UDC 517.9

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INTERVAL OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS*

ІНТЕРВАЛЬНІ КРИТЕРІЇ ОСЦИЛЯЦІЇ ДЛЯ НЕЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ

We present new interval oscillation criteria for certain classes of second order nonlinear differential equations, that are different from most known ones in the sense that they are based only on information on a sequence of subintervals of $[t_0, \infty)$ rather than on the whole half-line. We also present several examples that demonstrate wide possibilities of the results obtained.

Наведено нові інтервальні критерії осциляції для деяких класів диференціальних рівнянь другого порядку, відмінні від найбільш відомих у тому сенсі, що вони базуються на інформації стосовно лише деякої послідовності підінтервалів з $[t_0, \infty)$, а не цілої півосі, а також кілька прикладів, що демонструють широкі можливості одержаних результатів.

1. Introduction. In this paper we consider the oscillation behavior of solutions of the second order nonlinear differential equation

$$(r(t)y'(t))' + q(t)f(y(t))g(y'(t)) = 0, \tag{1.1}$$

where $t \geq t_0$, the functions r, q, f and g are to be specified in the following text.

We recall that a function $y: [t_0, t_1) \to (-\infty, \infty)$, $t_1 > t_0$, is called a solution of Eq. (1.1) if $y(t)$ satisfies Eq. (1.1) for all $t \in [t_0, t_1)$. In the sequel it will be always assumed that solutions of Eq. (1.1) exist for any $t_0 \geq 0$. A solution $y(t)$ of Eq. (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

When $r(t) \equiv 1$, Eq. (1.1) reduces to

$$y''(t) + q(t)f(y(t))q(y'(t)) = 0. \tag{1.2}$$

Eq. (1.1) has been studied by Grace and Lalli [1]. They mentioned that though stability, boundedness, and convergence of solutions of Eq. (1.2) to zero have been investigated in the papers of Burton and Grimmer [2], Graef and Spikes [3, 4], Lalli [5], and Wong and Burton [6]. Noting much has been known regarding the oscillatory behavior of Eq. (1.2) except for the result by Wong and Burton [6] (Theorem 4) regarding oscillatory behavior of Eq. (1.2) in connection with that of the corresponding linear equation

$$y''(t) + q(t)y(t) = 0. \tag{1.3}$$

Recently, Li and Agarwal [7] and Rogovchenko [8] presented new sufficient conditions which ensure oscillatory character of Eq. (1.2). They are different from those of [1] and are applicable to other classes of equations which are not covered by the results of [1]. However, except for the results of [7], all the mentioned above oscillation results involve the interval of q and hence require the information of q on the entire half- line $[t_0, \infty)$.

From the Sturm separation theorem, we see that oscillation is only an interval property, i. e., if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_0, \infty)$, as $a_i \to \to \infty$, such that for each i there exists a solution of Eq. (1.3) that has at least two zeros in $[a_i, b_i]$, then every solution of Eq. (1.3) is oscillatory.

Ei-Sayed [9] established an interval criterion for oscillation of a forced second-order equation, but the result is not very sharp because a comparison with equations of constant coefficient is used in the proof. Afterwards, Wong [10] proved a general

* Supported by the NNSF of China and the Foundation for University Key Teacher by Ministry of Education.

result for a linear forced equation and Li and Agarwal [11] established more general results for nonlinear forced equations.

In 1997, Huang [12] presented the following interval criteria for oscillation and nonoscillation of the second order linear differential Eq. (1.3), where $q(t) \geq 0$, $t \in [t_0, \infty)$.

Theorem A. (i) *If there exists* $t_0 > 0$ *such that for every* $n \in N$,

$$\int_{2^n t_0}^{2^{n+1} t_0} q(s)\,ds \leq \frac{\alpha_0}{2^{n+1} t_0}, \tag{1.4}$$

then every solution of Eq. (1.3) *is nonoscillatory, where* $\alpha_0 = 3 - 2\sqrt{2}$.

(ii) *If there exist* $t_0 > 0$ *and* $\alpha > \alpha_0$ *such that for every* $n \in N$,

$$\int_{2^n t_0}^{2^{n+1} t_0} q(s)\,ds \geq \frac{\alpha}{2^{n+1} t_0}, \tag{1.5}$$

then every solution of Eq. (1.3) *is oscillatory, where* $\alpha_0 = 3 - 2\sqrt{2}$.

As an application, Huang [12] obtained the following corollary.

***Corollary* A.** (i) *If*

$$\lim_{t \to \infty} t \int_t^{2t} q(s)\,ds = \alpha \leq \frac{\alpha_0}{2}, \tag{1.6}$$

then every solution of Eq. (1.3) *is nonoscillatory.*

(ii) *If*

$$\lim_{t \to \infty} t \int_t^{2t} q(s)\,ds = \alpha > \alpha_0, \tag{1.7}$$

then every solution of Eq. (1.3) *is oscillatory, where* $\alpha_0 = 3 - 2\sqrt{2}$.

We note that the above result seems surprisingly interesting because the interval $\left(\alpha_0 / 2^{n+1} t_0, \alpha_0 / 2^n t_0\right)$ is not covered by the conditions (1.4) and (1.5). In particular, if $q(t) = \gamma / t^2$, where $\gamma > 0$ is a constant, then

$$\lim_{t \to \infty} t \int_t^{2t} \frac{\gamma}{s^2} ds = \frac{\gamma}{2} < \frac{\alpha_0}{2} \to \gamma < 3 - 2\sqrt{2} < \frac{1}{4}$$

and

$$\lim_{t \to \infty} t \int_t^{2t} \frac{\gamma}{s^2} ds = \frac{\gamma}{2} = \alpha > \alpha_0 \to \gamma < 6 - 4\sqrt{2} < \frac{1}{4}.$$

This implies that Huang's result remains open for $\gamma \in \left(3 - 2\sqrt{2}, 6 - 4\sqrt{2}\right)$. That is to say, Huang's oscillation criterion is not sharp. In fact, the Euler equation

$$y''(t) + \frac{\gamma}{t^2} y(t) = 0$$

is oscillatory if $\gamma > 1/4$, and nonoscillatory if $\gamma \leq 1/4$ [13, 14].

We remark that Li and Agarwal [7] and Kong [15] employed the technique in the work of Philos [16] and obtained several interval oscillation results for the second order nonlinear equation (1.2) and linear Eq. (1.3). However, they can not be applied to the nonlinear differential Eq. (1.1).

Motivated by the ideas of Li and Agarwal [7, 17] in this paper we obtain, by using a generalized Riccati technique, several new interval criteria for oscillation, that is, criteria given by the behavior of Eq. (1.1) (or of r, q, f and g) only on a sequence of subintervals of $[t_0, \infty)$. Our results involve Kamenev's type condition and improve and extend the results of Huang [12], Kamenev [18] and Philos [16]. Finally, several examples that dwell upon the sharp conditions of our results are also included. Other related oscillation results can refer to [4, 14, 18 – 22].

Hereinafter, we assume that

(H1) the function $r: [t_0, \infty) \to (0, \infty)$ is continuous;

(H2) the function $q: [t_0, \infty) \to R$ is continuous and $q(t) \not\equiv 0$ on any ray $[T, \infty)$ for some $T \ge t_0$;

(H3) the function $f: R \to R$ is continuous and $y f(y) > 0$ for $y \ne 0$;

(H4) the function $g: R \to R$ is continuous and $g(y) \ge K > 0$ for $y \ne 0$.

We say that a function $H = H(t,s)$ belongs to a function class X, denoted by $H \in X$, if $H \in C(D, R_+)$, where $D = \{(t,s): -\infty < s \le t < \infty\}$, which satisfies

$$H(t,t) = 0, \qquad H(t,s) > 0, \ \text{for} \ t > s, \tag{1.8}$$

and has partial derivatives $\partial H / \partial t$ and $\partial H / \partial s$ on D such that

$$\frac{\partial H}{\partial s} = h_1(t,s) H(t,s)^{1/2} \ \text{ and } \ \frac{\partial H}{\partial s} = -h_2(t,s) H(t,s)^{1/2}, \tag{1.9}$$

where $h_1, h_2 \in L_{\text{loc}}(D, R)$.

2. Oscillation results for $f(x)$ with monotonity. In this section we always assume the following condition holds.

$$\text{(H5) there exists } f'(y) \text{ for } y \in R \text{ and } f'(y) \ge \mu > 0 \text{ for } y \ne 0. \tag{2.1}$$

First, we establish two lemmas, which will be useful for establishing oscillation criteria for Eq. (1.1).

Lemma 2.1. *Let assumptions* (H1) – (H5) *hold and suppose that* y *is a solution of Eq.* (1.1) *such that* $|y(t)| > 0$ *on* $[c, b)$. *For any* $v \in C^1([t_0, \infty), (0, \infty))$, *let*

$$u(t) = v(t) \frac{r(t) y'(t)}{f(y(t))} \tag{2.2}$$

on $[c, b)$. *Then for any* $H \in X$,

$$\int_c^b H(b,s) K v(s) q(s)\, ds \le H(b,c) u(c) + + \frac{1}{4\mu} \int_c^b r(s) v(s) \left[h_2(b,s) - \frac{v'(s)}{v(s)} \sqrt{H(b,s)} \right]^2 ds. \tag{2.3}$$

Proof. From (1.1) and (2.2) we have for $s \in [c, b)$

$$u'(t) = -v(t) q(t) g(y'(t)) - \frac{f'(y(t))}{r(t) v(t)} u^2(t) + \frac{v'(t)}{v(t)} u(t). \tag{2.4}$$

In view of $f'(y) \ge \mu > 0$ and $g(y') \ge K > 0$, we obtain

$$u'(t) + K v(t) q(t) + \frac{\mu}{r(t) v(t)} u^2(t) - \frac{v'(t)}{v(t)} u(t) \le 0. \tag{2.5}$$

Multiplying (2.5) by $H(t,s)$, integrating it with respect to s from c to t for $t \in [c, b)$, and using (1.8) and (1.9) we get that

$$\int_c^t H(t,s)Kv(s)q(s)\,ds \le -\int_c^t H(t,s)u'(s)\,ds -$$

$$-\int_c^t H(t,s)\frac{\mu u^2(s)}{r(s)v(s)}\,ds + \int_c^t H(t,s)\frac{v'(s)}{v(s)}u(s)\,ds =$$

$$= H(t,c)u(c) -$$

$$-\int_c^t \left\{ h_2(t,s)\sqrt{H(t,s)}\,u(s) - H(t,s)\frac{v'(s)}{v(s)}u(s) + H(t,s)\frac{\mu u^2(s)}{r(s)v(s)} \right\} ds =$$

$$= H(t,c)u(c) -$$

$$-\int_c^t \left\{ \sqrt{\frac{\mu H(t,s)}{r(s)v(s)}}u(s) + \frac{1}{2}\sqrt{\frac{r(s)v(s)}{\mu}}\left[h_2(t,s) - \frac{v'(s)}{v(s)}\sqrt{H(t,s)} \right] \right\}^2 ds +$$

$$+ \frac{1}{4\mu}\int_c^t r(s)v(s)\left[h_2(t,s) - \frac{v'(s)}{v(s)}\sqrt{H(t,s)} \right]^2 ds \le$$

$$\le H(t,c)u(c) + \frac{1}{4\mu}\int_c^t r(s)v(s)\left[h_2(t,s) - \frac{v'(s)}{v(s)}\sqrt{H(t,s)} \right]^2 ds.$$

Letting $t \to b^-$ in the above, we obtain (2.3). The proof is complete.

Lemma 2.2. *Let assumptions* (H1) – (H5) *hold and suppose that* y *is a solution of Eq.* (1.1) *such that* $|y(t)| > 0$ *on* $(a, c]$. *For any* $v \in C^1([t_0, \infty), (0, \infty))$, *let* $u(t)$ *be defined by* (2.2) *on* $(a, c]$. *Then for any* $H \in X$,

$$\int_a^c H(s,a)Kv(s)q(s)\,ds \le -H(c,a)u(c) +$$

$$+ \frac{1}{4\mu}\int_a^c r(s)v(s)\left[h_1(s,a) - \frac{v'(s)}{v(s)}\sqrt{H(s,a)} \right]^2 ds. \qquad (2.6)$$

Proof. Similar to the proof of Lemma 2.1, we multiply (2.5) by $H(s,t)$, integrate it with respect to s from c to t for $t \in (a, c]$, and use (1.8) and (1.9), then we get that

$$\int_t^c H(s,t)Kv(s)q(s)\,ds \le -\int_c^t H(s,t)u'(s)\,ds -$$

$$-\int_t^c H(s,t)\frac{\mu u^2(s)}{v(s)r(s)}\,ds + \int_t^c H(s,t)\frac{v'(s)}{v(s)}u(s)\,ds =$$

$$= -H(c,t)u(c) +$$

$$+\int_t^c \left\{ h_1(s,t)\sqrt{H(s,t)}\,u(s) - H(s,t)\frac{\mu u^2(s)}{r(s)v(s)} + H(s,t)\frac{v'(s)}{v(s)}u(s) \right\} ds =$$

$$= -H(c,t)u(c) - \int_t^c \frac{1}{r(s)v(s)}\left\{ [\sqrt{\mu H(s,t)}\,u(s)]^2 - \right.$$

$$- \Big(h_1(s,t)\sqrt{H(s,t)}r(s)v(s) + H(s,t)v'(s)r(s)\Big)u(s) +$$

$$+ \frac{1}{4\mu}r^2(s)v^2(s)\left[h_1(s,t) + \frac{v'(s)}{v(s)}\sqrt{H(s,t)}\right]^2\Bigg\}ds +$$

$$+ \frac{1}{4\mu}\int_t^c r(s)v(s)\left[h_1(s,t) + \frac{v'(s)}{v(s)}\sqrt{H(s,t)}\right]^2 ds =$$

$$= -H(c,t)u(c) -$$

$$-\int_t^c \frac{1}{v(s)r(s)}\left\{\sqrt{\mu H(s,t)}u(s) - \frac{1}{2\sqrt{\mu}}r(s)v(s)\left[h_1(s,t) + \frac{v'(s)}{v(s)}\sqrt{H(s,t)}\right]\right\}^2 ds +$$

$$+ \frac{1}{4\mu}\int_t^c r(s)v(s)\left[h_1(s,t) + \frac{v'(s)}{v(s)}\sqrt{H(s,t)}\right]^2 ds \le$$

$$\le -H(c,t)u(c) + \frac{1}{4\mu}\int_t^c r(s)v(s)\left[h_1(s,t) + \frac{v'(s)}{v(s)}\sqrt{H(s,t)}\right]^2 ds.$$

Letting $t \to a^-$ in the above, we obtain (2.6). The proof is complete.

The following theorem is an immediate result from Lemmas 2.1 and 2.2.

Theorem 2.1. *Assume that* (H1) – (H5) *hold and that for some* $c \in (a, b)$ *and for some* $H \in X$, $v \in C^1([t_0, \infty), (0, \infty))$,

$$\frac{1}{H(c,a)}\int_a^c H(s,a)Kv(s)q(s)\,ds + \frac{1}{H(b,c)}\int_c^b H(b,s)Kv(s)q(s)\,ds >$$

$$> \frac{1}{4\mu H(c,a)}\int_a^c r(s)v(s)\left[h_1(s,a) + \frac{v'(s)}{v(s)}\sqrt{H(s,a)}\right]^2 ds +$$

$$+ \frac{1}{4\mu H(b,c)}\int_c^b r(s)v(s)\left[h_2(b,s) - \frac{v'(s)}{v(s)}\sqrt{H(b,s)}\right]^2 ds. \qquad (2.7)$$

Then every solution of Eq. (1.1) *has at least one zero in* (a, b).

Proof. Suppose the contrary. Then there exists a solution $y(t)$ of Eq. (1.1) such that $|y(t)| > 0$ for $t \in (a, b)$. From Lemmas 2.1 and 2.2 we see that both (2.3) and (2.6) hold. By dividing (2.3) and (2.6) by $H(b,c)$ and $H(c,a)$, respectively, and then adding them, we have that

$$\frac{1}{H(c,a)}\int_a^c H(s,a)Kv(s)q(s)\,ds + \frac{1}{H(b,c)}\int_c^b H(b,s)Kv(s)q(s)\,ds \le$$

$$\le \frac{1}{4\mu H(c,a)}\int_a^c r(s)v(s)\left[h_1(s,a) + \frac{v'(s)}{v(s)}\sqrt{H(s,a)}\right]^2 ds +$$

$$+ \frac{1}{4\mu H(b,c)}\int_c^b r(s)v(s)\left[h_2(s,a) - \frac{v'(s)}{v(s)}\sqrt{H(b,s)}\right]^2 ds,$$

which contradicts the assumption (2.7) and completes the proof.

Theorem 2.2. *Assume that* (H1) – (H5) *hold. If, for each* $T \ge t_0$, *there exist* $H \in X$, $v \in C^1([t_0, \infty), (0, \infty))$ *and* $a, b, c \in R$ *such that* $T \le a < c < b$ *and* (2.7) *holds, then every solution of Eq.* (1.1) *is oscillatory.*

Proof. Pick up a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By assumption, for each $i \in N$, there exist $a_i, b_i, c_i \in R$ such that $T_i \le a_i < c_i < b_i$ and (2.7) holds, where a, b, c are replaced by a_i, b_i, c_i respectively. By Theorem 2.1, every solution $y(t)$ has at least one zero $t_i \in (a_i, b_i)$. Nothing that $t_i > a_i \ge T_i$, $i \in N$, we see that every solution has arbitrary large zeros. Thus, every solution of Eq. (1.1) is oscillatory. The proof is complete.

Theorem 2.3. *Assume that* (H1) – (H5) *hold. If*

$$\limsup_{t\to\infty} \int_l^t \left[H(s,l)Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s)\left(h_1(s,l) + \frac{v'(s)}{v(s)}\sqrt{H(s,l)}\right)^2 \right] ds > 0 \quad (2.8)$$

and

$$\limsup_{t\to\infty} \int_l^t \left[H(t,s)Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s)\left(h_2(t,s) - \frac{v'(s)}{v(s)}\sqrt{H(s,l)}\right)^2 \right] ds > 0, \quad (2.9)$$

for some $H \in X$, $v \in C^1([t_0,\infty),(0,\infty))$ *and for each* $l \ge t_0$, *then every solution of Eq.* (1.1) *is oscillatory.*

Proof. For any $T \ge t_0$, let $a = T$. In (2.8) we choose $l = a$. Then there exists $c > a$ such that

$$\int_a^c \left[H(s,a)Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s)\left(h_1(s,a) - \frac{v'(s)}{v(s)}\sqrt{H(s,a)}\right)^2 \right] ds > 0. \quad (2.10)$$

In (2.9) we choose $l = c$. Then there exists $b > c$ such that

$$\int_c^b \left[H(b,s)Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s)\left(h_2(b,s) - \frac{v'(s)}{v(s)}\sqrt{H(b,s)}\right)^2 \right] ds > 0. \quad (2.11)$$

Combining (2.10) and (2.11) we obtain (2.7). The conclusion thus comes from Theorem 2.2. The proof is complete.

For the case where $H := H(t-s) \in X$, we have that $h_1(t-s) = h_2(t-s)$ and denote them by $h(t-s)$. The subclass of X containing such $H(t-s)$ is denoted by X_0. Applying Theorem 2.2 to X_0, we obtain next theorem.

Theorem 2.4. *Assume that* (H1) – (H5) *hold. If for each* $T \ge t_0$, *there exist* $H \in X_0$, $v \in C^1([t_0,\infty),(0,\infty))$ *and* $a, c \in R$ *such that* $T \le a < c$ *and*

$$\int_a^c H(s-a)K[v(s)q(s) + v(2c-s)q(2c-s)]ds >$$

$$> \frac{1}{4\mu}\int_a^c [r(s)v(s) + r(2c-s)v(2c-s)]h^2(s-a)ds +$$

$$+ \frac{1}{2\mu}\int_a^c [r(2c-s)v'(2c-s) - r(s)v'(s)]h(s-a)\sqrt{H(s-a)}ds +$$

$$+ \frac{1}{4\mu}\int_a^c \left[\frac{(v'(s))^2 r(s)}{v(s)} + \frac{(v'(2c-s))^2 r(2c-s)}{v(2c-s)} \right] H(s-a)\,ds, \quad (2.12)$$

then every solution of Eq. (1.1) *is oscillatory.*

Proof. Let $b = 2c - a$. Then $H(b-c) = H(c-a) = H((b-a)/2)$, and for any $w \in L[a, b]$, we have

$$\int_c^b w(s)ds = \int_a^c w(2c-s)ds.$$

Hence

$$\int_c^b H(b-s)w(s)ds = \int_a^c H(s-a)w(2c-s)ds.$$

Thus that (2.12) holds implies that (2.7) holds for $H \in X_0$, $v \in C^1([t_0, \infty), (0, \infty))$ and therefore every solution of Eq. (1.1) is oscillatory by Theorem 2.2. The proof is complete.

From above oscillation criteria, we can obtain different sufficient conditions for oscillation of all solutions of Eq. (1.1) by different choices of $H(t, s)$.

Let

$$H(t, s) = (t-s)^\lambda, \quad t \geq s \geq t_0,$$

where $\lambda > 1$ is a constant.

Corollary 2.1. *Assume that* (H1) – (H5) *hold. Then every solution of Eq.* (1.1) *is oscillatory provided that for each* $l \geq t_0$ *and for some* $\lambda > 1$, *there exists a function* $v \in C^1([t_0, \infty), (0, \infty))$ *such that the following two inequalities hold:*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t (s-l)^\lambda \left[Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s)\left(\frac{\lambda}{s-l} - \frac{v'(s)}{v(s)}\right)^2 \right] ds > 0 \qquad (2.13)$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t (t-s)^\lambda \left[Kv(s)q(s) - \frac{1}{4\mu} r(s)v(s)\left(\frac{\lambda}{t-s} + \frac{v'(s)}{v(s)}\right)^2 \right] ds > 0. \qquad (2.14)$$

Define

$$R(t) = \int_l^t \frac{1}{r(s)} ds, \quad t \geq l \geq t_0, \qquad (2.15)$$

and let

$$H(t,s) = [R(t) - R(s)]^\lambda, \quad t \geq t_0, \qquad (2.16)$$

where $\lambda > 0$ is a constant.

By Theorem 2.3, we have the following oscillation criterion, which extends Theorem 2.3 (i) of Kong [15] and Theorem 2.5 of Li and Agarwal [7].

Theorem 2.5. *Assume that* (H1) – (H5) *hold and that* $\lim_{t\to\infty} R(t). = \infty$ *Then every solution of Eq.* (1.1) *is oscillatory provided that for each* $l \geq t_0$ *and some* $\lambda >$ > 1, *the following two inequalities hold:*

$$\limsup_{t\to\infty} \frac{\mu}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^\lambda Kq(s)ds > \frac{\lambda^2}{4(\lambda-1)} \qquad (2.17)$$

and

$$\limsup_{t\to\infty} \frac{\mu}{R^{\lambda-1}(t)} \int_l^t [R(t)-R(s)]^\lambda Kq(s)ds > \frac{\lambda^2}{4(\lambda-1)}. \tag{2.18}$$

The proof is similar to that of Theorem 2.5 of Li and Agarwal [7], we omit it here.

3. Oscillation results for $f(x)$ without monotonity. In this section we consider the oscillation of Eq. (1.1) when the function $f(y)$ is not monotonous. In this case we always assume the following condition holds:

$$\text{(H5}'\text{)}\quad f(y)/y \ge \mu_0 > 0 \quad \text{for } y \ne 0, \quad \text{where } \mu_0 \text{ is a constant.} \tag{3.1}$$

Lemma 3.1. *Let assumptions* (H1) – (H4) *and* (H5′) *hold and* y *be a solution of Eq.* (1.1) *such that* $y(t) > 0$ *on* $[c, b)$. *For any* $v \in C^1([t_0, \infty), (0, \infty))$, *let*

$$w(t) = v(t)\frac{r(t)y'(t)}{y(t)} \tag{3.2}$$

on $[c, b)$. *Then for any* $H \in X$,

$$\int_c^b H(b,s)K\mu_0 v(s)q(s)ds \le$$

$$\le H(b,c)w(c) + \frac{1}{4}\int_c^b r(s)v(s)\left[h_2(b,s) - \frac{v'(s)}{v(s)}\sqrt{H(b,s)}\right]^2 ds. \tag{3.3}$$

Proof. From (1.1) and (3.2) we have for $s \in [c, b)$

$$w'(t) = -v(t)q(t)\frac{f(y(t))}{y(t)}g(y'(t)) - \frac{1}{r(t)v(t)}w^2(t) + \frac{v'(t)}{v(t)}w(t). \tag{3.4}$$

In view of $f(y)/y \ge \mu_0 > 0$ and $g(y') \ge K > 0$, we obtain

$$w'(t) + K\mu_0 v(t)q(t) + \frac{1}{r(t)v(t)}w^2(t) - \frac{v'(t)}{v(t)}w(t) \le 0. \tag{3.5}$$

The rest of the proof is similar to that of Lemma 2.1. The proof is complete.

Lemma 3.2. *Let assumptions* (H1) – (H4) *and* (H5′) *hold and suppose that* y *is a solution of Eq.* (1.1) *such that* $y(t) > 0$ *on* $(a, c]$. *For any* $v \in C^1([t_0, \infty), (0, \infty))$, *let* $w(t)$ *be defined by* (3.2) *on* $(a, c]$. *Then for any* $H \in X$,

$$\int_a^c H(s,a)K\mu_0 v(s)q(s)ds \le$$

$$\le -H(c,a)w(c) + \frac{1}{4}\int_a^c r(s)v(s)\left[h_1(s,a) + \frac{v'(s)}{v(s)}\sqrt{H(s,a)}\right]^2 ds.$$

The following theorem is an immediate result from Lemmas 3.1 and 3.2.

Theorem 3.1. *Assume that* (H1) – (H4) *and* (H5′) *hold and that for some* $c \in (a, b)$ *and for some* $H \in X$, $v \in C^1([t_0, \infty), (0, \infty))$,

$$\frac{1}{H(c,a)}\int_a^c H(s,a)K\mu_0 v(s)q(s)ds + \frac{1}{H(b,c)}\int_c^b H(b,s)\mu_0 v(s)q(s)ds >$$

$$> \frac{1}{4H(c,a)} \int_a^c r(s)v(s)\left[h_1(s,a) + \frac{v'(s)}{v(s)}\sqrt{H(s,a)} \right]^2 ds +$$

$$+ \frac{1}{4H(b,c)} \int_c^b r(s)v(s)\left[h_2(b,s) - \frac{v'(s)}{v(s)}\sqrt{H(b,s)} \right]^2 ds.$$

Then every solution of Eq. (1.1) *has at least one zero in* (a, b).

Theorem 3.2. *Assume that* (H1) – (H4) *and* (H5′) *hold. If, for each* $T \ge t_0$, *there exist* $H \in X$, $v \in C^1([t_0, \infty), (0, \infty))$ *and* $a, b, c \in R$ *such that* $T \le a < c < b$ *and* (3.6) *holds, then every solution of Eq.* (1.1) *is oscillatory.*

Theorem 3.3. *Assume that* (H1) – (H4) *and* (H5′) *hold. If*

$$\limsup_{t\to\infty} \int_l^t \left[H(s,l)K\mu_0 v(s)q(s) - \frac{1}{4} r(s)v(s)\left(h_1(s,l) + \frac{v'(s)}{v(s)}\sqrt{H(s,l)} \right)^2 \right] ds > 0$$

and

$$\limsup_{t\to\infty} \int_l^t \left[H(t,s)K\mu_0 v(s)q(s) - \frac{1}{4} r(s)v(s)\left(h_2(t,s) - \frac{v'(s)}{v(s)}\sqrt{H(t,s)} \right)^2 \right] ds > 0$$

for some $H \in X$, $v \in C^1([t_0, \infty), (0, \infty))$ *and for each* $l \ge t_0$, *then every solution of Eq.* (1.1) *is oscillatory.*

Theorem 3.4. *Assume that* (H1) – (H4) *and* (H5′) *hold. If for each* $T \ge t_0$, *there exist* $H \in X_0$, $v \in C^1([t_0, \infty), (0, \infty))$ *and* $a, c \in R$ *such that* $T \le a < c$ *and*

$$\int_a^c H(s-a)K\mu_0[v(s)q(s) + v(2c-s)q(2c-s)]ds >$$

$$> \frac{1}{4}\int_a^c [r(s)v(s) + r(2c-s)v(2c-s)]h^2(s-a)ds +$$

$$+ \frac{1}{2}\int_a^c [r(2c-s)v'(2c-s) - r(s)v'(s)]h(s-a)\sqrt{H(s-a)}ds +$$

$$+ \frac{1}{4}\int_a^c \left[\frac{(v'(s))^2 r(s)}{v(s)} + \frac{(v'(2c-s))^2 r(2c-s)}{v(2c-s)} \right] H(s-a)ds,$$

then every solution of Eq. (1.1) *is oscillatory.*

Corollary 3.1. *Assume that* (H1) – (H4) *and* (H5′) *hold. Then every solution of Eq.* (1.1) *is oscillatory provided that for each* $l \ge t_0$ *and for some* $\lambda > 1$, *there exists a function* $v \in C^1([t_0, \infty), (0, \infty))$ *such that the following two inequalities hold*:

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t (s-l)^\lambda \left[K\mu_0 v(s)q(s) - \frac{1}{4} r(s)v(s)\left(\frac{\lambda}{s-l} - \frac{v'(s)}{v(s)} \right)^2 \right] ds > 0$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t (t-s)^\lambda \left[K\mu_0 v(s)q(s) - \frac{1}{4} r(s)v(s)\left(\frac{\lambda}{t-s} + \frac{v'(s)}{v(s)} \right)^2 \right] ds < 0.$$

Theorem 3.5. *Assume that* (H1) – (H4) *and* (H5′) *hold and that* $\lim_{t\to\infty} R(t) = \infty$. *Then every solution of Eq.* (1.1) *is oscillatory provided that for each* $l \ge t_0$ *and for some* $\lambda > 1$, *the following two inequalities hold*:

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^\lambda K\mu_0 q(s)\,ds > \frac{\lambda^2}{4(\lambda-1)}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t [R(t) - R(s)]^\lambda K\mu_0 q(s)\,ds > \frac{\lambda^2}{4(\lambda-1)}.$$

4. Examples. In this section we will show the applications of our oscillation criteria by two examples. Based on the results in Sections 2 and 3 we will see that the equations in the examples are oscillatory, whereas the oscillation cannot be demonstrated by the results of Huang [12], Kong [15] and Li and Agarwal [7].

1. Consider the nonlinear differential equation

$$\left(\frac{1}{2t} y'(t)\right)' + \frac{2t\gamma}{(t^2-1)^2} y(t)\left(1 + y^2(t)\right)\left[1 + (y'(t))^2\right] = 0, \quad t \ge 1. \tag{4.1}$$

Let $r(t) = 1/2t$, $f(y) = y(1+y^2)$ and $g(y) = 1 + y^2$. Then

$$K = 1, \quad R(t) = t^2 - 1,$$

$$f'(y) = 1 + 3y^2 \ge 1 = \mu, \quad g(y) = 1 + y^2 \ge 1.$$

Note that for $\lambda > 1$,

$$\lim_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^\lambda q(s)\,ds =$$

$$= \lim_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^\lambda \frac{2\gamma s}{(s^2-1)^2}\,ds =$$

$$= \lim_{t\to\infty} \frac{[R(t) - R(l)]^\lambda}{(\lambda-1) R^{\lambda-1}(t) R'(t)} \frac{2\gamma t}{(t^2-1)^2} = \frac{\gamma}{\lambda-1}. \tag{4.2}$$

Next, we will prove that

$$\int_l^t [R(t) - R(s)]^\lambda \frac{2\gamma s}{(s^2-1)^2}\,ds \ge \int_l^t [R(s) - R(l)]^\lambda \frac{2\gamma s}{(s^2-1)^2}\,ds. \tag{4.3}$$

Let

$$F(t) = \int_l^t \left\{[R(t) - R(s)]^\lambda - [R(s) - R(l)]^\lambda\right\} \frac{2\gamma s}{(s^2-1)^2}\,ds.$$

Then $F(l) = 0$ and for $t \ge l$,

$$F'(t) = \int_l^t \lambda [R(t) - R(s)]^{\lambda-1} R'(t) \frac{2\gamma s}{(s^2-1)^2}\,ds - [R(t) - R(s)]^\lambda \frac{2\gamma t}{(t^2-1)^2} \ge$$

$$\geq \int_l^t \lambda[R(t)-R(s)]^{\lambda-1}R'(s)\frac{2\gamma s}{(s^2-1)^2}ds - [R(t)-R(l)]^\lambda \frac{2\gamma t}{(t^2-1)^2} \geq$$

$$\geq \frac{2\gamma t}{(t^2-1)^2}\int_l^t \lambda[R(t)-R(s)]^{\lambda-1}R'(s)ds - [R(t)-R(l)]^\lambda \frac{2\gamma t}{(t^2-1)^2} = 0.$$

Hence $F(t) \geq F(l) = 0$ for $t \geq l$, i.e., (4.3) holds. By (4.2) and (4.3), for any $\gamma > 1/4$ there exists $\lambda > 1$ such that $\gamma/(\lambda-1) > \lambda^2/4(\lambda-1)$. This implies that (2.17) and (2.18) hold for the same λ. Applying Theorem 2.5, we find that (4.1) is oscillatory for $\gamma > 1/4$.

2. Consider the nonlinear differential equation

$$\left((1+\sin^2 t)y'(t)\right)' + \frac{2(1+\cos^2 t)(1+3\sin^2 t)}{(3+\cos^2 t)(1+\sin^2 t)}y(t)\left(\frac{1}{8}+\frac{1}{1+y^2(t)}\right)\left(1+(y'(t))^2\right) = 0, \tag{4.4}$$

where $t \geq 1$. Observe that

$$f(y) = y\left(\frac{1}{2}+\frac{1}{1+y^2}\right) \quad \text{and} \quad f'(y) = \frac{(y^2-3)^2}{8(1+y^2)^2}.$$

Clearly, theorem 2.5 cannot apply to Eq. (4.4). In spite of this, with $v(t) = t^2$ and $\lambda = 2$, we can prove the oscillatory character of Eq. (4.4) by Corollary 3.1. Because for all $y \in R \setminus \{0\}$,

$$\frac{f(y)}{y} = \frac{1}{2}+\frac{1}{1+y^2} \geq \frac{1}{2} = \mu_0,$$

and $g(y) = 1+y^2 \geq 1 = K$. Thus,

$$\lim_{t\to\infty}\frac{1}{t}\int_l^t (s-l)^2\left[\frac{1}{2}v(s)q(s) - \frac{1}{4}r(s)v(s)\left(\frac{2}{s-l}-\frac{v'(s)}{v(s)}\right)^2\right]ds =$$

$$= \lim_{t\to\infty}\frac{1}{t}\int_l^t (s-l)^2 s^2\left[\frac{1}{2}\frac{2(1+\cos^2 s)(1+3\sin^2 s)}{(3+\cos^2 s)(1+\sin^2 s)} - (1+\sin^2 s)\frac{l^2}{s^2(s-l)^2}\right]ds \geq$$

$$\geq \lim_{t\to\infty}\frac{1}{t}\int_l^t \left[\frac{1}{8}(s-l)^2 s^2 - 2l^2\right]ds = \infty$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_l^t (t-s)^2\left[\frac{1}{2}v(s)q(s) - \frac{1}{4}r(s)v(s)\left(\frac{2}{t-s}+\frac{v'(s)}{v(s)}\right)^2\right]ds =$$

$$= \lim_{t\to\infty}\frac{1}{t}\int_l^t (t-s)^2 s^2\left[\frac{1}{2}\frac{2(1+\cos^2 s)(1+3\sin^2 s)}{(3+\cos^2 s)(1+\sin^2 s)} - (1+\sin^2 s)\frac{l^2}{s^2(t-s)^2}\right]ds \geq$$

$$\geq \lim_{t\to\infty}\frac{1}{t}\int_l^t \left[\frac{1}{8}(t-s)^2 s^2 - 2t^2\right]ds = \infty.$$

Thereby, Eq. (4.4) is oscillatory by Corollary 3.1. Observe that $y(t) = \cos t$ is an oscillatory solution of Eq. (4.4).

3. Consider the nonlinear differential equation

$$(r(t)y'(t))' + q(t)\big(y(t)+y^3(t)\big)\big[1+(y'(t))^2\big] = 0, \tag{4.5}$$

where $q(t)$ is defined as following

$$q(t) = \begin{cases} 2(t-3n), & 3n \le t \le 3n+1, \\ 2(-t+3n+2), & 3n+1 < t \le 3n+2, \\ -n, & 3n+2 < t < 3n+3, \end{cases}$$

$n \in N_0 = \{0, 1, 2, \ldots\}$, and $r(t) = 1+\sin^2 t$. For any $T \ge 0$ there exists $n \in N_0$ such that $3n \ge T$. Let $a = 3n$, $c = 3n+1$, and $v(t) \equiv 1$. Pick up $H(t-s) = (t-s)^2$, then $h(t-s) = 2$. Since $f'(y) = 1+3y^2 \ge 1 = \mu$, $g(y) = 1+y^2 \ge 1 = K$, then

$$\int_a^c H(s-a)K[q(s)+q(2c-s)]\,ds =$$

$$= \int_{3n}^{3n+1} (s-3n)^2[2(s-3n)+2(6n+2-s-3n)]\,ds = 4\int_{3n}^{3n+1}(s-3n)^2\,ds = \frac{4}{3}$$

and

$$\frac{1}{4\mu}\int_a^c [r(s)v(s)+r(2c-s)v(2c-s)]h^2(s-a)\,ds +$$

$$+ \frac{1}{2\mu}\int_a^c \big[r(2c-s)v'(2c-s) - r(s)v'(s)\big]h(s-a)\sqrt{H(s-a)}\,ds +$$

$$+ \frac{1}{4\mu}\int_a^c \left[\frac{(v'(s))^2 r(s)}{v(s)} + \frac{(v'(2c-s))^2 r(2c-s)}{v(2c-s)}\right]H(s-a)\,ds =$$

$$= \frac{1}{4\mu}\int_{3n}^{3n+1} [r(s)+r(2c-s)]h^2(s-3n)\,ds =$$

$$= \int_{3n}^{3n+1} \frac{1}{4}\big[2+\sin^2 s+\sin^2(6n+2-s)\big]\,ds \le \frac{1}{4}\int_{3n}^{3n+1} 4\,ds = 1 < \frac{4}{3}.$$

This implies that (2.12) holds and hence every solution of Eq. (4.5) is oscillatory by Theorem 2.4. Note that in this equation we have $\int_0^\infty q(s) = -\infty$. However, the results of Kong [15] and Li and Agarwal [7] fail to apply to Eq. (4.5) since $r(t) = 1+\sin^2 t$ does not satisfy the condition $r(t) \equiv 1$.

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Received 17.03.2000,
after revision — 26.12.2000