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**F. Catino, M. M. Miccoli** (Univ. Lecce, Italy)

# THE UPPER SUBRINGS OF A RING*

# ВЕРХНІ ПІДКІЛЬЦЯ В КІЛЬЦІ

We describe maximal ideals of rings that are contained in the adjoint groups of their upper subrings.

Описано максимальні ідеали кілець, що містяться в приєднаних групах їхніх верхніх кілець.

**1. Introduction.** Let $R$ be an associative ring. For all $a, b \in R$ we set $a \circ b = ab - ba$ and $a * b = a + b + ab$. It is well-known that $(R, +, \circ)$ is a Lie ring and that $(R, *)$ is a monoid. We denote by $Q(R)$ the group of quasiregular elements of $R$, i.e. the group of invertible elements of $(R, *)$. If $a$ is an element of $Q(R)$ its inverse with respect to $*$ is denoted by $a^{-}$.

It is easily seen that for arbitrary subrings $A$ of an associative ring $R$ the equality $Q(A) = Q(R) \cap A$ does not hold but is also true for (one-sided) ideals $A$ of $R$.

We remark that if $Z_n(R)$ $(n \in \mathbf{N}_0$ is the $n$th center of Lie ring $(R, +, \circ)$, then $Z_n(R)$ is a subring of $R$ and $Q(Z_n) = Q(R) \cap Z_n)$ [1].

In this note about some subrings $A$ of an associative ring $R$ for which the equality $Q(A) = Q(R) \cap A$ holds, it is investigated. We will show that there exists a relation between these subrings and same particular ideals of $R$.

**2. The results.** Let $B$ be an additive submonoid of an associative ring $R$. Then

$$A := \{z \mid z \in R,\ z \circ R \subseteq B\} \tag{1}$$

is an associative subring of $R$ (cf. [2], Lemma 1). It is called a upper subring of $R$.

Relevant examples are the $n$th upper center $Z_n(R)$, $n \in \mathbf{N}_0$, of the associated Lie ring of $R$ and, also, the subrings of $R$

$$\overline{\gamma_n(R)} = \{z \mid z \in R,\ z \circ R \subseteq \gamma_{n+1}(R)\}, \quad n \in \mathbf{N},$$

called the closure of the lower central factor $\gamma_n(R) / \gamma_{n+1}(R)$ [3].

We remark that if $B$ is a Lie ideal of $R$, then $A$ is a Lie ideal of $R$ and $A/B$ is the center of Lie ring $R/B$.

Moreover we remark that, for upper subrings $A$ of an associative ring $A$, we have the equality $Q(A) = Q(R) \cap A$ holds (cf. [2], Lemma 1). It is easily seen that equality does not hold for arbitrary subrings $A$ but is also true for (one-sided) ideals $A$ of $R$.

Now, we see that an upper subring $A$ of $R$ is related to an ideal $F_A$ of $R$. In this case the subring $A$ is the center of an associative ring $R$, the ideal $F_A$ is the *strong center* of $R$ [4].

**Theorem 1.** *If $B$ is an additive submonoid of $R$ and $A$ as in* (1), *put*

$$F_A = \{z \mid z \in A,\ zR \subseteq A\}.$$

*Then $F_A$ is the largest ideal of $R$ which is contained in $A$.*

***Proof.*** Obviously $F_A$ is a right ideal of $R$. Now, let $z \in F_A$ and $y \in R$. For all $r \in R$ we have

$$yz \circ r = y \circ zr + z \circ ry \in B.$$

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Hence $yz \in A$. Moreover, for all $x, r \in R$ we have

$$yzx \circ r = y \circ zxr + zx \circ ry \in B.$$

Hence $yzx \in A$. Therefore, $F_A$ is an ideal of $R$. The remainder is an immediate consequence of the definition of $F_A$. The theorem is proved.

We remark that $F_A$ contains also any one-sided ideal of $R$ contained in $A$.

Moreover, we remark that if $B$ is a left ideal of $R$ and $A$ is as in (1), then $F_A \cap \cap B$ is the largest ideal of $R$ contained in $B$. In particular, if $B$ is a modular left ideal of $R$, then $F_A \cap B = (B : R)$ [5].

Clearly, $F_A \cap Q(A) = Q(F_A)$. In general we shall see that $Q(F_A)$ need not be an ideal of $R$. We introduce an ideal of $R$ contained in $Q(F_A)$ that is very similarly with the Jacobian radical of a ring.

**Theorem 2.** *If $B$ is an additive submonoid of $R$ and $A$ as in* (1), *we define*

$$J_A = \{z | z \in A, zR \subseteq Q(A)\}.$$

*Then $J_A$ is the largest ideal of $R$ which is contained in $Q(A)$.*

***Proof.*** Let us first prove that $J_A$ is an ideal of $R$. Let $a, b \in J_A$. Obviously, $a - -b \in A$. Moreover, if $r \in R$ then $ar \in Q(A)$ and $-br(1+(ar)^-) \in Q(A)$. It follows that

$$(a-b)r * \left(-br(1+(ar)^-) * ar\right)^- = 0.$$

This proves, by lemma 6.5 of [5], that $a, b \in J_A$.

Evidently $J_A$ is a right ideal of $R$. Now, if $z \in J_A$, then for all $x, y, r \in R$ we have

$$xz \circ r = x \circ zr + z \circ rx \in B,$$

$$(xz)y \circ r = x \circ zyr + zy \circ rx \in B.$$

Hence $xz \in A$ and $xyz \in A$. We have also that $zyx \in Q(A)$ and

$$xyz * \left(-xyz - x(zyx)^- zy\right) = -x\left((zyx)^- + zyx + zyx(zyx)^-\right)zy = 0.$$

Hence, by Lemma 6.5 of [5], $xzy \in Q(R) \cap A = Q(A)$. Therefore, $J_A$ is an ideal of $R$. Moreover, if $z \in J_A$, then $-z^2 \in Q(A)$. Then

$$z * \left((-z) * (-z^2)\right) = 0.$$

Thus, by Lemma 6.5 [5], $J_A \subseteq Q(A)$. The remainder is an immediate consequence of the definition of $J_A$. The theorem is proved.

We remark that $J_A$ contains also any one-side ideal of $R$ contained in $A$.

Moreover we remark that $J_A$ is different, in general, by Jacobson radical $J_A$ of $A$. But, clearly, if $A = R$ then $J_A = J(A)$.


1. *Laue H.* On the associated Lie ring and the adjoint group of a radical ring // Canad. Math. Bull. – 1984. – **27**. – P. 215–222.
2. *Catino F.* On a theorem of Gupta and Levin // Note Mat. – 1995. – **15**. – P. 131–137.
3. *Laue H.* On central chains in ring. – Lecce, 1990. – (Preprint / Univer. Lecce, № 6).
4. *Catino F., Miccoli M. M.* Note on strongly Lie nilpotent rings. – Lecce, 2000. – (Preprint / Univ. Lecce).
5. *McCoy N. H.* The theory of rings. – New York: Macmillan Comp., 1964.


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