{p_i^2}{2m} - \mu\right)I + \sum_{j-1}^{m}\left(\frac{2q_j^2}{2m} - 2\mu\right)I - H_{a,\Lambda}\right]\Phi_{(p)_l,(q)_m}\right)'_V. \tag{9.13}$$

According to (8.15′) the first term in (9.13) tends to zero as $V \to \infty$ (even without factor $1/V$). The second term in (9.13) can be represented as follows:

$$\frac{1}{V}\left(\frac{1}{V^{l+m}}\sum_{(p)_l,(q)_m}{}' \psi_l((p)_l)\chi_m((q)_m)\Phi_{(p)_l,(q)_m}, \right.$$

$$\left. \psi_l((p)_l)\chi_m((q)_m)\left[B^1_{(p)_l,(q)_m}\Phi_{(p)_l,(q)_m} + \sum_{i=1}^{m} v_{q_i}\Phi_{(p)_l,(\overset{i}{\check{q}})_m}\right]\right)'_V \tag{9.14}$$

and, according to (9.11), (9.12), it tends to zero as $V \to \infty$ uniformly with respect to l, m. To estimate the second term in (9.3) we use inequality similar to the last inequality (9.4).

X. Hilbert space of excited states. ***1. Ground state of the BCS Hamiltonian as vacuum for quasiparticles.*** Consider the following operators

$$\alpha_{\overline{k}} = u_{\overline{k}}a_{\overline{k}} + w_{\overline{k}}a^+_{-\overline{k}}, \qquad \alpha^+_{\overline{k}} = u_{\overline{k}}a^+_{\overline{k}} + w_k a_{-\overline{k}}, \tag{10.1}$$

where real functions $u_{\overline{k}}$, $w_{\overline{k}}$ satisfy conditions

$$u^2_{\overline{k}} + w^2_{\overline{k}} = 1, \quad u_{\overline{k}} = u_{-\overline{k}}, \quad w_{\overline{k}} = -w_{-\overline{k}}, \quad u_{\overline{k}} = u_k, \quad w_{\overline{k}} = w_k, \tag{10.2}$$

and $a_{\overline{k}}$, $a^+_{\overline{k}}$ satisfy canonical anticommutation relations.

It is easy to check that operators α_k, α^+_k also satisfy canonical anticommutation relations

$$\left\{\alpha_{\overline{k}}, \alpha^+_{\overline{k'}}\right\} = \delta_{\overline{k},\overline{k'}}, \quad \{\alpha_{\overline{k}}, \alpha_{\overline{k'}}\} = 0, \quad \left\{\alpha^+_{\overline{k}}, \alpha^+_{\overline{k'}}\right\} = 0. \tag{10.3}$$

We will say that the operator $\alpha_{\overline{k}}$, $\alpha^+_{\overline{k}}$ are respectively the operators of annihilation and creation of quasiparticles.

Let us show that the ground state Φ_0 is the vacuum for the operators $\alpha_{\overline{k}}$ if functions u_k, w_k are chosen as follows

$$w_k = \frac{-f_1^0(k)}{\sqrt{1 + \left(f_1^0(k)\right)^2}}, \qquad u_k = \frac{1}{\sqrt{1 + \left(f_1^0(k)\right)^2}}. \tag{10.4}$$

Indeed, we have $\left(k = (k, 1), -k = (-k, -1), \ \alpha_k = \alpha_{k,1}, a_k = a_{k,1}, a^+_{-k} = a^+_{-k,-1}\right)$

$$\alpha_k \Phi_0 = \left(u_k a_k + w_k a_{-k}^+\right) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1} f_1^0\left(k_1\right) a_{k_1}^+ a_{-k_1}^+ \cdots \sum_{k_n} f_1^0\left(k_n\right) a_{k_n}^+ a_{-k_n}^+ \mid 0\rangle =$$

$$= \left(u_k f_1^0(k) a_{-k}^+ + w_k a_{-k}^+\right) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1} f_1^0\left(k_1\right) a_{k_1}^+ a_{-k_1}^+ \cdots \sum_{k_n} f_1^0\left(k_n\right) a_{k_n}^+ a_{-k_n}^+ \mid 0\rangle =$$

$$= \left(u_k f_1^0(k) + w_k\right) a_{-k}^+ \Phi_0 = \left(\frac{f_1^0(k)}{\sqrt{1+f_1^0(k)^2}} - \frac{f_1^0(k)}{\sqrt{1+f_1^0(k)^2}}\right) a_{-k}^+ \Phi_0 = 0. \quad (10.5)$$

Obviously $\alpha_{-k}\Phi_0 = 0$.

It follows from (10.5) that the ground state Φ_0 is the vacuum for the operators α_k, α_k^+ of quasiparticles.

If $k \notin D$ then $\alpha_k = a_k$, $\alpha_k^+ = a_k^+$ and operators of creation and annihilation of quasiparticles coincide with corresponding operators of particles.

Remark. Note that we used, according to [2, 3], an odd potential $v_k = -v_{-k}$ and odd functions $f_n(k_1, \ldots, -k_i, \ldots, k_n) = -f_n(k_1, \ldots, k_i, \ldots, k_n)$, $f_1^0(-k) = -f_1^0(k)$. It can also be used, according to [4], an even potential $v_{-k} = v_k$ and even function $f_n(k_1, \ldots, -k_i, \ldots, k_n) = f_n(k_1, \ldots, k_i, \ldots, k_n)$, $f_1^0(-k) = f_1^0(k)$.

In the second case instead of (10.1) we have

$$\alpha_k = u_k a_k + w_k a_{-k}^+, \qquad \alpha_k^+ = u_k a_k^+ + w_k a_{-k},$$
$$\alpha_{-k} = u_k a_{-k} - w_k a_k^+, \qquad \alpha_{-k}^+ = u_k a_{-k}^+ - w_k a_k. \quad (10.1')$$

In what follows we will use both cases.

The ground states Φ_0 can also be represented as follows

$$\Phi_0 = \prod_k \left(1 + f_1^0(k)\, a_k^+ a_{-k}^+\right) \mid 0\rangle =$$

$$= \sum_{n=0}^{\infty} \sum_{k_1 \neq \ldots \neq k_n}' f_1^0\left(k_1\right) \ldots f_1^0\left(k_n\right) a_{k_1}^+ a_{-k_1}^+ \ldots a_{k_n}^+ a_{-k_n}^+ \mid 0\rangle,$$

$$\left(\Phi_0, \Phi_0\right)_V' = \prod_k \left(1 + \frac{1}{V}\left(f_1^0(k)\right)^2\right). \quad (10.6)$$

Consider the states

$$\varphi_{(p)_l,(q)_m} = \alpha_{p_1}^+ \ldots \alpha_{p_l}^+ \alpha_{q_1}^+ \alpha_{-q_1}^+ \ldots \alpha_{q_m}^+ \alpha_{-q_m}^+ \Phi_0 \frac{1}{\left(\Phi_0, \Phi_0\right)_V'}, \quad (10.7)$$

that are excitations created by quasiparticles where momenta $(p)_l$, $(q)_m$ satisfy the same conditions as for $\Phi_{(p)_l,(q)_m}$, Φ_{lm}.

It is easy to check that

$$\alpha_p^+ \Phi_0 = (u_p a_p^+ + w_p a_{-p})\Phi_0 = \sqrt{1 + (f_1^0(p))^2}\, a_p^+ \prod_{k \neq p} (1 + f_1^0(k) a_k^+ a_{-k}^+)|0\rangle,$$

$$\alpha_{-p}^+ \Phi_0 = (u_p a_{-p}^+ + w_p a_p)\Phi_0 = \sqrt{1 + (f_1^0(p))^2}\, a_{-p}^+ \prod_{k \neq p} (1 + f_1^0(k) a_k^+ a_{-k}^+)|0\rangle,$$

$$\alpha_q^+ \alpha_{-q}^+ \Phi_0 = \left(-f_1^0(q) + a_q^+ a_{-q}^+\right) \prod_{k \neq q} \left(1 + f_1^0(k) a_k^+ a_{-k}^+\right)|0\rangle. \quad (10.8)$$

By using formulae (10.8) we represent $\varphi_{(p)_l,(q)_m}$ as follows

$$\varphi_{(p)_l,(q)_m} = \prod_{i=1}^{l} \left(\sqrt{1+(f_1^0(p))^2} a_{p_i}^{+} \right) \times$$

$$\times \prod_{j=1}^{m} \left(-f_1^0(q_j) + a_{q_j}^{+} a_{-q_j}^{+} \right) \prod_{k \neq (p)_l \neq (q)_m} \left(1 + f_1^0(k) a_k^{+} a_{-k}^{+} \right) |0\rangle. \tag{10.9}$$

We define scalar product of $\varphi_{(p)_l,(q)_m}$ as before for $\Phi_{(p)_l,(q)_m}$. Namely, we use usual calculation with the operators of creation and annihilation a_k^{+}, a_{-k}^{+}, a_k, a_{-k}, and then, in the sums over $(p)_l$, $(q)_m$, $(k)_n$ in the expression obtained, we insert the factor V^{-l+m+n} or multiply each of $f_1^0(p_1), \ldots, f_1^0(p_l)$, $f_1^0(q_1), \ldots, f_1^0(q_m)$, $f_1^0(k_1), \ldots, f_1^0(k_n)$ by $V^{-1/2}$. For example,

$$\left(\alpha_p^{+} \Phi_0, \alpha_p^{+} \Phi_0 \right)_V' = \left(1 + \frac{1}{V} (f_1^0(p))^2 \right) \sum_{n=0}^{\infty} \frac{1}{V^n} \sum_{k_1 \neq \ldots \neq k_n \neq p} (f_1^0(k_1))^2 \ldots (f_1^0(k_n))^2.$$

Obviously scalar products of $\varphi_{(p)_l,(q)_m}$ with different form either $(p)_l$ or $(q)_m$ are equal to zero. For example,

$$\left(\Phi_0, \alpha_q^{+} \alpha_{-q}^{+} \Phi_0 \right)_V' = \left(-\frac{1}{V} f_1^0(q) + \frac{1}{V} f_1^0(q) \right) \prod_{k \neq p} \left(1 + \frac{1}{V} (f_1^0(k))^2 \right) = 0.$$

It is easy to check that states $\varphi_{(p)_l,(q)_m}$ are orthogonal. Further we have

$$\prod_{j=1}^{m} \left(-f_1^0(q_j) + a_{q_j}^{+} a_{-q_j}^{+} \right) =$$

$$= \sum_{s=0}^{m} \sum_{\substack{(i_1, \ldots, i_s) \\ (j_1, \ldots, j_{m-s})}} (-1)^s f_1^0(q_{i_1}) \ldots f_1^0(q_{i_s}) a_{q_{j_1}}^{+} a_{-q_{j_1}}^{+} \ldots a_{q_{j_{m-s}}}^{+} a_{-q_{j_{m-s}}}^{+}, \tag{10.10}$$

where summation is carried over all decompositions of the set of numbers $(1, \ldots, m)$ into two subsets $(i_1, \ldots, i_s)$, $(j_1, \ldots, j_{m-s})$.

Inserting this formula in (10.9) we see that $\varphi_{(p)_l,(q)_m}$ is the finite sum of 2^m terms $\Phi_{(p)_l,(q)_{m-s}}$, where $0 \leq s \leq m$ and therefore the Theorems 7 and 9 are also true for $\varphi_{(p)_l,(q)_m}$. It is sufficient to put $\varphi_{(p)_l,(q)_m}$ instead of $f_{(p)_l,(q)_m}$ in (8.9) and instead of $\Phi_{(p)_l,(q)_m}$ in (9.5).

Consider normalized states

$$\varphi_{lm} = \sum_{(p)_l,(q)_m} \psi_l((p)_l) \chi_m((q)_m) \varphi_{(p)_l,(q)_m} \tag{10.11}$$

with the same conditions imposed on ψ_l, χ_m as in Theorems 8′ and 10. It is easy to see that the Theorems 8′ and 10 hold for φ_{lm} because φ_{lm} consists from 2^m orthogonal terms (as it follows from (10.10)) and in order to estimate $(\varphi_{lm}, (H_\Lambda - H_{a,\Lambda})\varphi_{lm})$ it is sufficient to insert the additional factor $4^m \beta^{2m}$ ($\beta = \max\{1, f, \psi, \chi\}$) in series (9.11), (9.12). Obviously one obtains again convergent series.

To calculate $(\varphi_{lm}, (H_\Lambda - H_{a,\Lambda})\varphi_{lm})$, we use usual calculation with the operators a_k^{+}, a_{-k}^{+}, a_k, a_{-k}, and, in the sums over $(p)_l$, $(q)_m$, $(k)_n$ in the expression obtained, we insert the factor V^{-l+m+n}.

$$\Phi_0^a = \prod_k \left(1 + f^a(k) a_k^+ a_{-k}^+\right)|0\rangle, \qquad f^a(k) = \frac{-w_k}{u_k} \tag{10.19}$$

is the vacuum for the operators given by formulae (10.14). Indeed, by direct calculation one checks that

$$\alpha_p \Phi_0^a = 0, \qquad \alpha_{-p} \Phi_0^a = 0.$$

We say that the operators $\alpha_{\overline{k}}$, $\alpha_{\overline{k}}^+$ are the operators of annihilations and creations of the quasiparticles with vacuum Φ_0^a. They satisfy the same canonical anticommutation relation (10.3)

Consider the excited states

$$\varphi^a_{(p)_l,(q)_m} = \alpha^+_{\overline{p}_1} \ldots \alpha^+_{\overline{p}_l} \alpha^+_{q_1} \alpha^+_{-q_1} \ldots \alpha^+_{q_m} \alpha^+_{-q_m} \Phi_0^a \frac{1}{(\Phi_0^a, \Phi_0^a)'_V},$$

$$(\Phi_0^a, \Phi_0^a)'_V = \prod_k \left(1 + \frac{1}{V} f_0^a(k)^2\right). \tag{10.20}$$

They also are orthonormal as well as $\varphi_{(p)_l,(q)_m}$.

The excited states $\varphi_{(p)_l,(q)_m}$ and $\varphi^a_{(p)_l,(q)_m}$ for $(p)_l \subset D$, $(q)_m \subset D$ constitute as different orthonormal systems a finite-dimensional Hilbert space.

The vacuum Φ_0^a is the eigenvector of the Hamiltonian $H_{a,\Lambda}$ (10.10) with eigenvalue

$$E_0^a = V\left[\frac{1}{V}\sum_p \left[\varepsilon_p - \sqrt{\varepsilon_p^2 + c^2 v_p^2}\right] - g^{-1}c^2\right]. \tag{10.21}$$

The excited states $\varphi^a_{(p)_l,(q)_m}$ (10.18) also are the eigenvectors of $H_{a,\Lambda}$ with eigenvalues

$$E_0^a + \sum_{i=1}^{l} E_{p_i} + \sum_{i=1}^{m} 2E_{q_i},$$

$$H_{a,\Lambda}\varphi^a_{(p)_l,(q)_m} = \left[\sum_{i=1}^{l} E_{p_i} + 2\sum_{i=1}^{m} E_{q_i} + E_0^a\right]\varphi^a_{(p)_l,(q)_m}. \tag{10.22}$$

Thus we have two systems of the operators of annihilations and creations $(\alpha_{\overline{k}}, \alpha_{\overline{k}}^+)$ *of quasiparticles. The first one* (10.1) *is connected with the ground state* Φ_0 *of* H_Λ *as the vacuum, the second one* (10.14) *is connected with the ground state* Φ_0^a *of* $H_{a,\Lambda}$. *The both systems are not unitary equivalent* [5], *to each other in the thermodynamic limit because canonical transformations* (10.1), (10.14) *do not satisfy necessary conditions of unitary equivalence.*

3. ***Equation for the constant.*** The ground state was not yet defined completely because the constant

$$c = \frac{g}{V}\sum_p v_p f_1^0(p), \qquad f_1^0(p) = \frac{cv_p}{E_0(-2p^2/2m) + 2\mu}$$

was not yet determined.

As it is commonly accepted in quantum statistical mechanics the constant c should be determined from the condition of minimum of the average energy per volume E_0^a of the ground state Φ_0^a of the Hamiltonian $H_{\Lambda,a}$.

If one considers model with different from zero temperature then the constant c should be determined from the condition of minimum of the free energy per volume. It will be done in the third part of given work.

4. *Hamiltonian BCS on ground of the approximating Hamiltonian.* Consider vacuum state (10.19) of $H_{a,\Lambda}$

$$\Phi_0^a = \prod_k (1 + f^a(k) a_k^+ a_{-k}^+)|0\rangle =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1} f^a(k_1) a_{k_1}^+ a_{-k_1}^+ \ldots \sum_{k_n} f^a(k_n) a_{k_n}^+ a_{-k_n}^+ |0\rangle, \tag{10.23}$$

$$f^a(k) = -\frac{w_k}{u_k} = \sqrt{\frac{\sqrt{\varepsilon_k^2 + c^2 v_k^2} - \varepsilon_k}{\varepsilon_k + \sqrt{\varepsilon_k^2 + c^2 v_k^2}}}.$$

The state Φ_0^a satisfies the conditions of Theorem 9 and, thus,

$$\lim_{V\to\infty} \frac{1}{V} (\Phi_0^a, H_\Lambda \Phi_0^a)_{V'} = \lim_{V\to\infty} \frac{1}{V} (\Phi_0^a, H_{a,\Lambda} \Phi_0^a)'_V,$$

$$H_{a,\Lambda}\Phi_0^a = \left(H_0 + c_1 \sum_k v_k a_k^+ a_{-k}^+ + c_1 \sum_k v_k a_{-k} a_k - g^{-1} c_1^2 V\right) \Phi_0^a, \tag{10.24}$$

$$c_1 = \frac{g}{V} \sum_k v_k f^a(k). \tag{10.25}$$

In the right-hand side of (10.24) we have again $H_{a,\Lambda}$ but with constant c_1 instead of c. As known the constant c_1 should be determined from condition of minimum of the function equal to energy of ground state Φ_0^a with respect to the Hamiltonian $H_{a,\Lambda}$ (10.24)

$$\frac{1}{2V} \sum_p \left(\varepsilon_p - \sqrt{\varepsilon_p^2 + c_1^2 v_p^2}\right) - c_1^2 = f(c_1).$$

But we already know that constant c should be determined from condition of minimum of $f(c_1)$ with respect to c.

We have the condition of minimum

$$\frac{df(c_1)}{dc} = \frac{df(c_1)}{dc_1} \frac{dc_1}{dc} = 0.$$

According to (10.25)

$$\frac{dc_1}{dc} = \frac{g}{V} \sum_k \frac{v_k^2 \varepsilon_k}{\sqrt{\varepsilon_p^2 + c^2 v_p^2}\left(\varepsilon_k + \sqrt{\varepsilon_p^2 + c^2 v_p^2}\right)}.$$

Let us put $\mu = 0$, then $\varepsilon_k > 0$ and $\dfrac{dc_1}{dc} > 0$. This implies that $\dfrac{df(c_1)}{dc} = 0$ if and only if $\dfrac{df(c_1)}{dc_1} = 0$. Thus the condition of minimum of energy of the ground state Φ_0^a with respect to c reduce to condition $\dfrac{df(c_1)}{dc_1} = 0$, i.e. to minimum with respect to c_1 for sufficiently small μ.

The function $f^a(k)$ is bounded uniformly with respect to V and the Theorem 9 is true for $\varphi^a_{(p)_l,(q)_m}$ well as for

$$\varphi^a_{lm} = \sum_{(p)_l,(q)_m} \psi_l((p)_l)\chi_m((q)_m)\varphi^a_{(p)_l,(q)_m}$$

with constant c_1 in $H_{a,\Lambda}$. The constant c_1 should be determined as above.

XI. Thermodynamic equivalence of H_Λ and $H_{a,\Lambda}$. *1. Averages per volume of H_Λ and $H_{a,\Lambda}$.* It follows from the Theorems 9, 10 that

$$\lim_{V\to\infty} \frac{1}{V} \left(\Phi_{(p)_l,(q)_m}, (H_\Lambda - H_{a,\Lambda})\, \Phi_{(p)_l,(q)_m}\right)'_V = 0,$$
$$\lim_{V\to\infty} \frac{1}{V} \left(\Phi_{lm}, (H_\Lambda - H_{a,\Lambda})\, \Phi_{lm}\right)'_V = 0, \tag{11.1}$$

and even

$$\lim_{V\to\infty} \frac{1}{V} \left\|(H_\Lambda - H_{a,\Lambda})\, \Phi_{(p)_l,(q)_m}\right\|'_V = 0.$$

As it follows from Section X.1, the Theorems 9 and 10 are true for the system $\varphi_{(p)_l,(q)_m}$ and φ_{lm} and equalities (11.1) imply

$$\lim_{V\to\infty} \frac{1}{V} \left(\varphi_{(p)_l,(q)_m}, (H_\Lambda - H_{a,\Lambda})\varphi_{(p)_l,(q)_m}\right)'_V = 0,$$
$$\lim_{V\to\infty} \frac{1}{V} \left(\varphi_{lm}, (H_\Lambda - H_{a,\Lambda})\, \varphi_{lm}\right) = 0, \tag{11.2}$$
$$\lim_{V\to\infty} \frac{1}{V} \left\|(H_\Lambda - H_{a,\Lambda})\, \varphi_{(p)_l,(q)_m}\right\|'_V = 0.$$

As mentioned above system $\varphi_{(p)_l,(q)_m}$, $l \geq 0, m \geq 0$ are orthonormal bases in Hilbert space $\mathcal{H}^D_F$ created by operators $a^+_{k,1}$, $a^+_{k,-1}$ with $k \subset D$ and, thus, an arbitrary $f \subset \mathcal{H}^D_F$ can be represented as linear combination of $\varphi_{(p)_l,(q)_m}$. It follows from it and (11.2) that the following equalities are true

$$\lim_{V\to\infty} \frac{1}{V} (f, H_\Lambda f)'_V = \lim_{V\to\infty} \frac{1}{V} (f, H_{a,\Lambda} f)'_V,$$
$$\lim_{V\to\infty} \frac{1}{V} \left\| (H_\Lambda - H_{a,\Lambda})\, f \right\|'_V \tag{11.3}$$

for arbitrary $f \subset \mathcal{H}^D_F$ that are finite linear combinations of $\varphi_{(p)_l,(q)_m}$ or φ_{lm}. We have proved the following theorem.

Theorem 11. *The averages per volume* (11.3) *of the Hamiltonians H_Λ and $H_{a,\Lambda}$ coincide for arbitrary $f \subset \mathcal{H}^D_F$ that are finite linear combinations of $\varphi_{(p)_l,(q)_m}$ or φ_{lm}.*

Consider as orthonormal base the excited states $\varphi^a_{(p)_l,(q)_m}$ (10.20) *and denote by $\mathcal{H}^{D,a}_F$ the Hilbert space with this base. The Hilbert spaces $\mathcal{H}^D_F$ and $\mathcal{H}^{D,a}_F$ are not unitary equivalent in the thermodynamic limit because canonical transformations* (10.1) *and* (10.14) *do not satisfy necessary conditions of unitary equivalence* [5] *(operators of multiplication do not belong to Hilbert – Schmidt class). $\mathcal{H}^D_F$ and $\mathcal{H}^{D,a}_F$ are unitary equivalent for fixed V, because they are finite dimensional spaces.*

Note that Theorem 11 is also true for arbitrary $f \subset \mathcal{H}^{D,a}_F$ that are finite linear combinations of $\varphi^a_{(p)_l,(q)_m}$ or φ^a_{lm} because Theorems 9, 10 are also true for $\varphi^a_{(p)_l,(q)_m}$ and φ^a_{lm}, respectively,

$$\lim_{V\to\infty}\frac{1}{V}\left(\varphi^a_{(p)_l,(q)_m},(H_\Lambda-H_{a,\Lambda})\,\varphi^a_{(p)_l,(q)_m}\right)=0,$$
$$\lim_{V\to\infty}\frac{1}{V}\left(\varphi^a_{lm},(H_\Lambda-H_{a,\Lambda})\varphi^a_{lm}\right)=0, \qquad (11.4)$$
$$\lim_{V\to\infty}\frac{1}{V}\left\|(H_\Lambda-H_{a,\Lambda})\varphi^a_{(p)_l,(q)_m}\right\|=0.$$

If one chooses the excited states $\varphi^a_{(p)_l,(q)_m}$, $l\ge 0$, $m\ge 0$ then the averages

$$\left(\varphi^a_{(p)_l,(q)_m},H_{a,\Lambda}\varphi^a_{(p)_l,(q)_m}\right)'_V$$

can be calculated exactly because states $\varphi^a_{(p)_l,(q)_m}$ are eigenvectors of $H_{a,\Lambda}$. Taking into account (11.4), one can conclude from this that the averages

$$\lim_{V\to\infty}\frac{1}{V}\left(\varphi^a_{(p)_l,(q)_m},H_\Lambda\varphi^a_{(p)_l,(q)_m}\right)'_V$$

can also be calculated exactly.

Thus, equalities (11.3) hold for above described sets from $\mathcal{H}^D_F$ and $\mathcal{H}^{D,a}_F$. We say that the Hamiltonians H_Λ and $H_{\Lambda,a}$ are thermodynamic equivalent in this sense.

2. Two systems of eigenvectors. We have two systems of eigenvectors with correspondent eigenvalues. The first one $\Phi^n_{(p)_l,(q)_m}$ with eigenvalues $nE_0+\sum_{i=1}^l(p_i^2/2m-\mu)+$ $+\sum_{i=1}^m(2q_i^2/2m-2p)$. The second one $\varphi^a_{(p)_l,(q)_m}$ with eigenvalues $E_0^a+\sum_{i=1}^l E_{p_i}+$ $2\sum_{i=1}^m E_{q_i}$. The first one are asymptotic when $V\to\infty$, eigenvectors of H_Λ, the second one are eigenvectors of $H_{\Lambda,a}$. Note that $E^a_{0(V)}$ tends to infinity as $V\to\infty$ and all the spectra corresponding to the system Φ^a_0, $\varphi^a_{(p)_l,(q)_m}$ is shifted to infinity. At first sight, this contradicts equalities (11.2) and (11.4). Now we show that there are no contradictions.

In proving (11.2) as well as (11.4), we used representation of $H_{a,\Lambda}$ (10.10) through the operators A^+, A^- and $g^{-1}c^2VI$

$$H_{a,\Lambda}=\sum_{\overline{p}}a^+_{\overline{p}}a_{\overline{p}}\left(\frac{p^2}{2m}-\mu\right)+c\sum_p v_p a^+_p a^+_{-p}+c\sum_p v_p a_{-p}a_p-g^{-1}c^2VI=$$
$$=\sum_{\overline{p}}a^+_{\overline{p}}a_{\overline{p}}\left(\frac{p^2}{2m}-\mu\right)+A^++A^--g^{-1}c^2VI$$

and the fact that, according to (9.2)–(9.4), the operators A^- and $g^{-1}c^2VI$ asymptotically coincide

$$\lim_{V\to\infty}\frac{1}{V}\left(\Phi_{(p)_l,(q)_m},\left(A^--g^{-1}c^2IV\right)\Phi_{(p)_l,(q)_m}\right)'_V=$$
$$=\lim_{V\to\infty}\frac{1}{V}\left(\Phi_{(p)_l,(q)_m},\left[cB^1_{(p)_l,(q)_m}\Phi_{(p)_l,(q)_m}+c\sum_{i=1}^m v_{q_i}\Phi_{(p)_l,(\overset{i}{\overset{\vee}{q}}{}_0)_m}\right]\right)'_V=0.$$

In order to prove (11.4), we first consider the operator $H_{a,\Lambda}$ without the operator $-g^{-1}c^2VI$ diagonalized it and add the operator $-g^{-1}c^2VI$ after the diagonalization. We obtain (10.16)

$$H_{a,\Lambda}=\sum_{\overline{p}}E_p\alpha^\pm_{\overline{p}}\alpha_{\overline{p}}+V\left[\frac{1}{V}\sum_p\left[\varepsilon_p-\sqrt{\varepsilon_p^2+c^2v_p^2}\right]-g^{-1}c^2I\right]$$

and then proved that Φ^a_0 is the ground state of $H_{a,\Lambda}$ and $\varphi^a_{(p)_l,(q)_m}$ are the exited states.

Performing these calculations we did not use the asymptotic coincidence of A^- and $g^{-1}c^2VI$. In fact, if one uses asymptotic coincidence of A^- and $g^{-1}c^2VI$ on $\varphi_{(p)_l,(q)_m}$

then one obtains (11.2) and (8.20), (8.21), and if one uses the operator $H_{a,\Lambda} + g^{-1}c^2VI$ on $\varphi^a_{(p)_l,(q)_m}$ then one obtains (11.4) and (10.22).

It seems to us that in papers [2 – 4, 6 – 8] have been used implicit only system Φ^a_0, $\varphi^a_{(p)_l,(q)_m}$, $l+m \geq 0$. Namely in papers [2, 3, 6, 7] it has been considered the Hamiltonian

$$H_{\nu,\Lambda} = H_\Lambda + \nu \sum_{\overline{p}} \frac{v_p}{2}(a^+_{\overline{p}} a^+_{-\overline{p}} + a_{-\overline{p}} a_{\overline{p}}) \tag{11.5}$$

with sources proportional to the small parameter ν.

It is obviously that the Hamiltonian $H_{\nu,\Lambda}$ can not have eigenvectors in n-particle subspaces.

If eigenvectors of $H_{\nu,\Lambda}$ continuously depend on ν then in limit $\nu \to 0$, $V \to \infty$ the Hamiltonian H_Λ has only eigenvectors Φ^a_0, $\varphi^a_{(p)_l,(q)_m}$, $l+m > 0$.

There exists an open problem concerning physical consequences of existence of the second, asymptotics as $V \to \infty$, system of eigenvectors $\Phi_0, \varphi_{(p)_l,(q)_m}$, $l+m \geq 0$.

1. *Petrina D. Ya.* Spectrum and states of BCS Hamiltonian in finite domain I. Spectrum // Ukr. Mat. J. – 2000. – **52**, № 5. – P. 667 – 689.
2. *Bogolyubov N. N.* On the model Hamiltonian in the theory of superconductivity. – 1960. – (Preprint / JINR. R-511;60).
3. *Bogolyubov N. N., Zubarev D. N., Tserkovnikov Yu. A.* An asymptotically exact solution for the model Hamiltonian in the theory of superconductivity // Zh. Éksp. Teor. Fiz. – 1960. – **39**, № 1. – P. 120 – 129.
4. *Bardeen J., Cooper L. N., Schrieffer J. R.* Theory of superconductivity // Phys. Rev. – 1957.– **108**. – P. 1175 – 1204.
5. *Berezin F. A.* The method of second quantization [in Russian]. – Moscow: Nauka, 1965. – 235 p.
6. *Bogolyubov N. N. (Jr.)* A method for investigation of model Hamiltonians. – Moscow: Nauka, 1974. – 176 p.
7. *Bogolyubov N. N. (Jr.), Brankov J. G., Zagrebnov V. A., Kurbatov A. N., Tonchev N. S.* Method of approximating Hamiltonian in statistical physics. – Sofia: Izd. Bolg. Akad. Nauk, 1981. – 246 p.
8. *Haag R.* The mathematical structure of the Bardeen – Cooper – Schrieffer model // Nuovo Cim. – 1962. – **25**. – P. 287 – 299.

Received 05.05.2000