DOI: 10.37863/umzh.v73i7.441

UDC 517.9

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WEIGHTED LEBESGUE AND CENTRAL MORREY ESTIMATES FOR P-ADIC MULTILINEAR HAUSDORFF OPERATORS AND ITS COMMUTATORS

ЗВАЖЕНІ ОЦІНКИ ДЛЯ P-АДИЧНИХ БАГАТОЛІНІЙНИХ ГАУСДОРФОВИХ ОПЕРАТОРІВ ТА ЇХНІХ КОМУТАТОРІВ НА ПРОСТОРАХ ЛЕБЕГА І ЦЕНТРАЛЬНИХ ПРОСТОРАХ МОРРІ

We establish the sharp boundedness of p-adic multilinear Hausdorff operators on the product of Lebesgue and central Morrey spaces associated with both power weights and Muckenhoupt weights. Moreover, the boundedness for the commutators of p-adic multilinear Hausdorff operators on the such spaces with symbols in central BMO space is also obtained.

Встановлено точну обмеженість p-адичних багатолінійних гаусдорфових операторів на добутку просторів Лебега і центральних просторів Моррі, асоційованих як з вагами степенів, так і з вагами Макенхаупта. Також доведено обмеженість комутаторів p-адичних багатолінійних гаусдорфових операторів на таких просторах із символами в центральному BMO-просторі.

1. Introduction. The p-adic analysis in the past decades has received a lot of attention due to its important applications in mathematical physics as well as its necessity in sciences and technologies (see, e.g., [2 – 4, 10, 20 – 22, 28 – 31] and the references therein). It is known that the theory of functions from $\mathbb{Q}_p$ into $\mathbb{C}$ play an important role in p-adic quantum mechanics, the theory of p-adic probability in which real-valued random variables have to be considered to solve covariance problems. In recent years, there is an increasing interest in the study of harmonic analysis and wavelet analysis over the p-adic fields (see, e.g., [1, 4, 8, 18, 19, 22]).

It is crucial that the Hausdorff operator is one of the important operators in harmonic analysis. It is closely related to the summability of the classical Fourier series (see, for instance, [11, 13, 15] and the references therein). Let Φ be a locally integrable function on $\mathbb{R}^n$. The matrix Hausdorff operator $H_{\Phi,A}$ associated to the kernel function Φ is then defined by

$$H_{\Phi,A}(f)(x) = \int\limits_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f(A(y)x)dy, \quad x \in \mathbb{R}^n,$$

where $A(y)$ is an $n \times n$ invertible matrix for almost everywhere y in the support of Φ. It is worth pointing out that if the kernel function Φ is chosen appropriately, then the Hausdorff operator reduces to many classcial operators in analysis such as the Hardy operator, the Cesàro operator, the Riemann – Liouville fractional integral operator and the Hardy – Littlewood average operator.

In 2010, Volosivets [32] introduced the matrix Hausdorff operator on the p-adic numbers field as follows:

$$\mathcal{H}_{\varphi,A}(f)(x) = \int\limits_{\mathbb{Q}_p^n} \varphi(t) f(A(t)x)dt, \quad x \in \mathbb{Q}_p^n,$$

where $\varphi(t)$ is a locally integrable function on $\mathbb{Q}_p^n$ and $A(t)$ is an $n \times n$ invertible matrix for almost everywhere t in the support of φ. It is easy to see that if $\varphi(t) = \psi(t_1)\chi_{\mathbb{Z}_p^{*n}}(t)$ and $A(t) = t_1.I_n$ (I_n is an identity matrix) for $t = (t_1, t_2, \ldots, t_n)$, where $\psi : \mathbb{Q}_p \to \mathbb{C}$ is a measurable function, then $\mathcal{H}_{\varphi,A}$ reduces to the p-adic weighted Hardy – Littlewood average operator due to Rim and Lee [26].

In recent years, the theory of the Hardy operators, the Hausdorff operators over the p-adic numbers field has been significantly developed into different contexts, and they are actually useful for p-adic analysis (see, e.g., [5, 6, 14, 33]). It is known that the authors in [7] also introduced and studied a general class of multilinear Hausdorff operators on the real field defined by

$$\mathcal{H}_{\Phi,\vec{A}}(\vec{f})(x) = \int\limits_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \prod_{i=1}^m f_i(A_i(y)x)dy, \quad x \in \mathbb{R}^n,$$

for $\vec{f} = (f_1, \ldots, f_m)$ and $\vec{A} = (A_1, \ldots, A_m)$.

Motivated by above results, in this paper we shall introduce and study a class of p-adic multilinear (matrix) Hausdorff operators defined as follows.

Definition 1.1. *Let Φ be a measurable complex-valued function on $\mathbb{Q}_p^n$. The p-adic multilinear Hausdorff operator is defined by*

$$\mathcal{H}^p_{\Phi,\vec{A}}(\vec{f})(x) = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m f_i(A_i(y)x)dy, \quad x \in \mathbb{Q}_p^n,$$

where $\vec{f} = (f_1, \ldots, f_m)$ and $f_1, f_2, \ldots, f_m$ are measurable complex-valued functions on $\mathbb{Q}_p^n$.

Note that in this paper we will confine our attention to the case, where Φ is the nonnegative function.

Let b be a measurable function. The operator $\mathcal{M}_b$ is defined by $\mathcal{M}_b f(x) = b(x)f(x)$ for any measurable function f. If $\mathcal{H}$ is a linear operator on some measurable function space, the commutator of Coifman – Rochberg – Weiss type formed by $\mathcal{M}_b$ and $\mathcal{H}$ is defined by $[\mathcal{M}_b, \mathcal{H}]f(x) = (\mathcal{M}_b\mathcal{H} - \mathcal{H}\mathcal{M}_b)f(x)$. Similarly, the commutators of p-adic multilinear Hausdorff operator is defined as follows.

Definition 1.2. *Let $\Phi, \vec{A}$ be as above. The Coifman – Rochberg – Weiss type commutator of p-adic multilinear Hausdorff operator is defined by*

$$\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)(x) = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m (b_i(x) - b_i(A_i(y)x)) \prod_{i=1}^m f_i(A_i(y)x)dy,$$

where $x \in \mathbb{Q}_p^n$, $\vec{b} = (b_1, \ldots, b_m)$ and b_i are locally integrable functions on $\mathbb{Q}_p^n$ for all $i = 1, \ldots, m$.

The main purpose of this paper is to study the p-adic multilinear Hausdorff operators and its commutators on the p-adic numbers field. More precisely, we obtain the necessary and sufficient conditions for the boundedness of $\mathcal{H}^p_{\Phi,\vec{A}}$ and $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$ on the product of Lebesgue and central Morrey spaces with weights on p-adic field. In each case, we estimate the corresponding operator norms. Moreover, the boundedness of $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$ on the such spaces with symbols in central BMO space is also established. It should be pointed out that all our results are new even in the case of p-adic linear Hausdorff operators.

This paper is organized as follows. In Section 2, we present some notations and preliminaries about p-adic analysis as well as give some definitions of the Lebesgue and central Morrey spaces associated with power weights and Muckenhoupt weights. Main theorems are given and proved in Sections 3 and 4.

2. Some notations and definitions. For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. This field is the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: if $x = 0$, then $|0|_p = 0$; if $x \neq 0$ is an arbitrary rational number with the unique representation $x = p^\alpha \dfrac{m}{n}$, where m, n are not divisible by p, $\alpha = \alpha(x) \in \mathbb{Z}$, then $|x|_p = p^{-\alpha}$. This norm satisfies the following properties: $|x|_p \geq 0$ $\forall x \in \mathbb{Q}_p$ and $|x|_p = 0 \Leftrightarrow x = 0$; $|xy|_p = |x|_p|y|_p$ $\forall x, y \in \mathbb{Q}_p$; and $|x + y|_p \leq \max(|x|_p, |y|_p)$ $\forall x, y \in \mathbb{Q}_p$, and when $|x|_p \neq |y|_p$, we have $|x + y|_p = \max(|x|_p, |y|_p)$.

It is also known that any non-zero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^\alpha \left(x_0 + x_1 p + x_2 p^2 + \ldots\right),$$

where $\alpha = \alpha(x) \in \mathbb{Z}$, $x_k \in \{0, 1, \ldots, p-1\}$, $x_0 \neq 0$, $k = 0, 1, \ldots$. This series converges in the p-adic norm since $\left|x_k p^k\right|_p \leq p^{-k}$. The space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \ldots \times \mathbb{Q}_p$ consists of all points $x = (x_1, \ldots, x_n)$, where $x_i \in \mathbb{Q}_p$, $i = 1, \ldots, n$, $n \geq 1$. The p-adic norm of $\mathbb{Q}_p^n$ is defined by $|x|_p = \max_{1 \leq j \leq n} |x_j|_p$. Let A be an $n \times n$ matrix with entries $a_{ij} \in \mathbb{Q}_p$. For $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we denote

$$Ax = \left(\sum_{j=1}^{n} a_{1j} x_j, \ldots, \sum_{j=1}^{n} a_{nj} x_j\right).$$

By Lemma 2 in paper [33], the norm of A is $\|A\|_p := \max_{1 \leq i \leq n} \max_{1 \leq j \leq n} |a_{ij}|_p$. For simplicity of notation, we write $k_A = \log_p \|A\|_p$. It is clear to see that $k_A \in \mathbb{Z}$. It is easy to show that $|Ax|_p \leq \|A\|_p . |x|_p$ for any $x \in \mathbb{Q}_p^n$. In addition, if A is invertible, by the same arguments as the real setting (see also Lemma 3.1 [25] for the setting of the Heisenberg group), we get

$$\|A\|_p^{-n} \leq \left|\det\left(A^{-1}\right)\right|_p \leq \left\|A^{-1}\right\|_p^n. \tag{2.1}$$

Let $B_\alpha(a) = \left\{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\alpha\right\}$ be a ball of radius p^α with center at $a \in \mathbb{Q}_p^n$. Similarly, denote by $S_\alpha(a) = \left\{x \in \mathbb{Q}_p^n : |x - a|_p = p^\alpha\right\}$ the sphere with center at $a \in \mathbb{Q}_p^n$ and radius p^α. If $B_\alpha = B_\alpha(0)$, $S_\alpha = S_\alpha(0)$, then, for any $x_0 \in \mathbb{Q}_p^n$, we have $x_0 + B_\alpha = B_\alpha(x_0)$ and $x_0 + S_\alpha = S_\alpha(x_0)$. Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard theory that there exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to positive constant multiple and is translation invariant. This measure is unique by normalizing dx such that $\int_{B_0} dx = |B_0| = 1$, where $|B|$ denotes the Haar measure of a measurable subset B of $\mathbb{Q}_p^n$. By simple calculation, it is easy to obtain that $|B_\alpha(a)| = p^{n\alpha}$, $|S_\alpha(a)| = p^{n\alpha}\left(1 - p^{-n}\right) \simeq p^{n\alpha}$ for any $a \in \mathbb{Q}_p^n$. For $f \in L^1_{\mathrm{loc}}\left(\mathbb{Q}_p^n\right)$, we have

$$\int_{\mathbb{Q}_p^n} f(x)dx = \lim_{\alpha \to +\infty} \int_{B_\alpha} f(x)dx = \lim_{\alpha \to +\infty} \sum_{-\infty < \gamma \leq \alpha} \int_{S_\gamma} f(x)dx.$$

In particular, if $f \in L^1\left(\mathbb{Q}_p^n\right)$, we can write

$$\int\limits_{\mathbb{Q}_p^n} f(x)dx = \sum_{\alpha=-\infty}^{+\infty} \int\limits_{S_\alpha} f(x)dx \quad \text{and} \quad \int\limits_{\mathbb{Q}_p^n} f(tx)dx = \frac{1}{|t|_p^n} \int\limits_{\mathbb{Q}_p^n} f(x)dx,$$

where $t \in \mathbb{Q}_p \setminus \{0\}$. For a more complete introduction to the p-adic analysis, we refer the readers to [20, 31] and the references therein.

Let ω be a weighted function, that is a nonnegative locally integrable measurable function on $\mathbb{Q}_p^n$. The weighted Lebesgue space $L^q_\omega\left(\mathbb{Q}_p^n\right)$, $0 < q < \infty$, is defined to be the space of all measurable functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{L^q_\omega(\mathbb{Q}_p^n)} = \left(\int\limits_{\mathbb{Q}_p^n} |f(x)|^q \omega(x)dx \right)^{1/q} < \infty.$$

The space $L^q_{\omega,\mathrm{loc}}\left(\mathbb{Q}_p^n\right)$ is defined as the set of all measurable functions f on $\mathbb{Q}_p^n$ satisfying

$$\int\limits_K |f(x)|^q \omega(x)dx < \infty$$

for any compact subset K of $\mathbb{Q}_p^n$. The space $L^q_{\omega,\mathrm{loc}}\left(\mathbb{Q}_p^n \setminus \{0\}\right)$ is also defined in a similar way as the space $L^q_{\omega,\mathrm{loc}}\left(\mathbb{Q}_p^n\right)$. Throught the whole paper, we denote by C a positive constant that is independent of the main parameters, but can change from line to line. We also write $a \lesssim b$ to mean that there is a positive constant C, independent of the main parameters, such that $a \le Cb$. The symbol $f \simeq g$ means that f is equivalent to g (i.e., $C^{-1}f \le g \le Cf$). For any real number $\ell > 1$, denote by ℓ' conjugate real number of ℓ, i.e., $\dfrac{1}{\ell} + \dfrac{1}{\ell'} = 1$. Denote $\omega(B)^\lambda = \left(\int_B \omega(x)dx\right)^\lambda$ for $\lambda \in \mathbb{R}$. Remark that if $\omega(x) = |x|_p^\alpha$ for $\alpha > -n$, then we have

$$\omega(B_\gamma) = \int\limits_{B_\gamma} |x|_p^\alpha dx = \sum_{k\le\gamma} \int\limits_{S_k} p^{k\alpha} dx = \sum_{k\le\gamma} p^{k(\alpha+n)}\left(1 - p^{-n}\right) \simeq p^{\gamma(\alpha+n)}. \tag{2.2}$$

Next, let us give the definition of weighted λ-central Morrey spaces on p-adic numbers field as follows.

Definition 2.1. *Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The weighted λ-central Morrey p-adic spaces $\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ consists of all functions $f \in L^q_{\omega,\mathrm{loc}}\left(\mathbb{Q}_p^n\right)$ satisfying $\|f\|_{\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} < \infty$, where*

$$\|f\|_{\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int\limits_{B_\gamma} |f(x)|^q \omega(x)dx \right)^{1/q}.$$

Remark that $\dot{B}^{q,\lambda}_\omega\left(\mathbb{Q}_p^n\right)$ is a Banach space and reduces to $\{0\}$ when $\lambda < -\dfrac{1}{q}$.

Let us recall the definition of the weighted central BMO p-adic space of John – Nirenberg type.

Definition 2.2. *Let $1 \leq q < \infty$ and ω be a weight function. The weighted central bounded mean oscillation space $CMO^q_\omega\left(\mathbb{Q}_p^n\right)$ is defined as the set of all functions $f \in L^q_{\omega,\mathrm{loc}}\left(\mathbb{Q}_p^n\right)$ such that*

$$\|f\|_{CMO^q_\omega(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{\omega(B_\gamma)} \int\limits_{B_\gamma} |f(x) - f_{B_\gamma}|^q \omega(x) dx \right)^{\frac{1}{q}} < \infty,$$

where

$$f_{B_\gamma} = \frac{1}{|B_\gamma|} \int\limits_{B_\gamma} f(x)dx.$$

The theory of A_ℓ weight was first introduced by Benjamin Muckenhoupt on the Euclidean spaces in order to characterise the boundedness of Hardy – Littlewood maximal functions on the weighted L^ℓ spaces (see [24]). For A_ℓ weights on the p-adic fields, more generally, on the local fields or homogeneous type spaces, one can refer to [9, 16] for more details. Let us now recall the definition of A_ℓ weights.

Definition 2.3. *Let $1 < \ell < \infty$. It is said that a nonnegative locally integrable function $\omega \in \in A_\ell\left(\mathbb{Q}_p^n\right)$ if there exists a constant C such that, for all balls B, we have*

$$\left(\frac{1}{|B|} \int\limits_B \omega(x)dx \right) \left(\frac{1}{|B|} \int\limits_B \omega(x)^{-1/(\ell-1)} dx \right)^{\ell-1} \leq C.$$

It is said that a weight $\omega \in A_1\left(\mathbb{Q}_p^n\right)$ if there is a constant C such that, for all balls B, we get

$$\frac{1}{|B|} \int\limits_B \omega(x)dx \leq C \operatorname*{essinf}_{x\in B} \omega(x).$$

We denote by $A_\infty\left(\mathbb{Q}_p^n\right) = \bigcup_{1\leq\ell<\infty} A_\ell\left(\mathbb{Q}_p^n\right)$. Let us give the following standard result related to the Muckenhoupt weights.

Proposition 2.1. (i) *$A_\ell\left(\mathbb{Q}_p^n\right) \subsetneq A_q\left(\mathbb{Q}_p^n\right)$ for $1 \leq \ell < q < \infty$.*

(ii) *If $\omega \in A_\ell\left(\mathbb{Q}_p^n\right)$ for $1 < \ell < \infty$, then there is an $\varepsilon > 0$ such that $\ell - \varepsilon > 1$ and $\omega \in A_{\ell-\varepsilon}\left(\mathbb{Q}_p^n\right)$.*

We note that the class $A_\infty\left(\mathbb{Q}_p^n\right)$ is closely connected with the reverse Hölder condition. More precisely, if there exist $r > 1$ and a fixed constant C such that

$$\left(\frac{1}{|B|} \int\limits_B \omega(x)^r dx \right)^{1/r} \leq \frac{C}{|B|} \int\limits_B \omega(x)dx$$

for all balls $B \subset \mathbb{Q}_p^n$, then we say that ω satisfies the reverse Hölder condition of order r and write $\omega \in RH_r\left(\mathbb{Q}_p^n\right)$. According to Theorem 19 and Corollary 21 in [17], $\omega \in A_\infty\left(\mathbb{Q}_p^n\right)$ if and only if there exists some $r > 1$ such that $\omega \in RH_r\left(\mathbb{Q}_p^n\right)$. Moreover, if $\omega \in RH_r\left(\mathbb{Q}_p^n\right)$, $r > 1$, then $\omega \in RH_{r+\varepsilon}\left(\mathbb{Q}_p^n\right)$ for some $\varepsilon > 0$. We thus write $r_\omega = \sup\left\{r > 1 : \omega \in RH_r\left(\mathbb{Q}_p^n\right)\right\}$ to denote the critical index of ω for the reverse Hölder condition. It is worth noticing that an important example of $A_\ell\left(\mathbb{Q}_p^n\right)$ weight is the power function $|x|_p^\alpha$. By the similar arguments as Propositions 1.4.3 and 1.4.4 in [23], we obtain the following properties of power weights.

Proposition 2.2. *Let $x \in \mathbb{Q}_p^n$. Then we have:*
(i) $|\cdot|_p^\alpha \in A_1\left(\mathbb{Q}_p^n\right)$ *if and only if* $-n < \alpha \le 0$;
(ii) $|\cdot|_p^\alpha \in A_\ell\left(\mathbb{Q}_p^n\right)$ *for* $1 < \ell < \infty$ *if and only if* $-n < \alpha < n(\ell - 1)$.

Let us give the following standard characterization of A_ℓ weights which it is proved in the similar way as the real setting (see [12, 27] for more details).

Proposition 2.3. *Let* $\omega \in A_\ell\left(\mathbb{Q}_p^n\right) \cap RH_r\left(\mathbb{Q}_p^n\right)$, $\ell \ge 1$ *and* $r > 1$. *Then there exist constants* C_1, $C_2 > 0$ *such that*

$$C_1\left(\frac{|E|}{|B|}\right)^\ell \le \frac{\omega(E)}{\omega(B)} \le C_2\left(\frac{|E|}{|B|}\right)^{(r-1)/r}$$

for any measurable subset E of a ball B.

Proposition 2.4. *If* $\omega \in A_\ell\left(\mathbb{Q}_p^n\right)$, $1 \le \ell < \infty$, *then, for any* $f \in L^1_{\text{loc}}\left(\mathbb{Q}_p^n\right)$ *and any ball* $B \subset \mathbb{Q}_p^n$, *we have*

$$\frac{1}{|B|}\int\limits_B |f(x)|dx \le C\left(\frac{1}{\omega(B)}\int\limits_B |f(x)|^\ell \omega(x)dx\right)^{1/\ell}.$$

Let us recall the definition of the Hardy – Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{p^{n\gamma}} \int\limits_{B_\gamma(x)} |f(y)|dy.$$

It is useful to remark that the Hardy – Littlewood maximal operator $\mathcal{M}$ is bounded on $L^\ell_\omega\left(\mathbb{Q}_p^n\right)$ if and only if $\omega \in A_\ell\left(\mathbb{Q}_p^n\right)$ for all $\ell > 1$. Finally, we introduce a new maximal operator which is used in the sequel, that is,

$$\mathcal{M}^{\text{mod}}f(x) = \sup_{\substack{\gamma\in\mathbb{Z}\\ |x|_p \le p^\gamma}} \frac{1}{p^{n\gamma}} \int\limits_{B_\gamma(x)} |f(y)|dy.$$

3. Main results about the boundness of $\mathcal{H}^p_{\Phi,\vec{A}}$. Let us now assume that q and $q_i \in [1, \infty)$, α, α_i are real numbers such that $\alpha_i \in (-n, \infty)$ for $i = 1, 2, \ldots, m$ and

$$\frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_m} = \frac{1}{q}, \quad \frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2} + \ldots + \frac{\alpha_m}{q_m} = \frac{\alpha}{q}.$$

In this section, we will investigate the boundedness of multilinear Hausdorff operators on weighted Lebesgue spaces and weighted central Morrey spaces associated to the case of matrices having the important property as follows: there exists $\nu_{\vec{A}} \in \mathbb{N}$ such that

$$\|A_i(y)\|_p.\left\|A_i^{-1}(y)\right\|_p \le p^{\nu_{\vec{A}}} \quad \text{for all} \quad i = 1, \ldots, m \tag{3.1}$$

and for almost everywhere $y \in \mathbb{Q}_p^n$. Thus, by the property of invertible matrice, it is easy to show that

$$\|A_i(y)\|_p^\sigma \lesssim \left\|A_i^{-1}(y)\right\|_p^{-\sigma} \quad \text{for all} \quad \sigma \in \mathbb{R} \tag{3.2}$$

and

$$|A_i(y)x|_p^\sigma \gtrsim \left\|A_i^{-1}(y)\right\|_p^{-\sigma} |x|_p^\sigma \quad \text{for all} \quad \sigma \in \mathbb{R}, \quad x \in \mathbb{Q}_p^n \setminus \{0\}. \tag{3.3}$$

First main result of this paper is the following.

Theorem 3.1. *Let* $\omega_1(x) = |x|_p^{\alpha_1}, \ldots, \omega_m(x) = |x|_p^{\alpha_m}$ *and* $\omega(x) = |x|_p^{\alpha}$. *Then* $\mathcal{H}^p_{\Phi,\vec{A}}$ *is bounded from* $L^{q_1}_{\omega_1}\left(\mathbb{Q}_p^n\right) \times \ldots \times L^{q_m}_{\omega_m}\left(\mathbb{Q}_p^n\right)$ *to* $L^q_{\omega}\left(\mathbb{Q}_p^n\right)$ *if and only if*

$$\mathcal{C}_1 = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \left\|A_i^{-1}(y)\right\|_p^{\frac{\alpha_i+n}{q_i}} dy < \infty.$$

Furthermore, $\left\|\mathcal{H}^p_{\Phi,\vec{A}}\right\|_{L^{q_1}_{\omega_1}\left(\mathbb{Q}_p^n\right)\times\ldots\times L^{q_m}_{\omega_m}\left(\mathbb{Q}_p^n\right)\to L^q_{\omega}\left(\mathbb{Q}_p^n\right)} \simeq \mathcal{C}_1.$

Proof. Firstly, we will prove the sufficiency of the condition $\mathcal{C}_1 < \infty$. By applying the Minkowski inequality and the Hölder inequality, we have

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^q_{\omega}\left(\mathbb{Q}_p^n\right)} \le \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \|f_i(A_i(y).)\|_{L^{q_i}_{\omega_i}\left(\mathbb{Q}_p^n\right)} dy.$$

By using the change of variables, we get

$$\|f_i(A_i(y).)\|_{L^{q_i}_{\omega_i}\left(\mathbb{Q}_p^n\right)} \le \max\left\{\left\|A_i^{-1}(y)\right\|_p^{\alpha_i}, \|A_i(y)\|_p^{-\alpha_i}\right\}^{\frac{1}{q_i}} \left|\det A_i^{-1}(y)\right|_p^{\frac{1}{q_i}} \|f_i\|_{L^{q_i}_{\omega_i}\left(\mathbb{Q}_p^n\right)}.$$

Thus,

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^q_{\omega}\left(\mathbb{Q}_p^n\right)} \le$$

$$\le \left(\int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \max\left\{\left\|A_i^{-1}(y)\right\|_p^{\alpha_i}, \|A_i(y)\|_p^{-\alpha_i}\right\}^{\frac{1}{q_i}} \left|\det A_i^{-1}(y)\right|_p^{\frac{1}{q_i}} dy\right) \prod_{i=1}^m \|f_i\|_{L^{q_i}_{\omega_i}\left(\mathbb{Q}_p^n\right)}. \quad (3.4)$$

Note that, by (2.1) and (3.2), we obtain

$$\max\left\{\left\|A_i^{-1}(y)\right\|_p^{\alpha_i}, \|A_i(y)\|_p^{-\alpha_i}\right\}^{\frac{1}{q_i}} \left|\det A_i^{-1}(y)\right|_p^{\frac{1}{q_i}} \lesssim \left\|A_i^{-1}(y)\right\|_p^{\frac{(\alpha_i+n)}{q_i}}. \quad (3.5)$$

This shows that

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^q_{\omega}\left(\mathbb{Q}_p^n\right)} \lesssim \mathcal{C}_1 \prod_{i=1}^m \|f_i\|_{L^{q_i}_{\omega_i}\left(\mathbb{Q}_p^n\right)}.$$

Next, to prove that the condition $\mathcal{C}_1 < \infty$ is necessary, let us now take $\vec{f_r} = (f_{1r}, \ldots, f_{mr})$, where

$$f_{i,r}(x) = \begin{cases} 0, & \text{if} \quad |x|_p \le p^{-\nu_{\vec{A}}-1}, \\ |x|_p^{-\frac{n}{q_i}-\frac{\alpha_i}{q_i}-p^{-r}}, & \text{otherwise}, \end{cases}$$

for $i = 1, \ldots, m$ and $r \in \mathbb{Z}^+$. By a simple calculation, we have

$$\|f_{i,r}\|_{L^{q_i}_{\omega_i}\left(\mathbb{Q}_p^n\right)} = \left(\int\limits_{\mathbb{Q}_p^n} |x|_p^{-n-\alpha_i-q_ip^{-r}} \chi_{B^c_{-\nu_{\vec{A}}-1}}(x)|x|_p^{\alpha_i} dx\right)^{\frac{1}{q_i}} =$$

$$= \left(\sum_{k \geq -\nu_{\vec{A}}} \int_{S_k} p^{k(-n-q_i p^{-r})} dx \right)^{\frac{1}{q_i}} \simeq \left(\sum_{k \geq -\nu_{\vec{A}}} p^{k\left(-n-q_i p^{-r}\right)} p^{kn} \right)^{\frac{1}{q_i}} =$$

$$= \left(\sum_{k \geq -\nu_{\vec{A}}} p^{-kq_i p^{-r}} \right)^{\frac{1}{q_i}} = \frac{p^{\nu_{\vec{A}} \cdot p^{-r}}}{\left(1 - p^{-q_i p^{-r}}\right)^{\frac{1}{q_i}}}. \tag{3.6}$$

Next, we define two sets as follows:

$$S_x = \bigcap_{i=1}^{m} \left\{ y \in \mathbb{Q}_p^n : |A_i(y)x|_p \geq p^{-\nu_{\vec{A}}} \right\}$$

and

$$U_r = \left\{ y \in \mathbb{Q}_p^n : \|A_i(y)\|_p \geq p^{-r} \text{ for all } i = 1, \ldots, m \right\}.$$

From this we derive

$$U_r \subset S_x \quad \text{for all} \quad x \in \mathbb{Q}_p^n \setminus B_{r-1}. \tag{3.7}$$

In fact, by letting $y \in U_r$, we get $\|A_i(y)\|_p |x|_p \geq 1$ for all $x \in \mathbb{Q}_p^n \setminus B_{r-1}$. Thus, by applying the condition (3.1), one has

$$|A_i(y)x|_p \geq \left\|A_i^{-1}(y)\right\|_p^{-1} |x|_p = \frac{\|A_i(y)\|_p |x|_p}{\left\|A_i^{-1}(y)\right\|_p \|A_i(y)\|_p} \geq p^{-\nu_{\vec{A}}},$$

which confirms the relation (3.7). Now, by taking $x \in \mathbb{Q}_p^n \setminus B_{r-1}$ and using the relation (3.7), we obtain

$$\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}_r\right)(x) \geq \int_{S_x} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^{m} |A_i(y)x|_p^{-\frac{n}{q_i} - \frac{\alpha_i}{q_i} - p^{-r}} dy \geq \int_{U_r} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^{m} |A_i(y)x|_p^{-\frac{n}{q_i} - \frac{\alpha_i}{q_i} - p^{-r}} dy.$$

From this, by (3.3), one has

$$\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}_r\right)(x) \gtrsim \left(\int_{U_r} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^{m} \left\|A_i^{-1}(y)\right\|_p^{\frac{n+\alpha_i}{q_i} + p^{-r}} dy \right) |x|_p^{-\frac{(n+\alpha)}{q} - mp^{-r}} \times$$

$$\times \chi_{\mathbb{Q}_p^n \setminus B_{r-1}}(x) =: p^{rmp^{-r}} \mathcal{A}_r g(x),$$

where

$$\mathcal{A}_r = \int_{U_r} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^{m} \left\|A_i^{-1}(y)\right\|_p^{\frac{n+\alpha_i}{q_i}} p^{-rmp^{-r}} \prod_{i=1}^{m} \left\|A_i^{-1}(y)\right\|_p^{p^{-r}} dy$$

and

$$g(x) = |x|_p^{-\frac{(n+\alpha)}{q} - mp^{-r}} \chi_{\mathbb{Q}_p^n \setminus B_{r-1}}(x).$$

By estimating as (3.6) above, we also have $\|g\|_{L^q_\omega(\mathbb{Q}_p^n)} \simeq \dfrac{p^{-rmp^{-r}}}{\left(1-p^{-qmp^{-r}}\right)^{\frac{1}{q}}}$. As a consequence above, by (3.6), we find that

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}_r\right)\right\|_{L^q_\omega(\mathbb{Q}_p^n)} \gtrsim \mathcal{A}_r \frac{\prod_{i=1}^m \|f_{i,r}\|_{L^{q_i}_{\omega_i}(\mathbb{Q}_p^n)}}{\left(1-p^{-qmp^{-r}}\right)^{\frac{1}{q}} \prod_{i=1}^m \dfrac{p^{\nu_{\vec{A}}p^{-r}}}{\left(1-p^{-q_ip^{-r}}\right)^{\frac{1}{q_i}}}} =:$$

$$=: \mathcal{A}_r \mathcal{T}_r \prod_{i=1}^m \|f_{i,r}\|_{L^{q_i}_{\omega_i}(\mathbb{Q}_p^n)},$$

where

$$\mathcal{T}_r = \frac{\prod_{i=1}^m \left(1-p^{-q_ip^{-r}}\right)^{\frac{1}{q_i}}}{\left(1-p^{-qmp^{-r}}\right)^{\frac{1}{q}} p^{m.\nu_{\vec{A}}.p^{-r}}}.$$

Note that from $\dfrac{1}{q_1} + \ldots + \dfrac{1}{q_m} = \dfrac{1}{q}$, it is clear to obtain that $\lim_{r\to+\infty} \mathcal{T}_r = a > 0$. Therefore, because $\mathcal{H}^p_{\Phi,\vec{A}}$ is bounded from $L^{q_1}_{\omega_1}\left(\mathbb{Q}_p^n\right) \times \ldots \times L^{q_m}_{\omega_m}\left(\mathbb{Q}_p^n\right)$ to $L^q_\omega\left(\mathbb{Q}_p^n\right)$, there exists $M > 0$ such that $\mathcal{A}_r \le M$ for sufficiently big r. On the other hand, by letting r sufficiently large, $y \in U_r$ and by (3.1), we get

$$p^{-rmp^{-r}} \prod_{i=1}^m \|A_i^{-1}(y)\|_p^{p^{-r}} \le \left(\prod_{i=1}^m \|A_i(y)\|_p \left\|A_i^{-1}(y)\right\|_p\right)^{p^{-r}} \le p^{\nu_{\vec{A}}.m.p^{-r}} \lesssim 1.$$

Hence, by the dominated convergence theorem of Lebesgue, we obtain

$$\int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \left\|A_i^{-1}(y)\right\|_p^{\frac{\alpha_i+n}{q_i}} dy < \infty.$$

Theorem 3.1 is proved.

Theorem 3.2. *Let $1 \le q^*, \zeta < \infty$ and $\omega \in A_\zeta$ with the finite critical index r_ω for the reverse Hölder inequality and $\omega(B_\gamma) \lesssim 1$ for all $\gamma \in \mathbb{Z}$. Assume that $q > q^*\zeta r_\omega/(r_\omega - 1)$, $\delta \in (1, r_\omega)$ and the following condition holds:*

$$\mathcal{C}_2 = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{q_i}} \|A_i(y)\|_p^{\frac{\zeta}{q_i}} \times$$

$$\times \left(\chi_{\{\|A_i(y)\|_p\le 1\}}(y)\|A_i(y)\|_p^{\frac{-n\zeta}{q_i}} + \chi_{\{\|A_i(y)\|_p>1\}}(y)\|A_i(y)\|_p^{\frac{-n(\delta-1)}{q_i\delta}}\right) dy < \infty.$$

Then we have that $\mathcal{H}^p_{\Phi,\vec{A}}$ is bounded from $L^{q_1}_\omega\left(\mathbb{Q}_p^n\right) \times \ldots \times L^{q_m}_\omega\left(\mathbb{Q}_p^n\right)$ to $L^{q^*}_\omega\left(\mathbb{Q}_p^n\right)$.

Proof. For any $R\in\mathbb{Z}$, by the Minkowski inequality, we get

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(B_R)}\leq\int\limits_{\mathbb{Q}^n_p}\frac{\Phi(y)}{|y|^n_p}\left(\int\limits_{B_R}\prod_{i=1}^m|f_i(A_i(y)x)|^{q^*}\omega(x)dx\right)^{\frac{1}{q^*}}dy.$$

From the inequality $q>q^*\zeta r_\omega/(r_\omega-1)$, there exists $r\in(1,r_\omega)$ such that $q=\zeta q^*r'$. Then, by the Hölder inequality and the reverse Hölder inequality, we obtain

$$\left(\int\limits_{B_R}\prod_{i=1}^m|f_i(A_i(y)x)|^{q^*}\omega(x)dx\right)^{\frac{1}{q^*}}\lesssim\left(\int\limits_{B_R}\prod_{i=1}^m|f_i(A_i(y)x)|^{\frac{q}{\zeta}}dx\right)^{\frac{\zeta}{q}}\omega(B_R)^{\frac{1}{q^*}}|B_R|^{-\frac{\zeta}{q}}.$$

Next, by using the Hölder inequality and the change of variables formula, and applying Proposition 2.4, we have

$$\left(\int\limits_{B_R}\prod_{i=1}^m|f_i(A_i(y)x)|^{\frac{q}{\zeta}}dx\right)^{\frac{\zeta}{q}}\lesssim\prod_{i=1}^m\left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{q_i}}\left|B_{R+k_{A_i}}\right|^{\frac{\zeta}{q_i}}\omega(B_{R+k_{A_i}})^{-\frac{1}{q_i}}\|f_i\|_{L^{q_i}_\omega(B_{R+k_{A_i}})},$$

where $k_{A_i}(y)=\log_p\|A_i(y)\|_p$. Thus, by $\dfrac{\left|B_{R+k_{A_i}}\right|}{|B_R|}\simeq\dfrac{p^{(R+k_{A_i})n}}{p^{Rn}}=\|A_i(y)\|^n_p$, we infer that

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(B_R)}\lesssim$$

$$\lesssim\omega(B_R)^{\frac{1}{q^*}}\int\limits_{\mathbb{Q}^n_p}\frac{\Phi(y)}{|y|^n_p}\prod_{i=1}^m\left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{q_i}}\|A_i(y)\|_p^{\frac{\zeta n}{q_i}}\omega(B_{R+k_{A_i}})^{-\frac{1}{q_i}}\|f_i\|_{L^{q_i}_\omega(B_{R+k_{A_i}})}dy. \tag{3.8}$$

On the other hand, by $q>q^*\geq1$ and $\omega(B_R)\lesssim1$ for all $R\in\mathbb{Z}$, we imply that $\omega(B_R)^{\frac{1}{q^*}}\lesssim$ $\lesssim\omega(B_R)^{\frac{1}{q}}$. Hence, by (3.8), we get

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(B_R)}\lesssim$$

$$\lesssim\left(\int\limits_{\mathbb{Q}^n_p}\frac{\Phi(y)}{|y|^n_p}\prod_{i=1}^m\left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{q_i}}\|A_i(y)\|_p^{\frac{\zeta n}{q_i}}\left(\frac{\omega(B_R)}{\omega(B_{R+k_{A_i}})}\right)^{\frac{1}{q_i}}dy\right)\prod_{i=1}^m\|f_i\|_{L^{q_i}_\omega(\mathbb{Q}^n_p)}.$$

Next, for $i=1,\ldots,m$, by using Proposition 2.3, we have

$$\left(\frac{\omega(B_R)}{\omega(B_{R+k_{A_i}})}\right)^{\frac{1}{q_i}}\lesssim\begin{cases}\left(\dfrac{|B_R|}{\left|B_{R+k_{A_i}}\right|}\right)^{\frac{\zeta}{q_i}}\lesssim p^{\frac{(R-R-k_{A_i})n\zeta}{q_i}}=\|A_i(y)\|_p^{\frac{-n\zeta}{q_i}}, & \text{if } \|A_i(y)\|_p\leq1,\\ \left(\dfrac{|B_R|}{\left|B_{R+k_{A_i}}\right|}\right)^{\frac{(\delta-1)}{q_i\delta}}\lesssim p^{\frac{(R-R-k_{A_i})n(\delta-1)}{q_i\delta}}=\|A_i(y)\|_p^{\frac{-n(\delta-1)}{q_i\delta}}, & \text{otherwise.}\end{cases} \tag{3.9}$$

Hence, by letting $R \to +\infty$ and applying the dominated convergence theorem of Lebesgue, we obtain

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(\mathbb{Q}^n_p)} \lesssim \mathcal{C}_2 \prod_{i=1}^m \|f_i\|_{L^{q_i}_\omega(\mathbb{Q}^n_p)}.$$

Theorem 3.2 is proved.

Theorem 3.3. *Let ω_i, ω be as Theorem 3.1 and $\lambda_i \in \left(\dfrac{-1}{q_i}, 0\right)$ for all $i = 1, \ldots, m$. Assume that*

$$(\alpha + n)\lambda = (\alpha_1 + n)\lambda_1 + \ldots + (\alpha_m + n)\lambda_m. \tag{3.10}$$

Then $\mathcal{H}^p_{\Phi,\vec{A}}$ is bounded from $\dot{B}^{q_1,\lambda_1}_{\omega_1}\left(\mathbb{Q}^n_p\right) \times \ldots \times \dot{B}^{q_m,\lambda_m}_{\omega_m}\left(\mathbb{Q}^n_p\right)$ to $\dot{B}^{q,\lambda}_{\omega}\left(\mathbb{Q}^n_p\right)$ if and only if

$$\mathcal{C}_3 = \int\limits_{\mathbb{Q}^n_p} \frac{\Phi(y)}{|y|^n_p} \prod_{i=1}^m \left\|A_i^{-1}(y)\right\|_p^{-(\alpha_i+n)\lambda_i} dy < \infty.$$

Furthermore, $\left\|\mathcal{H}^p_{\Phi,\vec{A}}\right\|_{\dot{B}^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}^n_p)\times\ldots\times\dot{B}^{q_m,\lambda_m}_{\omega_m}(\mathbb{Q}^n_p)\to\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^n_p)} \simeq \mathcal{C}_3$.

Proof. We will prove the sufficiency of the condition $\mathcal{C}_3 < \infty$. For $\gamma \in \mathbb{Z}$, by estimating as (3.4) and (3.5) above, we have

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^q_\omega(B_\gamma)} \lesssim \int\limits_{\mathbb{Q}^n_p} \frac{\Phi(y)}{|y|^n_p} \prod_{i=1}^m \left\|A_i^{-1}(y)\right\|_p^{\frac{(\alpha_i+n)}{q_i}} \prod_{i=1}^m \|f_i\|_{L^{q_i}_{\omega_i}(B_{\gamma+k_{A_i}})} dy.$$

This implies that

$$\frac{1}{\omega(B_\gamma)^{\frac{1}{q}+\lambda}} \left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^q_\omega(B_\gamma)} \lesssim \int\limits_{\mathbb{Q}^n_p} \frac{\Phi(y)}{|y|^n_p} \left(\prod_{i=1}^m \left\|A_i^{-1}(y)\right\|_p^{\frac{(\alpha_i+n)}{q_i}}\right) \times$$

$$\times \mathcal{B}_i(y) \left(\prod_{i=1}^m \frac{1}{\omega_i(B_{\gamma+k_{A_i}})^{\frac{1}{q_i}+\lambda_i}} \|f_i\|_{L^{q_i}_{\omega_i}(B_{\gamma+k_{A_i}})}\right) dy, \tag{3.11}$$

where

$$\mathcal{B}_i(y) = \frac{\prod_{i=1}^m \omega_i(B_{\gamma+k_{A_i}})^{\frac{1}{q_i}+\lambda_i}}{\omega(B_\gamma)^{\frac{1}{q}+\lambda}}.$$

On the other hand, by hypothesis (3.10), we immediately get

$$\sum_{i=1}^m (\alpha_i + n)\left(\frac{1}{q_i} + \lambda_i\right) = (\alpha + n)\left(\frac{1}{q} + \lambda\right).$$

Consequently, by the estimation (2.2) and (3.1), we have

$$\mathcal{B}_i(y) \lesssim \frac{p^{\sum_{i=1}^m (\gamma + k_{A_i})(\alpha_i+n)\left(\frac{1}{q_i}+\lambda_i\right)}}{p^{\gamma(\alpha+n)\left(\frac{1}{q}+\lambda\right)}} =$$

$$= \frac{p^{\sum_{i=1}^m \gamma(\alpha_i+n)\left(\frac{1}{q_i}+\lambda_i\right)} p^{\sum_{i=1}^m k_{A_i}(\alpha_i+n)\left(\frac{1}{q_i}+\lambda_i\right)}}{p^{\gamma(\alpha+n)\left(\frac{1}{q}+\lambda\right)}} \lesssim \prod_{i=1}^m \left\|A_i^{-1}(y)\right\|_p^{-(\alpha_i+n)\left(\frac{1}{q_i}+\lambda_i\right)}.$$

Hence, by (3.11), one has

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)} \lesssim \mathcal{C}_3 \prod_{i=1}^m \|f_i\|_{\dot{B}^{q_i,\lambda_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

Conversely, suppose that $\mathcal{H}^p_{\Phi,\vec{A}}$ is bounded from $\dot{B}^{q_1,\lambda_1}_{\omega_1}\left(\mathbb{Q}_p^n\right)\times\ldots\times\dot{B}^{q_m,\lambda_m}_{\omega_m}\left(\mathbb{Q}_p^n\right)$ to $\dot{B}^{q,\lambda}_{\omega}\left(\mathbb{Q}_p^n\right)$. Let us choose the function $\vec{g}=(g_1,\ldots,g_m)$, where

$$g_i(x) = |x|_p^{(\alpha_i+n)\lambda_i}$$

for $i=1,\ldots,m$. Then, by (2.2), it is not difficult to show that

$$\|g_i\|_{\dot{B}^{q_i,\lambda_i}_{\omega_i}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \frac{1}{\omega_i(B_\gamma)^{\frac{1}{q_i}+\lambda_i}} \left(\int\limits_{B_\gamma} |x|_p^{(\alpha_i+n)\lambda_i q_i+\alpha_i} dx\right)^{\frac{1}{q_i}} \simeq$$

$$\simeq \sup_{\gamma\in\mathbb{Z}} \frac{p^{\gamma(\,(\alpha_i+n)\lambda_i q_i+\alpha_i+n)\frac{1}{q_i}}}{p^{\gamma(\alpha_i+n)\left(\frac{1}{q_i}+\lambda_i\right)}} = 1,$$

and, similarly, we also have

$$\|\,|\cdot|_p^{(\alpha+n)\lambda}\|_{\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)} \simeq 1. \tag{3.12}$$

Next, by choosing g_i's and using (3.3) and (3.10), we get

$$\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{g}\right)(x) \gtrsim \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \left\|A_i^{-1}(y)\right\|_p^{-(\alpha_i+n)\lambda_i} |x|_p^{(\alpha_i+n)\lambda_i} dy = \mathcal{C}_3 |x|_p^{(\alpha+n)\lambda}.$$

Thus, by (3.12), it follows that

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{g}\right)\right\|_{\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)} \gtrsim \mathcal{C}_3 \|\,|\cdot|_p^{(\alpha+n)\lambda}\|_{\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)} \gtrsim \mathcal{C}_3 \prod_{i=1}^m \|g_i\|_{\dot{B}^{q_i,\lambda_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

This gives that $\mathcal{C}_3 < \infty$.

Theorem 3.3 is proved.

Theorem 3.4. *Let* $1 \leq q^*, \zeta < \infty$, $\lambda_i \in \left(-\dfrac{1}{q_i}, 0\right)$ *for all* $i = 1, \ldots, m$ *and* $\omega \in A_\zeta$ *with the finite critical index* r_ω *for the reverse Hölder inequality. Assume that* $q > q^*\zeta r_\omega/(r_\omega - 1)$, $\delta \in (1, r_\omega)$ *and the following two conditions are true:*

$$\lambda = \lambda_1 + \cdots + \lambda_m, \tag{3.13}$$

$$\mathcal{C}_4 = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{q_i}} \|A_i(y)\|_p^{\frac{\zeta}{q_i}} \times$$

$$\times \left(\chi_{\{\|A_i(y)\|_p \leq 1\}}(y) \|A_i(y)\|_p^{n\zeta\lambda_i} + \chi_{\{\|A_i(y)\|_p > 1\}}(y) \|A_i(y)\|_p^{\frac{n\lambda_i(\delta-1)}{\delta}}\right) dy < \infty.$$

Then $\mathcal{H}^p_{\Phi,\vec{A}}$ *is bounded from* $\dot{B}^{q_1,\lambda_1}_\omega(\mathbb{Q}_p^n) \times \ldots \times \dot{B}^{q_m,\lambda_m}_\omega(\mathbb{Q}_p^n)$ *to* $\dot{B}^{q^*,\lambda}_\omega(\mathbb{Q}_p^n)$.

Proof. For $\gamma \in \mathbb{Z}$, by estimating as (3.8) above and using the relation (3.13), we obtain

$$\frac{1}{\omega(B_\gamma)^{\frac{1}{q^*}+\lambda}} \left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(B_\gamma)} \lesssim$$

$$\lesssim \left(\int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{q_i}} \|A_i(y)\|_p^{\frac{\zeta n}{q_i}} \left(\frac{\omega(B_{\gamma+k_{A_i}})}{\omega(B_\gamma)}\right)^{\lambda_i} dy\right) \prod_{i=1}^m \|f_i\|_{\dot{B}^{q_i,\lambda_i}_\omega(\mathbb{Q}_p^n)}.$$

In addition, for $i = 1, \ldots, m$, by using Proposition 2.3 again and $\lambda_i < 0$, we infer

$$\left(\frac{\omega(B_{\gamma+k_{A_i}})}{\omega(B_\gamma)}\right)^{\lambda_i} \lesssim \begin{cases} \left(\dfrac{|B_{\gamma+k_{A_i}}|}{|B_\gamma|}\right)^{\zeta\lambda_i} \lesssim p^{(\gamma+k_{A_i}-\gamma)n\zeta\lambda_i} = \|A_i(y)\|_p^{n\zeta\lambda_i}, & \text{if } \|A_i(y)\|_p \leq 1, \\ \left(\dfrac{|B_{\gamma+k_{A_i}}|}{|B_\gamma|}\right)^{\frac{\lambda_i(\delta-1)}{\delta}} \lesssim p^{\frac{(\gamma+k_{A_i}-\gamma)n\lambda_i(\delta-1)}{\delta}} = \|A_i(y)\|_p^{\frac{n\lambda_i(\delta-1)}{\delta}}, & \text{otherwise}. \end{cases} \tag{3.14}$$

Thus, we have

$$\left\|\mathcal{H}^p_{\Phi,\vec{A}}\left(\vec{f}\right)\right\|_{\dot{B}^{q^*,\lambda}_\omega(\mathbb{Q}_p^n)} \lesssim \mathcal{C}_4 \prod_{i=1}^m \|f_i\|_{\dot{B}^{q_i,\lambda_i}_\omega(\mathbb{Q}_p^n)}.$$

Theorem 3.4 is proved.

4. Main results about the boundness of $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$. Before stating next results, we introduce some notations which will be used throughout this section. Let $q, q_i \in [1, \infty)$ and α, α_i, r_i be real numbers such that $r_i \in (1, \infty)$, $\alpha_i \in \left(-n, \dfrac{nr_i}{r_i'}\right)$, $i = 1, 2, \ldots, m$. Suppose that

$$\left(\frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_m}\right) + \left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_m}\right) = \frac{1}{q},$$

$$\left(\frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2} + \ldots + \frac{\alpha_m}{q_m}\right) + \left(\frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2} + \ldots + \frac{\alpha_m}{r_m}\right) = \frac{\alpha}{q}.$$

Lemma 4.1. *Let $\omega(x) = |x|_p^{\alpha}$, $\omega_i(x) = |x|_p^{\alpha_i}$ and $b_i \in CMO_{\omega_i}^{r_i}\left(\mathbb{Q}_p^n\right)$ for all $i = 1, \ldots, m$. Then, for any $\gamma \in \mathbb{Z}$, we have*

$$\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{L^q_\omega(B_\gamma)} \lesssim p^{\sum_{i=1}^m \frac{\gamma(n+\alpha_i)}{r_i}} \mathcal{B}_{\vec{r},\vec{\omega}} \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \psi_i(y)\mu_i(y)\|f_i\|_{L^{q_i}_{\omega_i}(B_{\gamma+k_{A_i}})} dy,$$

where

$$\psi_i(y) = 1 + \left(\max\left\{\left\|A_i^{-1}(y)\right\|_p^{\alpha_i}, \|A_i(y)\|_p^{-\alpha_i}\right\} |\det A_i^{-1}(y)|_p\right)^{\frac{1}{r_i}} \|A_i(y)\|_p^{\frac{(n+\alpha_i)}{r_i}} +$$

$$+ \left|\log_p \|A_i(y)\|_p\right| + 2\,\frac{\|A_i(y)\|_p^n}{|\det A_i(y)|_p},$$

$$\mu_i(y) = \left(\max\left\{\left\|A_i^{-1}(y)\right\|_p^{\alpha_i}, \|A_i(y)\|_p^{-\alpha_i}\right\} \left|\det A_i^{-1}(y)\right|_p\right)^{\frac{1}{q_i}} \quad \text{and} \quad \mathcal{B}_{\vec{r},\vec{\omega}} = \prod_{i=1}^m \|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

Proof. In what follows, we will write b_{i,B_γ} instead of $(b_i)_{B_\gamma}$ for convenience. By the Minkowski inequality and the Hölder inequality, for any $\gamma \in \mathbb{Z}$, we get

$$\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{L^q_\omega(B_\gamma)} \lesssim \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^{r_i}_{\omega_i}(B_\gamma)} \|f_i(A_i(y)\cdot)\|_{L^{q_i}_{\omega_i}(B_\gamma)}\, dy. \tag{4.1}$$

To prove this lemma, we need to show that the following inequality holds:

$$\|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^{r_i}_{\omega_i}(B_\gamma)} \lesssim p^{\frac{\gamma(\alpha_i+n)}{r_i}} \psi_i(y)\|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)} \quad \text{for all} \quad i = 1, \ldots, m. \tag{4.2}$$

We put $I_{1,i} = \|b_i(\cdot) - b_{i,B_\gamma}\|_{L^{r_i}_{\omega_i}(B_\gamma)}$, $I_{2,i} = \|b_i(A_i(y)\cdot) - b_{i,A_i(y)B_\gamma}\|_{L^{r_i}_{\omega_i}(B_\gamma)}$ and $I_{3,i} = \|b_{i,B_\gamma} - - b_{i,A_i(y)B_\gamma}\|_{L^{r_i}_{\omega_i}(B_\gamma)}$. It is obvious that

$$\|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^{r_i}_{\omega_i}(B_\gamma)} \le I_{1,i} + I_{2,i} + I_{3,i} \quad \text{for all} \quad i = 1, \ldots, m. \tag{4.3}$$

By the definition of the space $CMO^{r_i}_{\omega_i}\left(\mathbb{Q}_p^n\right)$ and the estimation (2.2), we have

$$I_{1,i} \le \omega_i(B_\gamma)^{\frac{1}{r_i}} \|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)} \lesssim p^{\frac{\gamma(\alpha_i+n)}{r_i}} \|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)}. \tag{4.4}$$

To estimate $I_{2,i}$, we deduce that

$$I_{2,i} \le \omega_i(B_\gamma)^{\frac{1}{r_i}} \left|b_{i,A_i(y)B_\gamma} - b_{i,B_{\gamma+k_{A_i}}}\right| + \left(\int\limits_{B_\gamma} \left|b_i(A_i(y)x) - b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i} \omega_i(x)dx\right)^{\frac{1}{r_i}}, \tag{4.5}$$

where $k_{A_i}(y) = \log_p \|A_i(y)\|_p$. Note that, by the formula for change of variables, we get

$$|A_i(y)B_\gamma| = \int\limits_{A_i(y)B_\gamma} dx = \int\limits_{B_\gamma} |\det A_i(y)|_p dz \simeq |\det A_i(y)|_p p^{\gamma n}. \tag{4.6}$$

Thus, by using the Hölder inequality and (2.2), it is clear to see that

$$\left|b_{i,A_i(y)B_\gamma} - b_{i,B_{\gamma+k_{A_i}}}\right| \leq$$

$$\leq \frac{1}{|A_i(y)B_\gamma|}\left(\int\limits_{B_{\gamma+k_{A_i}}} \left|b_i(x) - b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i}\omega_i(x)dx\right)^{\frac{1}{r_i}}\left(\int\limits_{B_{\gamma+k_{A_i}}} \omega_i^{\frac{-r_i'}{r_i}}dx\right)^{\frac{1}{r_i'}} \lesssim$$

$$\lesssim \frac{p^{\frac{(\gamma+k_{A_i})(n+\alpha_i)}{r_i}} p^{(\gamma+k_{A_i})\left(\frac{-\alpha_i}{r_i}+\frac{n}{r_i'}\right)}}{|\det A_i(y)|_p p^{\gamma n}}\|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)} = \frac{\|A_i(y)\|_p^n}{|\det A_i(y)|_p}\|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)}. \tag{4.7}$$

It is easy to see that

$$\left(\int\limits_{B_\gamma} \left|b_i(A_i(y)x) - b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i}\omega_i(x)dx\right)^{\frac{1}{r_i}} \leq$$

$$\leq \left(\max\left\{\left\|A_i^{-1}(y)\right\|_p^{\alpha_i}, \|A_i(y)\|_p^{-\alpha_i}\right\}\left|\det A_i^{-1}(y)\right|_p \omega_i(B_{\gamma+k_{A_i}})\right)^{\frac{1}{r_i}} \times$$

$$\times\left(\frac{1}{\omega_i(B_{\gamma+k_{A_i}})}\int\limits_{B_{\gamma+k_{A_i}}} \left|b_i(z) - b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i}\omega_i(z)dz\right)^{\frac{1}{r_i}}. \tag{4.8}$$

This leads to

$$\left(\int\limits_{B_\gamma} \left|b_i(A_i(y)x) - b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i}\omega_i(x)dx\right)^{\frac{1}{r_i}} \lesssim$$

$$\lesssim p^{\frac{\gamma(n+\alpha_i)}{r_i}}\left(\max\left\{\left\|A_i^{-1}(y)\right\|_p^{\alpha_i}, \|A_i(y)\|_p^{-\alpha_i}\right\}\left|\det A_i^{-1}(y)\right|_p\right)^{\frac{1}{r_i}}\|A_i(y)\|_p^{\frac{(n+\alpha_i)}{r_i}}\|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

Therefore, by (4.5) and (4.7), we have

$$I_{2,i} \lesssim \left(\frac{\|A_i(y)\|_p^n}{|\det A_i(y)|_p} + \left(\max\left\{\left\|A_i^{-1}(y)\right\|_p^{\alpha_i}, \|A_i(y)\|_p^{-\alpha_i}\right\}\left|\det A_i^{-1}(y)\right|_p\right)^{\frac{1}{r_i}}\|A_i(y)\|_p^{\frac{(n+\alpha_i)}{r_i}}\right) \times$$

$$\times p^{\frac{\gamma(n+\alpha_i)}{r_i}}\|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)}. \tag{4.9}$$

Next, we consider the term $I_{3,i}$. We obtain

$$I_{3,i} \leq \omega_i(B_\gamma)^{\frac{1}{r_i}}\left|b_{i,B_\gamma} - b_{i,A_i(y)B_\gamma}\right|. \tag{4.10}$$

Fix $y \in \mathbb{Q}_p^n$. We set

$$S_{k_{A_i}} = \begin{cases} \{j \in \mathbb{Z} \colon 1 \le j \le k_{A_i}\}, & \text{if} \quad k_{A_i} \ge 1, \\ \{j \in \mathbb{Z} \colon k_{A_i} + 1 \le j \le 0\}, & \text{otherwise.} \end{cases}$$

As mentioned above, we obtain

$$\left|b_{i,B_\gamma} - b_{i,A_i(y)B_\gamma}\right| \le \sum_{j \in S_{k_{A_i}}} \left|b_{i,B_{\gamma+j-1}} - b_{i,B_{\gamma+j}}\right| + \left|b_{i,B_{\gamma+k_{A_i}}} - b_{i,A_i(y)B_\gamma}\right|. \tag{4.11}$$

Combining the Hölder inequality, the definition of the space $CMO^{r_i}_{\omega_i}\left(\mathbb{Q}_p^n\right)$ and (2.2), one has

$$\left|b_{i,B_{\gamma+j-1}} - b_{i,B_{\gamma+j}}\right| \le \frac{\omega_i(B_{\gamma+j})^{\frac{1}{r_i}}}{|B_{\gamma+j}|} \left(\int\limits_{B_{\gamma+j}} \omega_i^{\frac{-r_i'}{r_i}} dx \right)^{\frac{1}{r_i'}} \times$$

$$\times \left(\frac{1}{\omega_i(B_{\gamma+j})} \int\limits_{B_{\gamma+j}} \left|b_i(z) - b_{i,B_{\gamma+j}}\right|^{r_i} \omega_i(x) dz \right)^{\frac{1}{r_i}} \lesssim$$

$$\lesssim \frac{p^{(\gamma+j)\frac{(\alpha_i+n)}{r_i}}}{p^{(\gamma+j)n}} p^{(\gamma+j)\left(\frac{-\alpha_i}{r_i} + \frac{n}{r_i'}\right)} \|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)} = \|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

Thus,

$$\left|b_{i,B_\gamma} - b_{i,A_i(y)B_\gamma}\right| \lesssim |k_{A_i}| \, \|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)} + \left|b_{i,B_{\gamma+k_{A_i}}} - b_{i,A_i(y)B_\gamma}\right|. \tag{4.12}$$

In addition, by the Hölder inequality again and (4.6), we get

$$\left|b_{i,B_{\gamma+k_{A_i}}} - b_{i,A_i(y)B_\gamma}\right| \le \frac{1}{|A_i(y)B_\gamma|} \int\limits_{A_i(y)B_\gamma} \left|b_i(x) - b_{i,B_{\gamma+k_{A_i}}}\right| dx \le$$

$$\le \frac{\omega_i(B_{\gamma+k_{A_i}})^{\frac{1}{r_i}}}{|A_i(y)B_\gamma|} \left(\int\limits_{B_{\gamma+k_{A_i}}} \omega_i^{-\frac{r_i'}{r_i}} dx \right)^{\frac{1}{r_i'}} \left(\frac{1}{\omega_i(B_{\gamma+k_{A_i}})} \int\limits_{B_{\gamma+k_{A_i}}} \left|b_i(x) - b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i} \omega_i(x) dx \right)^{\frac{1}{r_i}} \lesssim$$

$$\lesssim \frac{p^{(\gamma+k_{A_i})\frac{(\alpha_i+n)}{r_i}}}{|\det A_i(y)|_p p^{\gamma n}} p^{(\gamma+k_{A_i})\left(\frac{-\alpha_i}{r_i} + \frac{n}{r_i'}\right)} \|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)} = \frac{\|A_i(y)\|_p^n}{|\det A_i(y)|_p} \|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

Consequently, by (4.10) – (4.12), it follows that

$$I_{3,i} \lesssim p^{\frac{\gamma(n+\alpha_i)}{r_i}} \left(|\log_p \|A_i(y)\|_p| + \frac{\|A_i(y)\|_p^n}{|\det A_i(y)|_p} \right) \|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

This together with (4.3), (4.4) and (4.9) follow us to have the proof of the inequality (4.2). Finally, by estimating as (4.8), we immediately have

$$\|f_i(A_i(y)\cdot)\|_{L^{q_i}_{\omega_i}(B_\gamma)} \le \left(\max\left\{\left\|A_i^{-1}(y)\right\|_p^{\alpha_i}, \|A_i(y)\|_p^{-\alpha_i}\right\} |\det A_i^{-1}(y)|_p\right)^{\frac{1}{q_i}} \|f_i\|_{L^{q_i}_{\omega_i}\left(B_{\gamma+k_{A_i}}\right)} =$$

$$= \mu_i(y)\|f_i\|_{L^{q_i}_{\omega}\left(B_{\gamma+k_{A_i}}\right)}.$$

In view of (4.1) and (4.2), the proof of this lemma is ended.

Lemma 4.2. *Let* $1 \le q^*, r_1^*, \ldots, r_m^*, q_1^*, \ldots, q_m^*, \zeta < \infty$, $\omega \in A_\zeta$ *with the finite critical index* r_ω *for the reverse Hölder condition,* $\delta \in (1, r_\omega)$, $\lambda_i \in \left(\dfrac{-1}{q_i^*}, 0\right)$, $\zeta \le r_i^*$ *and* $b_i \in CMO_\omega^{r_i^*}(\mathbb{Q}_p^n)$ *for all* $i = 1, \ldots, m$. *Assume that the following condition holds:*

$$\frac{1}{q^*} > \left(\frac{1}{r_1^*} + \ldots + \frac{1}{r_m^*} + \frac{1}{q_1^*} + \ldots + \frac{1}{q_m^*}\right)\zeta\,\frac{r_\omega}{r_\omega - 1}. \tag{4.13}$$

Then we have

$$\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(B_\gamma)} \lesssim$$

$$\lesssim \omega(B_\gamma)^{\frac{1}{q^*}} \mathcal{B}_{\vec{r}^*,\omega}\left(\int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \psi_i^*(y)\mu_i^*(y) \frac{1}{\omega(B_{\gamma+k_{A_i}})^{\frac{1}{q_i^*}}} \|f_i\|_{L^{q_i^*}_\omega\left(B_{\gamma+k_{A_i}}\right)} dy\right) \quad \textit{for all} \ \ \gamma \in \mathbb{Z}.$$

Here,

$$\psi_i^*(y) = 1 + \frac{2\|A_i(y)\|_p^n}{|\det A_i(y)|_p} + \left|\det A_i^{-1}(y)\right|^{\frac{\zeta}{r_i^*}} \|A_i(y)\|_p^{\frac{n\zeta}{r_i^*}} + |\log_p \|A_i(y)\|_p|,$$

$$\mu_i^*(y) = \left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{q_i^*}} \|A_i(y)\|_p^{\frac{n\zeta}{q_i^*}} \quad \textit{and} \quad \mathcal{B}_{\vec{r}^*,\omega} = \prod_{i=1}^m \|b_i\|_{CMO_\omega^{r_i^*}(\mathbb{Q}_p^n)}.$$

Proof. By virtue of the inequality (4.13), there exist $r_1, \ldots, r_m, q_1, \ldots, q_m$ such that

$$\frac{1}{q_i} > \frac{\zeta}{q_i^*}\frac{r_\omega}{r_\omega - 1}, \quad \frac{1}{r_i} > \frac{\zeta}{r_i^*}\frac{r_\omega}{r_\omega - 1} \text{ for all } i = 1, \ldots, m, \text{ and } \frac{1}{q_1} + \ldots + \frac{1}{q_m} + \frac{1}{r_1} + \ldots + \frac{1}{r_m} = \frac{1}{q^*}.$$

As mentioned above, for any $\gamma \in \mathbb{Z}$, by the same argument (4.1), we also get

$$\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(B_\gamma)} \lesssim \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^{r_i}_\omega(B_\gamma)} \|f_i(A_i(y)\cdot)\|_{L^{q_i}_\omega(B_\gamma)}\,dy. \tag{4.14}$$

In particular, we need to show the following result:

$$\|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^{r_i}_\omega(B_\gamma)} \lesssim \omega(B_\gamma)^{\frac{1}{r_i}} \psi_i^*(y)\|b_i\|_{CMO_\omega^{r_i^*}(\mathbb{Q}_p^n)} \tag{4.15}$$

for all $i = 1, \ldots, m$. Indeed, we see that

$$\|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^{r_i}_\omega(B_\gamma)} \le \left\|b_i(\cdot) - b_{i,B_\gamma}\right\|_{L^{r_i}_\omega(B_\gamma)} + \left\|b_i(A_i(y)\cdot) - b_{i,A_i(y)B_\gamma}\right\|_{L^{r_i}_\omega(B_\gamma)} +$$

$$+\left\|b_{i,B_\gamma}-b_{i,A_i(y)B_\gamma}\right\|_{L^{r_i}_\omega(B_\gamma)}:=J_{1,i}+J_{2,i}+J_{3,i}. \tag{4.16}$$

By virtue of the inequality $r_1<r_1^*$, it is easy to show that

$$J_{1,i}\le\omega(B_\gamma)^{\frac{1}{r_i}}\|b_i\|_{CMO^{r_i^*}_\omega(\mathbb{Q}_p^n)}. \tag{4.17}$$

Next, by estimating as (4.5) above, we get

$$J_{2,i}\le\omega(B_\gamma)^{\frac{1}{r_i}}|b_{i,A_i(y)B_\gamma}-b_{i,B_{\gamma+k_{A_i}}}|+\left(\int\limits_{B_\gamma}|b_i(A_i(y)x)-b_{i,B_{\gamma+k_{A_i}}}|^{r_i}\omega(x)dx\right)^{\frac{1}{r_i}}. \tag{4.18}$$

By having the inequality $\zeta\le r_i^*$ and applying Proposition 2.4 and (4.6), we infer that

$$\left|b_{i,B_{\gamma+k_{A_i}}}-b_{i,A_i(y)B_\gamma}\right|\le\frac{1}{|A_i(y)B_\gamma|}\int\limits_{B_{\gamma+k_{A_i}}}|b_i(x)-b_{i,B_{\gamma+k_{A_i}}}|dx\lesssim$$

$$\lesssim\frac{p^{(\gamma+k_{A_i})n}}{|\det A_i(y)|_p\,.p^{\gamma n}}\|b_i\|_{CMO^{\zeta}_\omega(\mathbb{Q}_p^n)}\le\frac{\|A_i(y)\|_p^n}{|\det A_i(y)|_p}\|b_i\|_{CMO^{r_i^*}_\omega(\mathbb{Q}_p^n)}. \tag{4.19}$$

By $\dfrac{1}{r_i}>\dfrac{\zeta}{r_i^*}\dfrac{r_\omega}{r_\omega-1}$, there exists $\beta_i\in(1,r_\omega)$ satisfying $\dfrac{r_i^*}{\zeta}=r_i\beta_i'$. Thus, by combining the Hölder inequality and the reverse Hölder condition again, we have

$$\left(\int\limits_{B_\gamma}\left|b_i(A_i(y)x)-b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i}\omega(x)dx\right)^{\frac{1}{r_i}}\lesssim$$

$$\lesssim|B_\gamma|^{\frac{-\zeta}{r_i^*}}\omega(B_\gamma)^{\frac{1}{r_i}}\left(\int\limits_{B_\gamma}\left|b_i(A_i(y)x)-b_{i,B_{\gamma+k_{A_i}}}\right|^{\frac{r_i^*}{\zeta}}dx\right)^{\frac{\zeta}{r_i^*}}.$$

According to Proposition 2.4, we get

$$\left(\int\limits_{B_\gamma}\left|b_i(A_i(y)x)-b_{i,B_{\gamma+k_{A_i}}}\right|^{\frac{r_i^*}{\zeta}}dx\right)^{\frac{\zeta}{r_i^*}}\le$$

$$\le\left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{r_i^*}}\left(\int\limits_{B_{\gamma+k_{A_i}}}\left|b_i(z)-b_{i,B_{\gamma+k_{A_i}}}\right|^{\frac{r_i^*}{\zeta}}dz\right)^{\frac{\zeta}{r_i^*}}\le$$

$$\le\left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{r_i^*}}\frac{|B_{\gamma+k_{A_i}}|^{\frac{\zeta}{r_i^*}}}{\omega(B_{\gamma+k_{A_i}})^{\frac{1}{r_i^*}}}\left(\int\limits_{B_{\gamma+k_{A_i}}}\left|b_i(z)-b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i^*}\omega(z)dz\right)^{\frac{1}{r_i^*}}.$$

In view of (2.2), one has $\dfrac{|B_{\gamma+k_{A_i}}|}{|B_\gamma|} \simeq \|A_i(y)\|_p^n$. From this we give

$$\left(\int\limits_{B_\gamma} \left|b_i(A_i(y)x) - b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i} \omega(x)dx\right)^{\frac{1}{r_i}} \lesssim$$

$$\lesssim \omega(B_\gamma)^{\frac{1}{r_i}} \left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{r_i^*}} \|A_i(y)\|_p^{\frac{n\zeta}{r_i^*}} \left(\frac{1}{\omega(B_{\gamma+k_{A_i}})} \int\limits_{B_{\gamma+k_{A_i}}} \left|b_i(z) - b_{i,B_{\gamma+k_{A_i}}}\right|^{r_i^*} \omega(z)dz\right)^{\frac{1}{r_i^*}} \lesssim$$

$$\lesssim \omega(B_\gamma)^{\frac{1}{r_i}} \left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{r_i^*}} \|A_i(y)\|_p^{\frac{n\zeta}{r_i^*}} \|b_i\|_{CMO_\omega^{r_i^*}(\mathbb{Q}_p^n)}. \tag{4.20}$$

As a consequence, by (4.18) and (4.19), we infer that

$$J_{2,i} \lesssim \omega(B_\gamma)^{\frac{1}{r_i}} \left(\frac{\|A_i(y)\|_p^n}{\left|\det A_i(y)\right|_p} + \left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{r_i^*}} \|A_i(y)\|_p^{\frac{n\zeta}{r_i^*}}\right) . \|b\|_{CMO_\omega^{r_i^*}(\mathbb{Q}_p^n)}. \tag{4.21}$$

Now, we will estimate $J_{3,i}$. By a same argument as (4.10), (4.11) and (4.19), we have

$$J_{3,i} \le \omega(B_\gamma)^{\frac{1}{r_i}} \left(\sum_{j\in S_{k_{A_i}}} \left|b_{i,B_{\gamma+j-1}} - b_{i,B_{\gamma+j}}\right| + \left|b_{i,B_{\gamma+k_{A_i}}} - b_{i,A_i(y)B_\gamma}\right|\right) \lesssim$$

$$\lesssim \omega(B_\gamma)^{\frac{1}{r_i}} \left(\sum_{j\in S_{k_{A_i}}} \|b_i\|_{CMO_\omega^{r_i^*}(\mathbb{Q}_p^n)} + \frac{\|A_i(y)\|_p^n}{|\det A_i(y)|_p} \|b_i\|_{CMO_\omega^{r_i^*}(\mathbb{Q}_p^n)}\right) \le$$

$$\le \omega(B_\gamma)^{\frac{1}{r_i}} \left(|\log_p \|A_i(y)\|_p| + \frac{\|A_i(y)\|_p^n}{|\det A_i(y)|_p}\right) \|b_i\|_{CMO_\omega^{r_i^*}(\mathbb{Q}_p^n)}.$$

This together with (4.17) and (4.21) yields that the inequality (4.15) is finished.

In other words, by estimating as (4.20) above, we get

$$\|f_i(A_i(y)\cdot)\|_{L_\omega^{q_i}(B_\gamma)} \lesssim \omega(B_\gamma)^{\frac{1}{q_i}} \left|\det A_i^{-1}(y)\right|_p^{\frac{\zeta}{q_i^*}} \|A_i(y)\|_p^{\frac{n\zeta}{q_i^*}} \omega(B_{\gamma+k_{A_i}})^{\frac{-1}{q_i^*}} \|f_i\|_{L_\omega^{q_i^*}\left(B_{\gamma+k_{A_i}}\right)} =$$

$$= \omega(B_\gamma)^{\frac{1}{q_i}} \mu_i^*(y) \omega(B_{\gamma+k_{A_i}})^{\frac{-1}{q_i^*}} \|f_i\|_{L_\omega^{q_i^*}\left(B_{\gamma+k_{A_i}}\right)}.$$

Hence, by (4.14) and (4.15), we conclude that the proof of this lemma is finished.

Theorem 4.1. *Let the assumptions of Lemma* 4.1 *hold and*

$$\mathcal{C}_5 = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \psi_i(y)\mu_i(y)dy < \infty.$$

Then, for any $\gamma \in \mathbb{Z}$, we have that $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$ is bounded from $L^{q_1}_{\omega_1}\left(\mathbb{Q}_p^n\right) \times \ldots \times L^{q_m}_{\omega_m}\left(\mathbb{Q}_p^n\right)$ to $L^q_\omega(B_\gamma)$.

Proof. For any $\gamma \in \mathbb{Z}$, by using Lemma 4.1, we infer that

$$\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{L^q_\omega(B_\gamma)} \lesssim p^{\sum_{i=1}^m \frac{\gamma(n+\alpha_i)}{r_i}} \mathcal{B}_{\vec{r},\vec{\omega}} \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \psi_i(y)\mu_i(y)\|f_i\|_{L^{q_i}_{\omega_i}(B_{\gamma+k_{A_i}})}dy.$$

Thus, we have $\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{L^q_\omega(B_\gamma)} \lesssim \mathcal{C}_5\mathcal{B}_{\vec{r},\vec{\omega}} \prod_{i=1}^m \|f_i\|_{L^{q_i}_{\omega_i}(\mathbb{Q}_p^n)}$.

Theorem 4.1 is proved.

Theorem 4.2. *Let the assumptions of Lemma* 4.2 *hold. Suppose that* $\omega(B_\gamma) \lesssim 1$ *for all* $\gamma \in \mathbb{Z}$ *and*

$$\mathcal{C}_6 = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \psi_i^*(y)\mu_i^*(y)\times$$

$$\times\left(\chi_{\{\|A_i(y)\|_p\leq 1\}}(y)\|A_i(y)\|_p^{\frac{-n\zeta}{q_i^*}} + \chi_{\{\|A_i(y)\|_p>1\}}(y)\|A_i(y)\|_p^{\frac{-n(\delta-1)}{q_i^*\delta}}\right)dy < \infty.$$

Then we have that $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$ *is bounded from* $L^{q_1^*}_\omega\left(\mathbb{Q}_p^n\right) \times \ldots \times L^{q_m^*}_\omega\left(\mathbb{Q}_p^n\right)$ *to* $L^{q^*}_\omega\left(\mathbb{Q}_p^n\right)$.

Proof. In view of Lemma 4.2, for any $R \in \mathbb{Z}$, it is clear to see that

$$\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(B_R)} \lesssim$$

$$\lesssim \omega(B_R)^{\frac{1}{q^*}}\mathcal{B}_{\vec{r^*},\omega}\left(\int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \psi_i^*(y)\mu_i^*(y)\frac{1}{\omega\left(B_{R+k_{A_i}}\right)^{\frac{1}{q_i^*}}}\|f_i\|_{L^{q_i^*}_\omega\left(B_{R+k_{A_i}}\right)}dy\right).$$

Next, by $\dfrac{1}{q^*} > \dfrac{1}{q_1^*} + \ldots + \dfrac{1}{q_m^*}$ and the assumption $\omega(B_R) \lesssim 1$ for any $R \in \mathbb{Z}$, we have $\omega(B_R)^{\frac{1}{q^*}} \leq$ $\leq \prod_{i=1}^m \omega(B_R)^{\frac{1}{q_i^*}}$. Thus,

$$\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(B_R)} \lesssim$$

$$\lesssim \mathcal{B}_{\vec{r^*},\omega}\left(\int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \psi_i^*(y)\mu_i^*(y)\left(\frac{\omega(B_R)}{\omega(B_{R+k_{A_i}})}\right)^{\frac{1}{q_i^*}}dy\right)\prod_{i=1}^m \|f_i\|_{L^{q_i^*}_\omega\left(\mathbb{Q}_p^n\right)} \lesssim$$

$$\lesssim \mathcal{C}_6\mathcal{B}_{\vec{r^*},\omega}\prod_{i=1}^m \|f_i\|_{L^{q_i^*}_\omega\left(\mathbb{Q}_p^n\right)}.$$

Consequence, by letting $R \to +\infty$ and applying dominated convergence theorem of Lebesgue, we have

$$\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}(\vec{f}\,)\right\|_{L^{q^*}_\omega\left(\mathbb{Q}_p^n\right)} \lesssim \mathcal{C}_6\mathcal{B}_{\vec{r^*},\omega}\prod_{i=1}^m \|f_i\|_{L^{q_i^*}_\omega\left(\mathbb{Q}_p^n\right)}.$$

Theorem 4.2 is proved.

Theorem 4.3. *Let* $1 < \zeta < \infty$, $1 \le q^*, q_i, r_i^* < \infty$, $-n < \alpha_i < n(\zeta - 1)$, $\omega(x) = |x|_p^{\alpha}$, $\omega_i(x) = |x|_p^{\alpha_i}$ *for all* $i = 1, \ldots, m$ *such that*

$$\frac{\alpha_1}{q_1} + \ldots + \frac{\alpha_m}{q_m} = \frac{\zeta\alpha}{q^*}, \quad \frac{1}{q_1} + \ldots + \frac{1}{q_m} = \frac{\zeta}{q^*}, \tag{4.22}$$

$$\frac{1}{q_1} + \ldots + \frac{1}{q_m} + \frac{1}{r_1^*} + \ldots + \frac{1}{r_m^*} = 1. \tag{4.23}$$

If $b_i \in CMO^{r_i^*}\left(\mathbb{Q}_p^n\right)$ *for all* $i = 1, \ldots, m$ *and*

$$\mathcal{C}_7 = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \Gamma_i(y) \|A_i(y)\|_p^{-\frac{(\zeta+n)}{\zeta q_i}} dy < \infty,$$

where

$$\Gamma_i(y) = \left(1 + |\log_p \|A_i(y)\|_p| + \frac{2\|A_i(y)\|_p^n}{|\det A_i(y)|_p} + \|A_i(y)\|_p^{\frac{n}{r_i^*}} \left|\det A_i^{-1}(y)\right|_p^{\frac{1}{r_i^*}}\right) \times$$

$$\times \left|\det A_i^{-1}(y)\right|_p^{\frac{1}{q_i}} \|A_i(y)\|_p^{\frac{n}{q_i}}, \tag{4.24}$$

then we have

$$\left\|\mathcal{M}^{\mathrm{mod}}\left(\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\right)\left(\vec{f}\right)\right\|_{L^{q^*}_{\omega}(\mathbb{Q}_p^n)} \lesssim \mathcal{C}_7 \left(\prod_{i=1}^m \|b_i\|_{CMO^{r_i^*}(\mathbb{Q}_p^n)}\right) \prod_{i=1}^m \|f_i\|_{L^{\zeta q_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

Proof. For the sake of simplicity, we denote $\mathcal{B}_{\vec{r}^*} = \prod_{i=1}^m \|b_i\|_{CMO^{r_i^*}(\mathbb{Q}_p^n)}$. Now, let $x \in \mathbb{Q}_p^n$ and fix a ball B_γ such that $x \in B_\gamma$. In view of (4.23), by using the Hölder inequality, we have

$$\frac{1}{|B_\gamma|} \int\limits_{B_\gamma} \left|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)(z)\right| dz \le$$

$$\le \frac{1}{|B_\gamma|} \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \|f_i(A_i(y)\cdot)\|_{L^{q_i}(B_\gamma)} \prod_{i=1}^m \|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^{r_i^*}(B_\gamma)} dy.$$

For $i = 1, \ldots, m$, by estimating as (4.2) above, we get

$$\|b_i(\cdot) - b_i(A_i(y)\cdot)\|_{L^{r_i^*}(B_\gamma)} \lesssim$$

$$\lesssim |B_\gamma|^{\frac{1}{r_i^*}} \left(1 + |\log_p \|A_i(y)\|_p| + \frac{2\|A_i(y)\|_p^n}{|\det A_i(y)|_p} + \|A_i(y)\|_p^{\frac{n}{r_i^*}} \left|\det A_i^{-1}(y)\right|_p^{\frac{1}{r_i^*}}\right) \|b_i\|_{CMO^{r_i^*}(\mathbb{Q}_p^n)}.$$

By $x \in B_\gamma$, we imply that $\|A_i(y)\|_p^{-1} x \in B_{\gamma + k_{A_i}}$. Thus, by definition of the Hardy – Littlewood maximal operator, one has

$$\|f_i(A_i(y)\cdot)\|_{L^{q_i}(\mathbb{Q}_p^n)} = \left|\det A_i^{-1}(y)\right|_p^{\frac{1}{q_i}} \left(\int\limits_{A_i(y)B_\gamma} |f_i(t)|^{q_i} dt\right)^{\frac{1}{q_i}} \le$$

$$\le \left|\det A_i^{-1}(y)\right|_p^{\frac{1}{q_i}} \left(\int\limits_{B_{\gamma+k_{A_i}}} |f_i(t)|^{q_i} dt \right)^{\frac{1}{q_i}} \lesssim$$

$$\lesssim |B_\gamma|^{\frac{1}{q_i}} \left|\det A_i^{-1}(y)\right|_p^{\frac{1}{q_i}} \|A_i(y)\|_p^{\frac{n}{q_i}} \left(\mathcal{M}\left(|f_i|^{q_i}\right)\left(\|A_i(y)\|_p^{-1}.x\right)\right)^{\frac{1}{q_i}}.$$

As mentioned above, we give

$$\frac{1}{|B_\gamma|} \int\limits_{B_\gamma} \left|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)(z)\right| dz \lesssim \mathcal{B}_{\vec{r}^*} \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \Gamma_i(y) \left(\mathcal{M}\left(|f_i|^{q_i}\right)\left(\|A_i(y)\|_p^{-1}x\right)\right)^{\frac{1}{q_i}} dy.$$

Hence, we infer that

$$\mathcal{M}^{\mathrm{mod}}\left(\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right)(x) \lesssim \mathcal{B}_{\vec{r}^*} \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \Gamma_i(y) \left(\mathcal{M}\left(|f_i|^{q_i}\right)\left(\|A_i(y)\|_p^{-1}.x\right)\right)^{\frac{1}{q_i}} dy.$$

Thus, by using the assumption (4.22), the Minkowski inequality and the Hölder inequality, we obtain

$$\left\|\mathcal{M}^{\mathrm{mod}}\left(\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right)\right\|_{L^{q^*}_\omega(\mathbb{Q}_p^n)} \le$$

$$\le \mathcal{B}_{\vec{r}^*} \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \Gamma_i(y) \prod_{i=1}^m \left(\int\limits_{\mathbb{Q}_p^n} \mathcal{M}\left(|f_i|^{q_i}\right)^\zeta \left(\|A_i(y)\|_p^{-1}x\right) \omega_i dx \right)^{\frac{1}{\zeta q_i}} dy. \tag{4.25}$$

For $i = 1, \ldots, m,$ by Proposition 2.2, we have $\omega_i \in A_\zeta$. From this, by virtue of the boundedness of the Hardy – Littlewood maximal operator on the Lebesgue spaces with the Muckenhoupt weights, we have

$$\left(\int\limits_{\mathbb{Q}_p^n} M\left(|f_i|^{q_i}\right)^\zeta \left(\|A_i(y)\|_p^{-1}x\right) \omega_i dx \right)^{\frac{1}{\zeta q_i}} =$$

$$= \left(\int\limits_{\mathbb{Q}_p^n} M\left(|f_i|^{q_i}\right)^\zeta (z) | \, \|A_i(y)\|_p z|_p^{\alpha_i} | \, \|A_i(y)\|_p^n|_p dz \right)^{\frac{1}{\zeta q_i}} =$$

$$= \|A_i(y)\|_p^{\frac{-(\alpha_i+n)}{\zeta q_i}} \left(\int\limits_{\mathbb{Q}_p^n} M\left(|f_i|^{q_i}\right)^\zeta (z) \omega_i(z) dz \right)^{\frac{1}{\zeta q_i}} \lesssim \|A_i(y)\|_p^{\frac{-(\alpha_i+n)}{\zeta q_i}} \|f_i\|_{L^{\zeta q_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

This together with (4.25) yields that the proof of this theorem is completed.

In what follows, we set

$$\mathcal{C}_8 = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \left\|A_i^{-1}(y)\right\|_p^{-(\alpha_i+n)\lambda_i} |\log_p \|A_i(y)\|_p| dy.$$

Theorem 4.4. *Suppose the hypothesis in Lemma* 4.1 *holds. Let* $\lambda_i \in \left(\dfrac{-1}{q_i}, 0\right)$ *for all* $i = 1, \ldots, m$ *and conditions* (3.1) *and* (3.10) *be true. Assume that*

$$\operatorname{supp}(\Phi) \subset \bigcap_{i=1}^{m} \left\{ y \in \mathbb{Q}_p^n : \|A_i(y)\|_p < 1 \right\}. \tag{4.26}$$

(i) *If* $\mathcal{C}_8 < \infty$, *then* $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$ *is bounded from* $\dot{B}^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}_p^n) \times \ldots \times \dot{B}^{q_m,\lambda_m}_{\omega_m}(\mathbb{Q}_p^n)$ *to* $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)$.

(ii) *If* $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$ *is bounded from* $\dot{B}^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}_p^n) \times \ldots \times \dot{B}^{q_m,\lambda_m}_{\omega_m}(\mathbb{Q}_p^n)$ *to* $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)$ *for all* $\vec{b} = (b_1, \ldots, b_m) \in CMO^{r_1}_{\omega_1}(\mathbb{Q}_p^n) \times \ldots \times CMO^{r_m}_{\omega_m}(\mathbb{Q}_p^n)$, *then* $\mathcal{C}_8 < \infty$.

Proof. Firstly, we prove the part (i) of the theorem. For any $R \in \mathbb{Z}$, by Lemma 4.1, we get

$$\frac{1}{\omega(B_R)^{\frac{1}{q}+\lambda}} \left\| \mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right) \right\|_{L^q_\omega(B_R)} \leq$$

$$\leq \mathcal{B}_{\vec{r},\vec{\omega}} \left(\int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^{m} \psi_i(y)\mu_i(y) \frac{p^{\sum_{i=1}^m \frac{R(n+\alpha_i)}{r_i}} \prod_{i=1}^m \omega_i\left(B_{R+k_{A_i}}\right)^{\frac{1}{q_i}+\lambda_i}}{\omega(B_R)^{\frac{1}{q}+\lambda}} dy \right) \prod_{i=1}^{m} \|f_i\|_{\dot{B}^{q_i,\lambda_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

Now, by (2.2) and (3.10), we calculate

$$\frac{p^{\sum_{i=1}^m \frac{R(n+\alpha_i)}{r_i}} \prod_{i=1}^m \omega_i(B_{R+k_{A_i}})^{\frac{1}{q_i}+\lambda_i}}{\omega(B_R)^{\frac{1}{q}+\lambda}} \simeq \prod_{i=1}^{m} \|A_i(y)\|_p^{(\alpha_i+n)\left(\frac{1}{q_i}+\lambda_i\right)}.$$

Hence, one has

$$\left\| \mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right) \right\|_{\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)} \lesssim$$

$$\lesssim \mathcal{B}_{\vec{r},\vec{\omega}} \left(\int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^{m} \psi_i(y)\mu_i(y) \|A_i(y)\|_p^{(\alpha_i+n)\left(\frac{1}{q_i}+\lambda_i\right)} dy \right) \prod_{i=1}^{m} \|f_i\|_{\dot{B}^{q_i,\lambda_i}_{\omega_i}(\mathbb{Q}_p^n)}. \tag{4.27}$$

Note that, by the hypothesis (4.26), we see that $|\log_p \|A_i(y)\|_p| \geq 1$ for all $y \in \operatorname{supp}(\Phi)$. As mentioned above, by (2.1) and (3.1), we make

$$\psi_i(y)\mu_i(y)\|A_i(y)\|_p^{(\alpha_i+n)\left(\frac{1}{q_i}+\lambda_i\right)} \lesssim$$

$$\lesssim \left(1 + \left\|A_i^{-1}(y)\right\|_p^{\frac{(\alpha_i+n)}{r_i}} \left\|A_i^{-1}(y)\right\|_p^{\frac{-(\alpha_i+n)}{r_i}} + |\log_p \|A_i(y)\|_p| + 2p^{\nu_{\vec{A}}} \right) \times$$

$$\times \left\|A_i^{-1}(y)\right\|_p^{\frac{(\alpha_i+n)}{q_i}} \left\|A_i^{-1}(y)\right\|_p^{-(\alpha_i+n)\left(\frac{1}{q_i}+\lambda_i\right)} \lesssim |\log_p \|A_i(y)\|_p| \|A_i(y)\|_p^{-(\alpha_i+n)\lambda_i}.$$

As an application, by (4.27), we obtain

$$\left\| \mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right) \right\|_{\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^n)} \lesssim \mathcal{C}_8 \mathcal{B}_{\vec{r},\vec{\omega}} \prod_{i=1}^{m} \|f_i\|_{\dot{B}^{q_i,\lambda_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

To give the proof for the part (ii) of the theorem, for $i = 1, \ldots, m$, let us choose $b_i(x) = \log_p |x|_p$ for all $x \in \mathbb{Q}_p^n \setminus \{0\}$, and $f_i(x) = |x|_p^{(n+\alpha_i)\lambda_i}$ for all $x \in \mathbb{Q}_p^n$. Now, we need to prove that

$$\|b_i\|_{CMO^{r_i}_{\omega_i}(\mathbb{Q}_p^n)} < \infty \quad \text{for all} \quad i = 1, \ldots, m. \tag{4.28}$$

In fact, for any $R \in \mathbb{Z}$, we see that $b_{i,R} = p^{-Rn} \sum_{\gamma \le R} \gamma p^{\gamma n}(1 - p^{-n}) = R - \dfrac{1}{p^n - 1}$. Thus, we get

$$\frac{1}{\omega_i(B_R)} \int\limits_{B_R} |b_i(x) - b_{i,B_R}|^{r_i} \omega_i dx = p^{-R(\alpha_i+n)} \sum_{\gamma \le R} \int\limits_{S_\gamma} \left| \gamma - \left(R - \frac{1}{p^n - 1} \right) \right|^{r_i} p^{\gamma \alpha_i} dx \lesssim$$

$$\lesssim p^{-R(\alpha_i+n)} \sum_{\ell \le 0} \left| \ell + \frac{1}{p^n - 1} \right|^{r_i} p^{(R+\ell)(\alpha_i+n)} \le \sum_{\ell \le 0} \left(|\ell|^{r_i} + \frac{1}{(p^n - 1)^{r_i}} \right) p^{\ell(\alpha_i+n)} < \infty$$

uniformly for $R \in \mathbb{Z}$. As an application, it immediately follows that the inequality (4.28) holds.

By choosing b_i and f_i, we get

$$\mathcal{H}^p_{\Phi,\vec{A},\vec{b}} \left(\vec{f} \right)(x) = \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m |A_i(y)x|_p^{(\alpha_i+n)\lambda_i} \left(\log_p \frac{|x|_p}{|A_i(y)x|_p} \right) dy.$$

By the hypothesis (4.26), we have $\|A_i(y)\|_p < 1$ for all $y \in \operatorname{supp}(\Phi)$. Thus, $|A_i(y)x|_p < |x|_p$. This gives that $0 < |\log_p \|A_i(y)\|_p| = \log_p \dfrac{1}{|A_i(y)|_p} \le \log_p \dfrac{|x|_p}{|A_i(y)x|_p}$. Consequently, by (3.1) and (4.1), we lead to

$$\mathcal{H}^p_{\Phi,\vec{A},\vec{b}} \left(\vec{f} \right)(x) \gtrsim \left(\int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m \|A_i(y)\|_p^{-(\alpha_i+n)\lambda_i} |\log_p \|A_i(y)\|_p| dy \right) |x|_p^{(\alpha+n)\lambda} = \mathcal{C}_8 |x|_p^{(\alpha+n)\lambda}.$$

From this, by (3.12) above, we infer that

$$\left\| \mathcal{H}^p_{\Phi,\vec{A},\vec{b}} \left(\vec{f} \right) \right\|_{\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} \gtrsim \mathcal{C}_8 \| |\cdot|_p^{(\alpha+n)\lambda} \|_{\dot{B}^{q,\lambda}_\omega(\mathbb{Q}_p^n)} \gtrsim \mathcal{C}_8 \prod_{i=1}^m \|f_i\|_{\dot{B}^{q_i,\lambda_i}_{\omega_i}(\mathbb{Q}_p^n)}.$$

Therefore, since $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$ is bounded from $\dot{B}^{q_1,\lambda_1}_{\omega_1}\left(\mathbb{Q}_p^n\right) \times \ldots \times \dot{B}^{q_m,\lambda_m}_{\omega_m}\left(\mathbb{Q}_p^n\right)$ to $\dot{B}^{q,\lambda}_{\omega}\left(\mathbb{Q}_p^n\right)$, it implies that $\mathcal{C}_8 < \infty$.

Theorem 4.4 is proved.

Now, we consider $A_i(y) = s_i(y) I_n$ for $i = 1, \ldots, m$. By the similar arguments, we then obtain the following useful result.

***Corollary* 4.1.** *Suppose that the hypothesis in Lemma* 4.1 *and* (3.10) *hold. Let*

$$\mathcal{C}_9 := \int\limits_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \prod_{i=1}^m |s_i(y)|_p^{(\alpha_i+n)\lambda_i} |\log_p |s_i(y)|_p| dy.$$

Then the following statements are equivalent:

(i) $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$ *is bounded from* $\dot{B}^{q_1,\lambda_1}_{\omega_1}\left(\mathbb{Q}_p^n\right)\times\ldots\times\dot{B}^{q_m,\lambda_m}_{\omega_m}\left(\mathbb{Q}_p^n\right)$ *to* $\dot{B}^{q,\lambda}_{\omega}\left(\mathbb{Q}_p^n\right)$ *for any* $\vec{b}=$ $=(b_1,\ldots,b_m)\in CMO^{r_1}_{\omega_1}\left(\mathbb{Q}_p^n\right)\times\ldots\times CMO^{r_m}_{\omega_m}\left(\mathbb{Q}_p^n\right)$.

(ii) $\mathcal{C}_9<\infty$.

Theorem 4.5. *Suppose that the hypothesis in Lemma* 4.2 *holds. Let* $\lambda_i\in\left(\dfrac{-1}{q_i^*},0\right)$ *for all* $i=1,\ldots,m$ *and condition* (3.13) *in Theorem* 3.4 *hold. Then, if*

$$\mathcal{C}_{10}=\int\limits_{\mathbb{Q}_p^n}\frac{\Phi(y)}{|y|_p^n}\prod_{i=1}^m\psi_i^*(y)\mu_i^*(y)\times$$

$$\times\left(\chi_{\{\|A_i(y)\|_p\le1\}}(y)\|A_i(y)\|_p^{n\zeta\lambda_i}+\chi_{\{\|A_i(y)\|_p>1\}(y)}\|A_i(y)\|_p^{\frac{n\lambda_i(\delta-1)}{\delta}}\right)dy<\infty,$$

we have that $\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}$ *is bounded from* $\dot{B}^{q_1^*,\lambda_1}_{\omega}\left(\mathbb{Q}_p^n\right)\times\ldots\times\dot{B}^{q_m^*,\lambda_m}_{\omega}\left(\mathbb{Q}_p^n\right)$ *to* $\dot{B}^{q^*,\lambda}_{\omega}\left(\mathbb{Q}_p^n\right)$.

Proof. For any $R\in\mathbb{Z}$, by Lemma 4.2 and (3.13), we infer

$$\frac{1}{\omega(B_R)^{\frac{1}{q^*}+\lambda}}\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{L^{q^*}_\omega(B_R)}\lesssim$$

$$\lesssim\mathcal{B}_{\vec{r}^*,\omega}\left(\int\limits_{\mathbb{Q}_p^n}\frac{\Phi(y)}{|y|_p^n}\prod_{i=1}^m\psi_i^*(y)\mu_i^*(y)\left(\frac{\omega(B_{R+k_{A_i}})}{\omega(B_R)}\right)^{\lambda_i}dy\right)\prod_{i=1}^m\|f_i\|_{\dot{B}^{q_i^*,\lambda_i}_{\omega}(\mathbb{Q}_p^n)}.$$

From this, by (3.14), we have

$$\left\|\mathcal{H}^p_{\Phi,\vec{A},\vec{b}}\left(\vec{f}\right)\right\|_{\dot{B}^{q^*,\lambda}_{\omega}(\mathbb{Q}_p^n)}\lesssim\mathcal{C}_{10}\mathcal{B}_{\vec{r}^*,\,\omega}\prod_{i=1}^m\|f_i\|_{\dot{B}^{q_i^*,\lambda_i}_{\omega}(\mathbb{Q}_p^n)}.$$

Theorem 4.5 is proved.

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Received 21.10.18,
after revision — 31.03.19