UDC 517.9

**L. R. Berrone** (Univ. Nac. Rosario, Argentina)

# RESOLVENT KERNELS WHICH CONSTITUTE AN APPROXIMATION OF THE IDENTITY AND LINEAR HEAT TRANSFER PROBLEMS

# РЕЗОЛЬВЕНТНІ ЯДРА, ЩО Є АПРОКСИМАЦІЄЮ ОДИНИЦІ, ТА ЛІНІЙНІ ЗАДАЧІ ТЕПЛООБМІНУ

Sufficient conditions are obtained for a Volterra integral equation with kernel depending on an increasing parameter $\alpha$ to admit an approximation of the identity in $\alpha$ as resolvent kernel. In this case, the solution to the integral equation tends to zero when $\alpha$ tends to infinity and estimates in $L^\infty$ of this convergence are established. These results are applied in obtaining estimates of the convergence of linear heat transfer boundary conditions to Dirichlet ones, when the heat transfer coefficient tends to infinity.

Отримані достатні умови, при яких інтегральне рівняння Вольтерра з ядром, що залежить від зростаючого параметра $\alpha$, допускає наближення одиниці відносно $\alpha$ у вигляді резольвентного ядра. У цьому випадку розв'язок інтегрального рівняння прямує до нуля, коли $\alpha$ прямує до нескінченності, і отримані оцінки цієї збіжності в $L^\infty$. За допомогою цих результатів одержані оцінки збіжності лінійних граничних умов Діріхле, коли коефіцієнт теплообміну прямує до нескінченності.

**1. Introduction and preliminaries.** In this work we are interested in the behavior of the solution to the parametric linear Volterra integral equation

$$u_\alpha(t) = F(t) - \int_0^t k_\alpha(t-s)\, u_\alpha(s)\, ds, \qquad t > 0, \quad \alpha > 0, \tag{1}$$

when the parameter $\alpha$ tends to infinity. For every $\alpha > 0$, equation (1) is of convolution type and can be compactly written in the form $u_\alpha(t) = F(t) - (k_\alpha * u_\alpha)(t)$, $t > 0$, with $f * g$ denoting the convolution of the two functions $k_\alpha$ and $u_\alpha$. As is well known, the solution to this equation can be expressed by means of the resolvent kernel $\Gamma_\alpha(t)$; namely, if for every $\alpha > 0$, $\Gamma_\alpha(t)$ solves the equation

$$\Gamma_\alpha(t) = k(t) - \int_0^t k_\alpha(t-s)\, \Gamma_\alpha(s)\, ds, \qquad t > 0, \tag{2}$$

then the solution (1) admits the following representation

$$u_\alpha(t) = F(t) - \int_0^t \Gamma_\alpha(t-s)\, F(s)\, ds, \qquad t > 0, \quad \alpha > 0. \tag{3}$$

In the case in which for every $t > 0$, $u_\alpha(t) \to 0$ when $\alpha \uparrow +\infty$, it is deduced from (3) that

$$(\Gamma_\alpha * F)(t) = \int_0^t \Gamma_\alpha(t-s)\, F(s)\, ds \;\to\; F(t)$$

and we expect $\{\Gamma_\alpha\}$ to be something like an approximation of the identity. It is opportune to define the exact sense of what is meant by "approximation of the identity".

**Definition 1.** *A family* $\{f_\alpha\colon \alpha > 0\}$ *of real continuous functions defined on* $(0, +\infty)$ *verifying*

$A_1$) $f_\alpha(t) \ge 0, \quad t > 0$;

$A_2$) $\int_0^{+\infty} f_\alpha(s)\,ds = 1, \quad \alpha > 0$;

$A_3$) *for every* $\delta > 0, \quad \lim_{\alpha \uparrow \infty} \int_\delta^{+\infty} f_\alpha(s)\,ds = 0$;

*is said to be an approximation of the identity or, in short, an approximate identity.*

The following convergence properties of convolutions with an approximate identity $\{f_\alpha \colon \alpha > 0\}$ are classically known [1–3]:

(i) if $F \in C^0(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ then $f_\alpha * F \to F$ in $C^0(\mathbb{R}^+)$ when $\alpha \uparrow +\infty$;

(ii) if $F$ is picewise-continuous and locally bounded on $\mathbb{R}_0^+$, then $f_\alpha * F \to F$ uniformly on compact subsets of the intervals of continuity of $F$ when $\alpha \uparrow +\infty$.

In what follows, we relax the restriction $A_2$ on the family $\{f_\alpha \colon \alpha > 0\}$ preserving convergence properties (i) and (ii) for the convolution $f_\alpha * F$.

**Lemma 1.** *Assume that conditions* $A_1$ *and* $A_3$ *are satisfied by the family* $\{f_\alpha \colon \alpha > 0\}$. *Furthermore, suppose that* $\mu(\alpha) = \int_0^{+\infty} f_\alpha(s)\,ds$ *satisfies the property*

$A_2^*$) $0 < \mu(\alpha) \to 1$ *when* $\alpha \uparrow \infty$.

*Then, the convergence properties of the convolution* $f_\alpha * F$ *stated by* (i) *and* (ii) *are preserved.*

Let us call *a generalized approximate identity* to a family of function $\{f_\alpha \colon \alpha > 0\}$ satisfying properties $A_1$, $A_2^*$, and $A_3$.

***Proof.*** It is sufficient to note that the family $\{f_\alpha / \mu(\alpha) \colon \alpha > 0\}$ is an approximate identity and that

$$|F(t) - (f_\alpha * F)(t)| \le \left| F(t) - \frac{1}{\mu(\alpha)} (f_\alpha * F)(t) \right| + \\ + |1 - \mu(\alpha)| \left| \frac{1}{\mu(\alpha)} (f_\alpha * F)(t) \right|, \qquad t > 0. \tag{4}$$

In fact, take for instance a piecewise-continuous function $F$ locally bounded on $\mathbb{R}_0^+$. Then, the first term on the right hand side of (4) converges to zero uniformly on compact subsets of the intervals of continuity of $F$ by (ii). For the second term we have

$$|1 - \mu(\alpha)| \left| \frac{1}{\mu(\alpha)} (f_\alpha * F)(t) \right| \le |1 - \mu(\alpha)| \, \|F\|_{\infty,t} \to 0$$

uniformly on compact subsets of $\mathbb{R}^+$ by $A_2^*$. The convergence $f_\alpha * F \to F$ in case (i) is proved analogously.

In Section 2 a set of conditions on the kernels $\{k_\alpha\}$ of equation (1) is established in order that family of resolvent kernels $\{\Gamma_\alpha\}$ is a generalized approximate identity.

An estimate of the rate of convergence to zero of the solution $u_\alpha$ is also given. In Section 3, these tools are employed for the study of the behavior of the solution to problems

$$\begin{aligned} u_t - u_{xx} &= 0, \quad x > 0, \quad t > 0, \\ u(x, 0) &= \theta_0(x), \quad x > 0, \\ u_x(0, t) &= \alpha(u(0, t) - g(t)), \quad t > 0; \end{aligned} \tag{5}$$

and

$$\begin{aligned} u_t - u_{xx} &= 0, \quad 0 < x < 1, \quad t > 0, \\ u(x, 0) &= \theta_0(x), \quad 0 < x < 1, \\ u_x(0, t) &= \alpha(u(0, t) - g(t)), \quad t > 0, \\ u(1, t) &= b(t), \quad t > 0, \end{aligned} \tag{6}$$

when the parameter $\alpha \uparrow +\infty$. Standard procedures to analyse this bahavior involve asymptotic expansions and scaling techniques, hence the developed in Section 2 can be considered as an alternative way to accomplish this analysis in the one-dimensional case.

For future use, we recall the expression of the fundamental solution to the one-dimensional heat equation,

$$K(x, t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}},$$

and also those for the Green and Neumann functions, respectively

$$\begin{aligned} G(x, \xi, t) &= K(x - \xi, t) - K(x + \xi, t), \\ N(x, \xi, t) &= K(x - \xi, t) + K(x + \xi, t). \end{aligned}$$

**2. Families of kernels whose associated resolvents are generalized approximate identities.** In this section, appropriate tools are developed which enable us to decide when a family of Volterra integral equations of the form (1) possesses a family of resolvent kernels $\{\Gamma_\alpha : \alpha > 0\}$ which is a generalized approximate identity. First of all, we consider the following non-parametric linear Volterra integral equation

$$u(t) = F(t) - \int_0^t k(t - s)\, u(s)\, ds, \quad t > 0, \tag{7}$$

where the kernel $k \in C^0(\mathbb{R}^+) \cap L^1(0, 1)$ and $F$ ia a piecewise-continuous function defined on $\mathbb{R}^+$. It is known that (7) admits a unique piecewise-continuous solution $u(t)$, $t > 0$ (see [1] and references in [4]). Hence, the resolvent kernel corresponding to equation (7); that is, the solution $\Gamma$ to the equation

$$\Gamma(t) = k(t) - \int_0^t k(t - s)\, \Gamma(s)\, ds, \quad t > 0, \tag{8}$$

is a continuous function of $t \in \mathbb{R}^+$. Moreover $\Gamma \in L^1(0, 1)$ [1, 5]. Aside from this, in the applications of Section 3 it will be sufficient to consider kernels of the form $k(t) = k_0(t)/t^\gamma$, $t > 0$, with $k_0 \in C^0[0, +\infty]$ and $0 \le \gamma < 1$, for which the classical theory of Volterra integral equations can be successfully applied (see, for example [6, 7]). Under additional assumptions on $k$, two lemmas which establish useful properties of the resolvent kernel $\Gamma$ are proved. The first one ensures the non-negativity of $\Gamma$ when $k$ is positive and logarithmically convex.

**Lemma 2.** *If the kernel* $k \in C^0(\mathbb{R}^+) \cap L^1(0, 1)$ *satisfies the conditions*

(i) $k(t) > 0,\ t > 0,$ *and* $\lim\limits_{t \downarrow 0} k(t) > 0$;

(ii) *for each* $T > 0$ *the function* $t \mapsto \dfrac{k(t)}{k(t+T)}$ *is non-decreasing for* $t > 0$;

*then the resolvent kernel* $\Gamma(t)$ *satisfies*

$$\Gamma(t) \geq 0, \qquad t > 0.$$

It is shown in Remark 1 below that condition (ii) is equivalent to the convexity of $t \mapsto \ln k(t)$. Then $t$ itself is a convex function and existence of $\lim\limits_{t \downarrow 0} k(t)$ in (i) is not so restrictive as it might seem. Note that (i) does not exclude the case $\lim\limits_{t \downarrow 0} k(t) = +\infty$.

***Proof.*** First, the case in which $0 < k(t) < +\infty$ is to be considered. In this case, $k$ is continuous up to $0$ and, taking into account that $\Gamma \in C^0(\mathbb{R}^+) \cap L^1(0, 1)$, we see that

$$\left| \int_0^t k(t-s)\,\Gamma(s)\,ds \right| \leq \| k \|_{\infty, t} \|\Gamma\|_{1, t} \to 0 \quad \text{as} \quad t \downarrow 0,$$

whence

$$\lim_{t \downarrow 0} \Gamma(t) = \lim_{t \downarrow 0} \left( k(t) - \int_0^t k(t-s)\,\Gamma(s)\,ds \right) = \lim_{t \downarrow 0} k(t) > 0.$$

Thus, there exist a $\delta_0 > 0$ such that $\Gamma(t) > 0,\ t \in (0, \delta_0)$. Now, if we assume that $\Gamma(t_1) < 0$ for a certain $t_1 > 0$; then, the continuity of $\Gamma$ guarantees that the first point $t_0 > 0$ exists such that $\Gamma(t) \geq 0$ on $(0, t_0)$, $\Gamma(t_0) = 0$, and $\Gamma(t) < 0$ on the interval $(t_0, t_0 + \delta)$ for an appropriate $\delta > 0$. By choosing $t \in (t_0, t_0 + \delta)$ and applying the hypothesis (i) and (ii), we have

$$0 > \Gamma(t) = k(t) - \int_0^t k(t-s)\,\Gamma(s)\,ds =$$

$$= k(t) - \int_0^{t_0} k(t-s)\,\Gamma(s)\,ds - \int_{t_0}^{t} k(t-s)\,\Gamma(s)\,ds \geq$$

$$\geq k(t) - \int_0^{t_0} k(t-s)\,\Gamma(s)\,ds = \frac{k(t)}{k(t_0)} \left( k(t_0) - \int_0^{t_0} \frac{k(t-s)\,k(t_0)}{k(t)}\,\Gamma(s)\,ds \right) \geq$$

$$\geq \frac{k(t)}{k(t_0)} \left( k(t_0) - \int_0^{t_0} k(t_0 - s)\,\Gamma(s)\,ds \right) = \frac{k(t)}{k(t_0)}\,\Gamma(t_0) = 0.$$

Thus we arrive at a contradiction and $\Gamma(t) \geq 0,\ t > 0$, as the lemma asserts.

Now, assume that $\lim_{t\downarrow 0} k(t) = +\infty$ and consider the sequence $\Gamma_n$ of resolvent kernels corresponding to the translated kernels $k_n(t) = k\left(t+\frac{1}{n}\right)$; i.e., for every $n \in \mathbb{N}$, we take $\Gamma_n$ to be the solution to

$$\Gamma_n(t) = k\left(t+\frac{1}{n}\right) - \int_0^t k\left(t+\frac{1}{n}-s\right)\Gamma_n(s)\,ds, \qquad t>0.$$

Since $k_n$, $n\in\mathbb{N}$, satisfies conditions (i), (ii) and $0 < \lim_{t\downarrow 0} k_n(t) = k\left(\frac{1}{n}\right) < +\infty$, the first part of the proof applies to give $\Gamma_n \geq 0$ for every $n \in \mathbb{N}$. On the other hand, $k_n \to k$ when $n \uparrow +\infty$, the convergence being understood in the sense of $L^1_{\mathrm{loc}}[0,+\infty)$, the space of functions which are locally integrable on $[0,+\infty)$. Hence, $0 \leq \Gamma_n \to \Gamma$ when $n\uparrow +\infty$ in $L^1_{\mathrm{loc}}[0,+\infty)$ (cf. Theorem 3.1 in [1, p. 42]) and then $\Gamma \geq 0$. This concludes the proof.

In the next lemma, an important inequality is established for the function

$$t \mapsto \int_0^t \Gamma(s)\,ds.$$

**Lemma 3.** *Let the conditions*

(i) $k(t) > 0, \quad t>0;$

(ii) $t \mapsto k(t)$ *is non-increasing on* $(0,+\infty)$;

(iii) $\Gamma(t) \geq 0, \quad t>0;$

*hold. Then the resolvent kernel* $\Gamma(t)$ *becomes an integrable function on* $(0,+\infty)$ *and*

$$\frac{\int_0^t k(s)\,ds}{1+\int_0^t k(s)\,ds} \leq \int_0^t \Gamma(s)\,ds \leq 1, \qquad t>0. \tag{9}$$

*Furthermore, if in addition the kernel* $k(t)$ *satisfies the condition*

(iv) $\int_0^{+\infty} k(s)\,ds = +\infty;$

*then* $\int_0^t \Gamma(s)\,ds = 1$.

***Proof.*** From the hypothesis (i)–(iii) and equation (8), we obtain

$$\int_0^t \Gamma(s)\,ds = \int_0^t \Gamma(t-s)\,ds \leq$$

$$\leq \int_0^t \Gamma(t-s)\frac{k(s)}{k(t)}\,ds = 1-\frac{\Gamma(t)}{k(t)} \leq 1, \qquad t>0,$$

which is the right inequality in (9). To prove the remaining inequality, we define $\mu = \int_0^{+\infty}\Gamma(s)\,ds$ and consider the solution $x$ to the following Volterra integral equation

$$\mu_\alpha = \int_0^{+\infty} \Gamma_\alpha(s)\,ds, \qquad \alpha > 0.$$

Since Lemma 3 clearly holds for every $k_\alpha$, $\alpha > 0$, we have $0 < \mu_\alpha \le 1$, $\alpha > 0$, and we derive from inequalities (9)

$$0 \le \int_\delta^{+\infty} \Gamma_\alpha(s)\,ds = \mu_\alpha - \int_0^\delta \Gamma_\alpha(s)\,ds \le$$

$$\le \mu_\alpha - \frac{\int_0^\delta k_\alpha(s)\,ds}{1 + \int_0^\delta k_\alpha(s)\,ds} \le \frac{1}{1 + \int_0^\delta k_\alpha(s)\,ds}. \tag{13}$$

Hence, property $A_3$ follows from hypothesis (iv). Now, taking into account that $x \mapsto \dfrac{x}{1+x}$, $x > 0$, is an increasing function and using hypothesis (i), for $\delta > 0$ we obtain

$$\int_0^\delta k_\alpha(s)\,ds < \int_0^{+\infty} k_\alpha(s)\,ds.$$

Then, by using inequality (9), we deduce

$$\frac{\int_0^\delta k_\alpha(s)\,ds}{1 + \int_0^\delta k_\alpha(s)\,ds} \le \lim_{t \uparrow +\infty} \left( \frac{\int_0^t k_\alpha(s)\,ds}{1 + \int_0^t k_\alpha(s)\,ds} \right) \le \int_0^{+\infty} \Gamma_\alpha(s)\,ds \le 1, \qquad \alpha > 0.$$

Passing to the limit as $\alpha \uparrow +\infty$ and applying hypothesis (iv), we conclude that property $A_2^*$ also holds for the family $\{\Gamma_\alpha : \alpha > 0\}$, proving that this family is a generalized approximate identity. Finally, if property (v) is satisfied, then Lemma 3 ensures that $\int_0^{+\infty} \Gamma_\alpha(s)\,ds = 1$ for every $\alpha > 0$, so that $\{\Gamma_\alpha : \alpha > 0\}$ is an approximate identity.

Now, we combine the previous theorem and Lemma 1 to obtain the result on the limit behavior of the family $\{u_\alpha : \alpha > 0\}$ of solutions to equations (1) as $\alpha \uparrow +\infty$.

**Theorem 2.** *Let the family $\{k_\alpha : \alpha > 0\}$ of kernels of equations* (1) *satisfy the conditions of Theorem* 1 *and let the forcing function $F$ of these equations be piecewise-continuous and locally bounded on $\mathbb{R}_0^+$. Then for the family of solutions $\{u_\alpha : \alpha > 0\}$ one has $u_\alpha(t) \to 0$ uniformly on compact subsets of intervals of continuity of $F$. In particular, if $F$ is continuous and bounded on $\mathbb{R}^+$ then $u_\alpha(t) \to 0$ uniformly on compact subsets of $\mathbb{R}^+$.*

***Proof.*** By recalling the representation (3) for the solution $u_\alpha(t)$ of equation (1), the proof easily follows from Theorem 1 and Lemma 1.

The conclusions of Theorems 1 and 2 obviously hold provided there exists $\alpha_0 > 0$ such that conditions (i) – (iv) of Theorem 1 are verified only for $\alpha > \alpha_0$. In this way, Theorem 2 provides a useful criterion to decide the convergence to zero of the solutions $u_\alpha(t)$ to equation (7). Next, we deal with the problem of estimating the rate of convergence of $u_\alpha$. For this purpose, a useful notation is introduced. For a function $f$ defined on $\mathbb{R}^+$ and such that $f$ is bounded on every interval $(0, t)$, $t > 0$, $\|f\|_{\infty,t}$ denotes the sup norm of $f$ on the interval $(0, t)$. If $f$ is Hölder-continuous with exponent $0 < \beta \le 1$ in $[0, t]$ for every $t > 0$, $|f|_{\beta,t}$ indicates the Hölder seminorm of $f$ on $[0, t]$.

**Theorem 3.** *Suppose that the kernels $k_\alpha$ satisfy hypothesis* (i) – (iv) *from Theorem* 1. *If, for every $t > 0$, the forcing function $F$ of the equation* (1) *is bounded and Hölder-continuous with exponent $0 < \beta \le 1$ in the interval $[0, t]$, then the solution $u_\alpha$ to equation* (1) *satisfies*

$$|u_\alpha(t)| \le \inf_{\psi \in \Psi} \left( |F|_{\beta,t} (\psi(\alpha))^\beta + \frac{3\|F\|_{\infty,t}}{1 + \int_0^{\psi(\alpha)} k_\alpha(s)\, ds} \right), \tag{14}$$

*where the infimum is taken over the class of functions*

$$\Psi = \Bigg\{ \psi\colon \mathbb{R}^+ \to [0, t]\colon\ \psi \in C^0(\mathbb{R}^+);$$

$$\psi(0^+) = t;\ \ \psi(+\infty) = 0;\ \ \lim_{\alpha \uparrow \infty} \int_0^{\psi(\alpha)} k_\alpha(s)\, ds = +\infty \Bigg\}. \tag{15}$$

***Proof.*** First we show that the class (15) is not empty by explicitly constructing one of its members. For this purpose, by fixing $t > 0$ we define the sequence $\psi_0 = t/(n+1)$, $n \in \mathbb{N}_0$. In view of condition (iv) from Theorem 1, there exists a sequence $\{\alpha_n\colon n \in \mathbb{N}_0\} \subseteq \mathbb{R}_0^+$ such that $\alpha_0 = 0$, $\alpha_n < \alpha_{n+1}$, $n \in \mathbb{N}_0$, and

$$\int_0^{\psi_{n+1}} k_\alpha(s)\, ds \ge n, \quad \alpha_n \le \alpha < \alpha_{n+1}, \quad n \in \mathbb{N}_0.$$

Thus, the piecewise-linear function $\psi\colon \mathbb{R}_0^+ \to (0, t]$ such that $\psi(\alpha_n) = \psi_n$, $n \in \mathbb{N}_0$, belongs to the class $\Psi$. In fact, we obviously have $\psi \in C^0(\mathbb{R}^+)$ and $\psi(0^+) = \psi_0 == t$, $\psi(+\infty) = \lim_{n \uparrow \infty} \psi_n = 0$. Furthermore, for $\alpha_n \le \alpha < \alpha_{n+1}$ is

$$\int_0^{\psi(\alpha)} k_\alpha(s)\, ds \ge \int_0^{\psi_{n+1}} k_\alpha(s)\, ds \ge n;$$

and hence,

$$\lim_{\alpha \uparrow \infty} \int_0^{\psi(\alpha)} k_\alpha(s)\, ds = +\infty.$$

Now, to deduce the estimate (14) we consider a member $\psi$ of the class $\Psi$; then, taking into account that $0 < \mu_\alpha = \int_0^{+\infty} \Gamma_\alpha(s)\, ds \le 1$ and that $\Gamma_\alpha \ge 0$, from representation (3) of the solution $u_\alpha(t)$ we get

$$|u_\alpha(t)| = \left| F(t) - \int_0^t \Gamma_\alpha(t-s) F(s)\, ds \right| \le$$

$$\leq \int_0^t \Gamma_\alpha(s)|F(t) - F(t-s)|\,ds + |F(t)| \int_0^{+\infty} \Gamma_\alpha(s)\,ds + (1-\mu_\alpha)|F(t)| \leq$$

$$\leq \int_0^{\psi(\alpha)} \Gamma_\alpha(s)|F(t) - F(t-s)|\,ds + \int_{\psi(\alpha)}^t \Gamma_\alpha(s)|F(t) - F(t-s)|\,ds +$$

$$+ |F(t)| \left(1 - \int_0^{\psi(\alpha)} \Gamma_\alpha(s)\,ds\right) = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}). \tag{16}$$

Since $F(t)$ is Hölder-continuous with exponent $\beta$, we have the following estimate

$$(\mathrm{I}) \leq |F|_{\beta,t} \int_0^{\psi(\alpha)} \Gamma_\alpha(s)\, s^\beta\,ds \leq |F|_{\beta,t}\,(\psi(\alpha))^\beta \int_0^{\psi(\alpha)} \Gamma_\alpha(s)\,ds \leq$$

$$\leq |F|_{\beta,t}\,(\psi(\alpha))^\beta \int_0^{+\infty} \Gamma_\alpha(s)\,ds \leq |F|_{\beta,t}(\psi(\alpha))^\beta. \tag{17}$$

Moreover, since $F$ is bounded on every interval $[0, t]$, $t > 0$, we obtain

$$(\mathrm{II}) \leq 2\|F\|_{\infty,t} \int_{\psi(\alpha)}^t \Gamma_\alpha(s)\,ds \leq 2\|F\|_{\infty,t} \int_{\psi(\alpha)}^{+\infty} \Gamma_\alpha(s)\,ds,$$

but $\int_{\psi(\alpha)}^{+\infty} \Gamma_\alpha(s)\,ds \leq \left(1 + \int_0^{\psi(\alpha)} k_\alpha(s)\,ds\right)^{-1}$ by inequalities (13) in the proof of Theorem 1 ; therefore

$$(\mathrm{II}) \leq \frac{2\|F\|_{\infty,t}}{1 + \int_0^{\psi(\alpha)} k_\alpha(s)\,ds}. \tag{18}$$

As concerns (III), we get an estimate similar to (18) :

$$(\mathrm{III}) \leq \|F\|_{\infty,t} \left(1 - \int_0^{\psi(\alpha)} \Gamma_\alpha(s)\,ds\right) \leq \frac{\|F\|_{\infty,t}}{1 + \int_0^{\psi(\alpha)} k_\alpha(s)\,ds}. \tag{19}$$

In this way, from (16) – (19) we deduce

$$|u_\alpha(t)| \leq |F|_{\beta,t}(\psi(\alpha))^\beta + \frac{3\|F\|_{\infty,t}}{1 + \int_0^{\psi(\alpha)} k_\alpha(s)\,ds}.$$

Hence, by the arbitrariness of $\psi \in \Psi$, the estimate (14) follows.

In practice, explicit estimates for $|u_\alpha(t)|$ can be derived from the estimate (14) by considering appropriate subfamilies of $\Psi$. Some examples are provided in the next section. On the other hand, it is obvious that estimate (14) holds only for $\alpha > \alpha_0$ provided that conditions (i) – (iv) of Theorem 1 are imposed for $\alpha > \alpha_0$.

We close this section by making some observation on the hypothesis made to derive the results thus far exposed. In the next section, the usefulness of many of these observations will be shown.

***Remark* 1.** As it is pointed out by A. Friedman in [4], the condition (ii) of Lemma 2 is equivalent to the *log-convexity* of $k(t)$; that is, to the convexity of $t \mapsto \ln k(t)$. In fact, it is easily seen that both conditions on $k(t)$ are equivalent to the assumption

$k(t_1)k(t_2) \geq k(t_1+h)k(t_2-h)$, $t_1 < t_2$, $0 \leq h < t_2 - t_1$. In particular, if $k \in C^1(\mathbb{R}^+)$, the function $k(t)$ is log-convex if and only if $t \mapsto k'(t)/k(t)$ is non-decreasing. Moreover, if $k \in C^2(\mathbb{R}^+)$ we have $k(t)$ log-convex if and only if $k(t)k''(t) - (k'(t))^2 \geq 0$, $t > 0$. On the other hand, the conditions (i) – (iii) from Theorem 1 all hold for *completely monotone kernels* [5]; i.e., kernels $k \in C^\infty(\mathbb{R}^+)$ that satisfy $(-1)^n k^{(n)}(t) \geq 0$, $t > 0$, $n \in \mathbb{N}_0$. Since the kernels $t \mapsto t^{-\gamma}$, $0 < \gamma < 1$, are completely monotone, advantages of this fact are used in the applications of the next section. We observe that condition

(ii′) there exist $l = \lim_{t \uparrow \infty} \dfrac{k(t)}{k(t-1)}$ and $l \leq 1$;

in Theorem 1 can be supposed instead of condition (ii). Indeed, if in addition to (i) and (iii), condition (ii) is satisfied by $k(t)$, then the non-decreasing function $t \mapsto k(t)/k(t-1)$ satisfies $k(t)/k(t-1) \leq 1$, $t > 1$, hence the limit $l$ exists and $l \leq 1$. Conversely, if (ii′) is verified instead of (ii), for each $\alpha > 0$ we can define $L(a) = \lim_{t \uparrow \infty} (k(t)/k(t-a))$. In fact, it is easy to see that there exists $L(a+b)$ provided that there exist $L(a)$ and $L(b)$; furthermore, $L(a+b) = L(a)L(b)$. It follows from (ii′) that $L(r)$ exists for each $r \in \mathbb{Q}$ and, in view of the continuity of $(a, t) \mapsto k(t)/k(t-a)$, a standard argument shows that $L$ can be continuously extended to $\mathbb{R}^+$. Thus, $a \mapsto L(a)$ is a continuous function on $\mathbb{R}^+$ which satisfies the functional equation $L(a+b) = L(a)L(b)$, $a, b > 0$, and then $L(a) = (L(1))^a$, $a > 0$, (cf. [8]). Now, by choosing $0 < t_1 < t_2$, we obtain

$$\frac{k(t_1)}{k(t_2)} \geq \frac{k(t_2)}{k(t_2 + (t_2 - t_1))} \geq \ldots \geq \frac{k(t_2 + n(t_2 - t_1) - (t_2 - t_1))}{k(t_2 + n(t_2 - t_1))},$$

whence

$$\frac{k(t_1)}{k(t_2)} \geq \lim_{n \uparrow \infty} \frac{k(t_2 + n(t_2 - t_1) - (t_2 - t_1))}{k(t_2 + n(t_2 - t_1))} = \frac{1}{L(t_2 - t_1)} = \frac{1}{(L(1))^{t_2 - t_1}} \geq 1;$$

that is, $t \mapsto k(t)$ is a non-increasing function.

**3. Examples.** In this section we apply the results of the previous one to analyse the behaviour of solutions to problems (5) and (6) when the parameter $\alpha \uparrow +\infty$. These problems are normalized models of heat conduction in a semi-infinite (5) or a finite (6) slab with a linear heat transfer condition imposed at one of its ends. We note that the solutions to problems (5) and (6) respectively converge to the solutions of problems

$$\begin{aligned} u_t - u_{xx} &= 0, \quad x > 0, \quad t > 0, \\ u(x, 0) &= \theta_0(x), \quad x > 0, \\ u(0, t) &= g(t), \quad t > 0; \end{aligned} \tag{20}$$

and

$$\begin{aligned} u_t - u_{xx} &= 0, \quad 0 < x < 1, \quad t > 0, \\ u(x, 0) &= \theta_0(x), \quad 0 < x < 1, \\ u(0, t) &= g(t), \quad t > 0, \\ u(1, t) &= b(t), \quad t > 0. \end{aligned} \tag{21}$$

This is a well known fact [9 – 14] which has a clear physical meaning. The theorems established in the preceeding section enable us to give precise $L^{\infty}$ estimates for this convergence.

***Example* 1.** *The semi-infinite slab.* Here we assume that functions $\theta_0(x)$ and $g(t)$ arising in (5) (and (20)) are continuous and bounded for $x \geq 0$ and piecewise-continuous and bounded for $t > 0$, respectively. These assumptions are sufficient for problem (5) to admit a unique bounded solution (cf. [15]) which can be represented in the form

$$v_\alpha(x, t) = \int_0^{+\infty} N(x, \xi, t)\, \theta_0(\xi)\, d\xi - 2 \int_0^t K(x, t-\tau)\, \varphi(\tau)\, d\tau, \quad x > 0, \quad t > 0; \tag{22}$$

where $\varphi(t) = (\nu_\alpha)_x(0, t)$ satisfies the Volterra integral equation

$$\varphi(t) = \alpha \int_0^{+\infty} N(0, \xi, t)\, \theta_0(\xi)\, d\xi - \alpha\, g(t) - $$

$$- 2\alpha \int_0^t K(0, t-\tau)\, \varphi(\tau)\, d\tau, \quad t > 0. \tag{23}$$

The kernels $K(x, t)$ and $N(x, \xi, t)$ in (22) and (23) were defined in the introductory section. By defining $u_\alpha(t) = g(t) - v_\alpha(0, t)$, from the boundary conditions of the problem (5) we obtain

$$\varphi(t) = -\alpha\, u_\alpha(t); \tag{24}$$

thus, from (23) and (24) we finally arrive at the following Volterra integral equation for $u_\alpha(t)$

$$u_\alpha(t) = g(t) - \int_0^{+\infty} N(0, \xi, t)\, \theta_0(\xi)\, d\xi - 2\alpha \int_0^t K(0, t-\tau)\, u_\alpha(\tau)\, d\tau, \quad t > 0. \tag{25}$$

The equation (25) has the form (1) with

$$F(t) = g(t) - \int_0^{+\infty} N(0, \xi, t)\, \theta_0(\xi)\, d\xi, \quad k_\alpha(t) = 2\,\alpha\, K(0, t) = \frac{\alpha}{\sqrt{\pi t}}; \tag{26}$$

and the conditions (i) – (v) of Theorem 1 are satisfied. Particularly, conditions (i) – (iii) are straightforward because the kernels $k_\alpha(t)$, $\alpha > 0$, are completely monotone for $t > 0$.

Let $v(x, t)$ be the solution to problem (20) and denote by $v_\alpha(x, t)$ the solution to problem (5) for a given $\alpha > 0$. We are in position to prove the following result.

**Theorem 4.** *For a continuous and bounded $\theta_0(x)$, $x \geq 0$, and for a piecewise-continuous and bounded $g(t)$, $t \geq 0$, $v_\alpha(x, t)$ converges to $v(x, t)$ uniformly for $x \in \mathbb{R}^+$ and $t$ belonging to compact subsets of intervals of continuity of $g(t)$ as $\alpha \uparrow +\infty$. Furthermore, if the forcing function $F(t)$ given by (26) is Hölder-continuous with exponent $\beta$ in $[0, t]$ for each $t > 0$, then the difference $w_\alpha(x, t) = v(x, t) - v_\alpha(x, t)$ can be estimated as follows:*

$$|w_\alpha(x, t)| \leq \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) \inf_{\psi \in \Psi} \left( |F|_{\beta, t} (\psi(\alpha))^\beta + \frac{3\, \|F\|_{\infty, t}}{1 + \int_0^{\psi(\alpha)} k_\alpha(s)\, ds} \right),$$

$$x > 0, \quad t > 0, \quad \alpha > 0;$$

*where*

$$\Psi = \Big\{ \psi \colon \mathbb{R}^+ \to [0, t] \colon \ \psi \in C^0(\mathbb{R}^+);$$

$$\psi(0^+) = t; \ \psi(+\infty) = 0; \ \lim_{\alpha \uparrow \infty} \alpha \sqrt{\psi(\alpha)} = +\infty \Big\}. \tag{27}$$

***Proof.*** The difference $w_\alpha(x, t) = v(x, t) - v_\alpha(x, t)$ satisfies the problem

$$u_t - u_{xx} = 0, \quad x > 0, \quad t > 0,$$

$$u(x, 0) = 0, \quad x > 0,$$

$$u(0, t) = u_\alpha(t), \quad t > 0;$$

whose solution can be expressed (see [15]) in the form

$$w_\alpha(x, t) = -2 \int_0^t \frac{\partial K}{\partial x}(x, t - \tau)\, u_\alpha(\tau)\, d\tau, \quad x > 0, \quad t > 0.$$

Then, for $x > 0$, $t > 0$, we have

$$|w_\alpha(x, t)| \le \|u_\alpha\|_{\infty, t}\, 2 \int_0^t \left| \frac{\partial K}{\partial x}(x, t - \tau) \right| d\tau = \|u_\alpha\|_{\infty, t} \operatorname{erfc}\left( \frac{x}{2\sqrt{t}} \right). \tag{28}$$

Taking into account that $\operatorname{erfc}\left( \frac{x}{2\sqrt{t}} \right) < 1$, $x > 0$, $t > 0$, the convergence to zero of $w_\alpha(x, t)$ uniformly in $x \in \mathbb{R}^+$ and $t$ belonging to compact subsets of intervals of continuity of $g(t)$ follows from (28) and Theorem 2. On the other hand, by assuming $F(t)$ a Hölder-continuous function with exponent $\beta$ on each interval $[0, t]$, an application of Theorem 3 provides

$$\|u_\alpha\|_{\infty, t} \le \inf_{\psi \in \Psi} \left( |F|_{\beta, t} (\psi(\alpha))^\beta + \frac{3 \|F\|_{\infty, t}}{1 + \int_0^{\psi(\alpha)} k_\alpha(s)\, ds} \right), \tag{29}$$

where $\Psi$ is given by (27). The required estimate for $|w_\alpha(x, t)|$ follows from (28) and (29).

***Remark* 2.** When $\theta_0$ is a uniformly Hölder-continuous function on $[0, +\infty)$ with exponent $\lambda$, we have

$$\left| \int_0^{+\infty} N(0, \xi, t)\, \theta_0(\xi)\, d\xi - \theta_0(0) \right| \le 2 \int_0^{+\infty} K(\xi, t)\, |\theta_0(\xi) - \theta_0(0)|\, d\xi \le$$

$$\le \int_0^{+\infty} \frac{e^\lambda}{\sqrt{\pi t}} e^{-\xi^2/4t} |\theta_0|_\lambda\, d\xi = \frac{2^{1+\lambda}}{\sqrt{\pi}} |\theta_0|_\lambda \left( \int_0^{+\infty} \eta^\lambda e^{-\eta^2}\, d\eta \right) t^{\lambda/2},$$

so that the function

$$t \mapsto \int_0^{+\infty} N(0, \xi, t)\, \theta_0(\xi)\, d\xi$$

becomes uniformly Hölder-continuous on $[0, +\infty)$ with exponent $\lambda/2$. In this manner, by assuming $g(t)$ to be Hölder-continuous with exponent $\beta$ on every interval $[0, t]$, the forcing function $F(t)$ given by (26) satisfies

$$|F(t_1) - F(t_2)| \le |g|_{\beta,t} |t_1 - t_2|^{\beta} + C|t_1 - t_2|^{\lambda/2}, \quad t_1, t_2 \ge 0,$$

with a constant $C$ independent of $t$.

***Remark*** **3.** Since the function $\psi_p(\alpha) = \dfrac{t}{(1+\alpha)^p}$, $0 < p < 2$, belongs to the family (27), from estimate (29) in the proof of Theorem 4 we derive

$$|u_\alpha(t)| \le \inf_{0<p<2} \left( |F|_{\beta,t} \frac{t^{\beta}}{(1+\alpha)^{\beta p}} + \frac{3\|F\|_{\infty,t}}{1 + 2\sqrt{t/\pi}\,\dfrac{\alpha}{(1+\alpha)^{p/2}}} \right).$$

For the particular case of constant data; i.e., when $\theta_0(x) \equiv C_0$ and $g(t) \equiv C_1$ are constant, we have $F(t) \equiv C_1 - C_0$ and then

$$|u_\alpha(t)| \le \inf_{0<p<2} \left( \frac{3|C_1 - C_0|}{1 + 2\sqrt{t/\pi}\,\dfrac{\alpha}{(1+\alpha)^{p/2}}} \right) = \frac{3|C_1 - C_0|}{1 + 2\sqrt{t/\pi}\,\alpha}. \tag{30}$$

Now, we compare estimate (30) with that obtained via an exact calculation of the solution to (25). Indeed, the resolvent kernel corresponding to equation (25) can be computed by using the Abel method (see for example, [15 – 17]) or the Laplace transform,

$$\Gamma_\alpha(t) = \frac{\alpha}{\sqrt{\pi t}} - \alpha^2 e^{\alpha^2 t}\operatorname{erfc}(\alpha\sqrt{t}), \quad t > 0, \quad \alpha > 0;$$

whence the solution to (25) for the case $F(t) \equiv C_1 - C_0$ is given by

$$u_\alpha(t) = (C_1 - C_0)\left(1 - \int_0^t \left(\frac{\alpha}{\sqrt{\pi s}} - \alpha^2 e^{\alpha^2 s}\operatorname{erfc}(\alpha\sqrt{s})\right) ds\right) =$$

$$= (C_1 - C_0)\, e^{\alpha^2 t}\operatorname{erfc}(\alpha\sqrt{t}). \tag{31}$$

On the other hand, we have [18, p. 298]

$$\frac{2}{\sqrt{\pi}\,(x + \sqrt{x^2 + 2})} < e^{x^2}\operatorname{erfc}(x) \le \frac{2}{\sqrt{\pi}\,(x + \sqrt{x^2 + 4/\pi})}, \quad x > 0; \tag{32}$$

and therefore, it follows from (31) and (32) that

$$|u_\alpha(t)| \le \frac{2|C_1 - C_0|}{\sqrt{\pi}\,(\alpha\sqrt{t} + \sqrt{\alpha^2 t + 4/\pi})}. \tag{33}$$

Note that the asymptotical behavior as $\alpha \uparrow +\infty$ in the estimates (30) and (33) differ only by the constant factor $3\pi/2$.

***Example*** **2.** *The finite slab.* If $\theta_0$ is continuous on $[0, 1]$, the solution to problem (6) can be represented [15] in the form

$$v_\alpha(x, t) = v(x, t) - 2\int_0^t K(x, t-\tau)\,\varphi_1(\tau)\,d\tau +$$

$$+ 2\int_0^t \frac{\partial K}{\partial x}(x-1, t-\tau)\,\varphi_2(\tau)\,d\tau, \tag{34}$$

where

$$v(x,t) = \int_{-\infty}^{+\infty} K(x-\xi, t)\,\theta(\xi)\,d\xi,$$

and $\theta$ is a bounded continuous extension of $\theta_0$ to $\mathbb{R}$. In (34), $\varphi_1$ and $\varphi_2$ are piecewise-continuous solutions to the following system of Volterra integral equations

$$b(t) = v(1,t) - 2\int_0^t K(1, t-\tau)\,\varphi_1(\tau)\,d\tau + \varphi_2(t), \qquad t>0,$$

$$v_x(0,t) + \varphi_1(t) + 2\int_0^t \frac{\partial^2 K}{\partial x^2}(-1, t-\tau)\,\varphi_2(\tau)\,d\tau = \alpha\Bigg(v(0,t) -$$

$$- 2\int_0^t K(0, t-\tau)\,\varphi_1(\tau)\,d\tau + 2\int_0^t \frac{\partial K}{\partial x}(-1, t-\tau)\,\varphi_2(\tau)\,d\tau - g(t)\Bigg), \qquad t>0. \tag{35}$$

In order to simplify the exposition, we express these equations in a compact form by using convolutions. Thus, by defining $u_\alpha(t) = g(t) - v_\alpha(0,t)$, $t>0$, from (34) and (35) the following system of Volterra integral equations is derived

$$\alpha u_\alpha(t) + \varphi_1(t) + \left(2\frac{\partial^2 K}{\partial x^2}(-1,\cdot) * \varphi_2\right)(t) = -v_x(0,t),$$

$$u_\alpha(t) - \left(2K(0,\cdot) * \varphi_1\right)(t) + \left(2\frac{\partial K}{\partial x}(-1,\cdot) * \varphi_2\right)(t) = g(t) - v(0,t), \tag{36}$$

$$-\left(2K(1,\cdot) * \varphi_1\right)(t) + \varphi_2(t) = b(t) - v(1,t),$$

whence a single Volterra integral equation for $u_\alpha(t)$ can be deduced. Indeed, after a tedious but not difficult calculation, system (36) provides an equation of the form (1) for $u_\alpha(t)$, in which

$$k_\alpha(t) = 2\alpha K(0,t) + 4\left(K(1,\cdot) * \left(\frac{\partial^2 K}{\partial x^2}(-1,\cdot) - \alpha\frac{\partial K}{\partial x}(-1,\cdot)\right)\right)(t) =$$

$$= \frac{1}{\sqrt{\pi t}}\left(\alpha\left(1 - e^{-1/t}\right) + t^{-1} e^{-1/t}\right), \tag{37}$$

and

$$F(t) = g(t) - v(0,t) + \left(q_1 * \left(g - v(0,\cdot)\right)\right)(t) -$$

$$- \left(q_2 * v_x(0,\cdot)\right)(t) - \left(q_3 * \left(b - v(1,\cdot)\right)\right)(t), \tag{38}$$

where

$$q_1(t) = 4\left(K(1,\cdot) * \frac{\partial^2 K}{\partial x^2}(-1,\cdot)\right)(t) = \frac{1}{\sqrt{\pi}} t^{-3/2} e^{-1/t},$$

$$q_2(t) = 2\left(K(0,t) - 2K(1,\cdot) * \frac{\partial K}{\partial x}(-1,\cdot)\right)(t) = \frac{1}{\sqrt{\pi}} t^{-1/2}\left(1 - e^{-1/t}\right), \tag{39}$$

$$q_3(t) = 2\left(\left(\frac{\partial K}{\partial x}(-1,\cdot) + 2K(0,t)\right) * \frac{\partial^2 K}{\partial x^2}(-1,\cdot)\right)(t) = \frac{1}{\sqrt{\pi}} t^{-3/2} e^{-1/(4t)}.$$

The calculations involved in (37) and (39) have been accomplished by means of the Laplace transform. We note that $F$ is continuous and bounded on $\mathbb{R}^+$ provided that $g$ has the same properties. Now, we show that the kernels $k_\alpha(t)$ satisfy the restrictions of Theorem 1. To this end, we first note that $k_\alpha(t) > 0,\ t > 0$; that is, condition (i) of Theorem 1 is satisfied. Condition (iv) is quickly verified and it is easy to see that $\lim\limits_{t\uparrow\infty} \dfrac{k_\alpha(t)}{k_\alpha(t-1)} = 1,\ \alpha > 0$. By Remark 1, it is enough to prove that condition (iii) is also satisfied. This matter seems to be less trivial and deserves a more detailed consideration.

**Lemma 4.** *If* $\alpha > 1$, *then* $k_\alpha(t)\,k''_\alpha(t) - (k'_\alpha(t))^2 > 0,\ t > 0$.

***Proof.*** From (37) and (39) we see that

$$k_\alpha(t) = \alpha\, q_2(t) + q_1(t), \qquad t > 0, \quad \alpha > 0; \tag{40}$$

and therefore

$$k_\alpha(t)\,k''_\alpha(t) - (k'_\alpha(t))^2 = A(t)\alpha^2 + B(t)\alpha + C(t), \tag{41}$$

where, for each $t > 0$,

$$A(t) = q_2(t)\,q''_2(t) - (q'_2(t))^2,$$
$$B(t) = q_1(t)\,q''_2(t) + q_2(t)\,q''_1(t) - 2\,q'_1(t)\,q'_2(t),$$
$$C(t) = q_1(t)\,q''_1(t) - (q'_1(t))^2.$$

The right hand side of (41) is a quadratic polynomial in $\alpha$ whose discriminant $\delta(t) = B^2(t) - 4A(t)\,C(t) > 0,\ t > 0$; hence, to prove that this polynomial is positive for $\alpha > 1$, it is sufficient to show that $A(t) > 0,\ t > 0$, and that

$$\rho(t) = \frac{-B(t) + \sqrt{B^2(t) - 4A(t)\,C(t)}}{2A(t)} < 1, \qquad t > 0. \tag{42}$$

The simplest way to show that $A(t) > 0,\ t > 0$, consist, perhaps, in noting that $q_2(t)$ is a completely monotone function on $(0, +\infty)$; in fact, we have

$$q_2(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-st}\,\frac{1 - \cos(2\sqrt{s})}{\sqrt{s}}\,ds.$$

Thus, by the Hausdorff–Bernstein–Widder theorem (see [7]), $q_2(t)$ is completely monotone on $(0, +\infty)$. To show (42), after tedious calculations we find

$$\rho(t) = \frac{1}{t\left(t^2 e^{2/t} - 2e^{1/t}(t-1)^2 + t(t-4)\right)}\Big(-e^{1/t}(3t^2 - 4t + 1) + t(3t - 4) +$$
$$+ \sqrt{e^{2/t}(6t^4 - 20t^3 + 22t^2 - 8t + 1) - 4t^2 e^{1/t}(3t^2 - 7t + 4) + 6t^4 - 8t^3}\Big),$$

whence we deduce

$$\lim_{t\downarrow 0}\rho(t) = 0, \quad \rho(4/3) = 0; \quad \rho(t) > 0,\ 0 < t < 4/3; \quad \rho(t) < 0,\ t > 4/3;$$

$$\lim_{t\uparrow\infty}\rho(t) = -\frac{\sqrt{6}}{2}.$$

The proof is completed by observing that $0 < \max_{0 \le t \le 4/3} \rho(t) < 1$.

Note that, unlike to the kernels (26), which satisfy condition (v) from Theorem 1, the kernels (37) corresponding to the finite slab are intgrable on $\mathbb{R}^+$.

Now we prove a theorem similar to Theorem 4 for the finite slab. As before, we denote by $v_\alpha(x, t)$ the solution to problem (6) and by $v(x, t)$ that corresponding to problem (21).

**Theorem 5.** *For a continuous* $\theta_0(x)$, $0 \le x \le 1$, *and for continuous* $g(t)$ *and* $b(t)$, $t \ge 0$, *the solution to problem* (6) *converges to that of problem* (21) *when* $\alpha \uparrow +\infty$. *The convergence is uniform on subsets of the form* $(x, t) \in [0, 1] \times [0, T]$, $T > 0$. *Furthermore, if the forcing function* $F(t)$ *given by* (38), (39) *is Hölder-continuous with exponent* $\beta$ *in* $[0, t]$ *for each* $t > 0$; *then, the following estimate holds for the difference* $w_\alpha(x, t) = v(x, t) - v_\alpha(x, t)$:

$$|w_\alpha(x, t)| \le \inf_{\psi \in \Psi} \left( |F|_{\beta, t} (\psi(\alpha))^\beta + \frac{3 \|F\|_{\infty, t}}{1 + \int_0^{\psi(\alpha)} k_\alpha(s)\, ds} \right), \tag{43}$$

$$0 < x < 1, \quad t > 0, \quad \alpha > 1;$$

*where* $\Psi$ *is the class of function defined by* (27).

***Proof.*** As in the proof of Theorem 4, we see that $w_\alpha(x, t) = v(x, t) - v_\alpha(x, t)$ satisfies the problem

$$\begin{aligned} u_t - u_{xx} &= 0, \quad 0 < x < 1, \quad t > 0, \\ u(x, 0) &= 0, \quad 0 < x < 1, \\ u(0, t) &= u_\alpha(t), \quad t > 0, \\ u(1, t) &= 0, \quad t > 0; \end{aligned} \tag{44}$$

then by using the maximum principle and the Vyborny lemma [15] we conclude that the solution to (44) satisfies

$$\| w_\alpha \|_{\infty, t} = \sup_{x \in [0,1],\ \tau \in (0,t)} |w_\alpha(x, \tau)| \le \| u_\alpha \|_{\infty, t}.$$

Thus, the uniform convergence $w_\alpha \to 0$ on every subset $[0, 1] \times [0, T]$, $T > 0$, immediately follows from Theorem 2. Now, to prove the estimate (43) by application of Theorem 3, we only need to prove that

$$\lim_{\alpha \uparrow \infty} \int_0^{\psi(\alpha)} k_\alpha(s)\, ds = +\infty \quad \Leftrightarrow \quad \lim_{\alpha \uparrow \infty} (\alpha \sqrt{\psi(\alpha)}) = +\infty. \tag{45}$$

To this end, we can easily compute, for a given $\psi \in \Psi$,

$$\int_0^{\psi(\alpha)} k_\alpha(s)\, ds = \int_0^{\psi(\alpha)} \frac{1}{\sqrt{\pi t}} (\alpha (1 - e^{-1/t}) + t^{-1} e^{-1/t})\, ds =$$

$$= \frac{2}{\sqrt{\pi}} (1 - e^{-1/\psi(\alpha)}) \alpha \sqrt{\psi(\alpha)} + (1 + 2\alpha) \operatorname{erfc}\left( \frac{1}{\sqrt{\psi(\alpha)}} \right). \tag{46}$$

Since $\psi(\alpha) \to 0$ when $\alpha \uparrow +\infty$, from (46) we deduce

$$\lim_{\alpha\uparrow\infty}\int_0^{\psi(\alpha)} k_\alpha(s)\,ds = \lim_{\alpha\uparrow\infty}\left(\frac{2}{\sqrt{\pi}}\alpha\sqrt{\psi(\alpha)}\right) + \lim_{\alpha\uparrow\infty}\left(2\alpha\,\mathrm{erfc}\left(\frac{1}{\sqrt{\psi(\alpha)}}\right)\right), \quad (47)$$

but from inequalities (32) we obtain

$$\frac{2\alpha\sqrt{\psi(\alpha)}\,e^{-1/\psi(\alpha)}}{\sqrt{\pi}(1+\sqrt{1+2\psi(\alpha)})} < \alpha\,\mathrm{erfc}\left(\frac{1}{\sqrt{\psi(\alpha)}}\right) \le$$

$$\le \frac{2\alpha\sqrt{\psi(\alpha)}\,e^{-1/\psi(\alpha)}}{\sqrt{\pi}(1+\sqrt{1+4\psi(\alpha)/\pi})}, \qquad \alpha>0;$$

whence

$$\lim_{\alpha\uparrow\infty}\left(\alpha\,\mathrm{erfc}\left(\frac{1}{\sqrt{\psi(\alpha)}}\right)\right) = \lim_{\alpha\uparrow\infty}\left(\frac{\alpha\sqrt{\psi(\alpha)}\,e^{-1/\psi(\alpha)}}{\sqrt{\pi}}\right). \quad (48)$$

The equivalence (45) easily follows from (46), (47) and (48).

A particular case of Theorem 5 is presented in the following remark.

***Remark* 4.** By assuming $\theta_0(x)\equiv C_0$, $g(t)\equiv C_1$ and $b(t)\equiv C_2$, with $C_0$, $C_1$, $C_2\ge 0$, the forcing function $F$ given by (38), (39) can be written in the following way:

$$F(t) = C_1 - C_0 + (C_1 - C_0)\int_0^t \frac{1}{\sqrt{\pi}}t^{-3/2}e^{-1/t}\,dt -$$

$$-(C_2 - C_0)\int_0^t \frac{1}{\sqrt{\pi}}t^{-3/2}e^{-1/(4t)}\,dt =$$

$$= (C_1 - C_0)\left(1+\mathrm{erfc}\left(\frac{1}{\sqrt{t}}\right)\right) - 2(C_2 - C_0)\,\mathrm{erfc}\left(\frac{1}{2\sqrt{t}}\right), \qquad t>0.$$

Since $t^{-p}e^{-1/(qt)}\le (pq/e)^p$, $t>0$, $p,q>0$, it is easy to see that $\dot F$ is a Lipschitz and bounded function on $[0,+\infty)$ with

$$|F|_{1,t} \le \frac{1}{\sqrt{\pi}}\left(\frac{3}{2e}\right)^{3/2}|C_1 - C_0| + \frac{1}{\sqrt{\pi}}\left(\frac{6}{e}\right)^{3/2}|C_2 - C_0|, \qquad t>0,$$

and

$$\|F\|_{\infty,t} \le 2(|C_1 - C_0| + |C_2 - C_0|), \qquad t>0.$$

As in Remark 3 we introduce the functions $\psi_\alpha(\alpha) = \dfrac{t}{(1+\alpha)^p}$, $0<p<2$, which belong to the family (27); then, from Theorem 5 we derive, for $0<x<1$, $t>0$,

$$|w_\alpha(x,t)| \le$$

$$\le \inf_{0<p<2}\left\{\left(\frac{1}{\sqrt{\pi}}\left(\frac{3}{2e}\right)^{3/2}|C_1 - C_0| + \frac{1}{\sqrt{\pi}}\left(\frac{6}{e}\right)^{3/2}|C_2 - C_0|\right)\frac{t}{(1+\alpha)^p} + \right.$$

$$+ \frac{6(|C_1 - C_0| + |C_2 - C_0|)}{1 + \frac{2}{\sqrt{\pi}}(1 - e^{-(1+\alpha)^p/t})\alpha t^{1/2}(1+\alpha)^{-p/2} + (1+2\alpha)\,\mathrm{erfc}\left(\frac{(1+\alpha)^{p/2}}{t^{1/2}}\right)}\Bigg\}\Bigg],$$

$$\alpha > 1;$$

where we have employed the value of $\int_0^{\psi(\alpha)} k_\alpha(s)\,ds$ given by (46).

To conclude with we point out that the analysis performed in the previous sections on the convergence of the solutions to equations (1) can be reproduced without essential changes for convergence in $L^p$ instead of $L^\infty$. A version of Theorem 3 providing an estimate of the convergence in $L^p$ is also feasible, which allow to obtain estimates of convergence in $L^p$ to Dirichlet type boundary conditions of linear heat transfer boundary conditions when the coefficient of heat transfer becomes infinity.

**Acknowledgements.** This work was finished during a visit of the author to the Instituto di Matematica "Ullisse Dini", Univ. degli Studi di Firenze, Florence, Italy. Conversations with professors J. J. Eliçabe Urriol, A. Fasano and M. Primicerio have been very stimulating. Finally, the author is greatly indebted to an anonymous referee, who pointed out several errors in the draft version of the paper.

1. *Gripenberg G., Londen S. O., Staffans O.* Volterra integral and functional equations. – Cambridge: Cambridge Univ. Press, 1990.
2. *Rudin W.* Functional analysis . – New York: McGraw-Hill, 1973.
3. *Yosida K.* Functional analysis // Grundlehren math. Wiss. – 1978. – **123**.
4. *Friedman A.* On integral equations on Volterra type // J. Analysis Math. – 1963. – **11**. – P. 381–413.
5. *Miller R. K.* Nonlinear Volterra integral equations. – Menlo-Park: W. A. Benjamin, Inc., 1971.
6. *Goursat E.* Cours d'Analyse Mathématique. – Paris: Gauthier-Villars, 1923. – **3**.
7. *Widder D. V.* The Laplace transform. – Princeton: Princeton Univ. Press, 1946.
8. *Saaty T. L.* Modern nonlinear equations. – New York: Dover, 1981.
9. *Carslaw H. S., Jaeger J. C.* Conduction of heat in solids. – London: Oxford Univ. Press, 1959.
10. *Fourier J. L.* Sur le Réfroidissement Séculaire du Globe Terrestre // Oeuvres de Fourier. – Paris: Gauthier-Villars, 1890. – **2**. – P. 269–288.
11. *Lamé G.* Leçons sur la Théorie Analytique de la Chaleur. – Paris: Mallet-Bachelier, 1861.
12. *Tarzia D. A.* Una Familia de Problemas que Converge Hacia el Caso Estacionario del Problema de Stefan a Dos Fases // Math. Notae. – 1979/1980. – **27**. – P. 145–156.
13. *Tarzia D. A.* Etude de l'Inéquation Variationnelle Proposée par Duvaut pour le Problème de Stefan à Deux Phases, I // Boll. Unione mat. ital. – 1982. – **6**, № 1-B. – P. 865–883.
14. *Tarzia D. A.* Etude de l'Inéquation Variationnelle Proposée par Duvaut pour le Problème de Stefan à Deux Phases, II // Ibid. – 1983. – **6**, № 2-B. – P. 589–603.
15. *Cannon J. R.* The one-dimensional heat equation. – Menlo-Park: Addison-Wesley, 1984.
16. *Gorenflo R., Vesella S.* Abel integral equations // Lect. Notes Math. – 1991. – **1461**.
17. *Yosida K.* Lectures on differential and integral equations. – New York: Interscience, 1960.
18. *Handbook* of mathematical functions // Eds. M. Abramowitz, I. A. Stegun. – New York: Dover, 1972.

Received 27.11.96,
after revision – 10.09.98