UDC 517.9

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TIME-IRREVERSIBILITY AND EXISTENCE AND UNIQUENESS OF SOLUTIONS OF PROBLEMS IN LINEAR VISCOELASTICITY

НЕЗВОРОТНІСТЬ ЧАСУ ТА ІСНУВАННЯ І ЄДИНІСТЬ РОЗВ'ЯЗКІВ У ЗАДАЧАХ ЛІНІЙНОЇ В'ЯЗКО-ПРУЖНОСТІ

A problem of linear viscoelasticity for the case where the relation between Cauchy stress and strain tensors is described by a linear integral relation is studied. Theorems on existence and uniqueness of a solution of the problem are proved.

Вивчається задача теорії пружності для випадку, коли зв'язок між тензорами напруги та деформації Коші описується за допомогою лінійного інтегрального співвідношення. Доведено теореми про існування та єдиність розв'язку відповідних задач.

1. The question posed by Fichera in [1] on the *principle of fading memory* [2] is related indirectly to the controversial time-reversal hypothesis. On this matter Fichera formulates significant counterexamples [3, 4] of relaxation functions $\mathbf{G}(s)$, $s \in [0, +\infty)$, for which the quasi-static problem for a linear viscoelastic body fails to have solutions, or it may have more than one solution.

Here we characterize the behaviour of these functions at infinity, by observing that both primitives of the Boltzmann function $\mathbf{G}'(s)$, $\mathbf{G}(s)-\mathbf{G}_0$ and $\mathbf{G}(s)-\mathbf{G}_\infty$, where $\mathbf{G}_0$ and $\mathbf{G}_\infty$ denote respectively the instantaneous and equilibrium elastic moduli, are summable in R^+ and that a reversal in the time direction does not change in this case the behaviour of the relaxation function at infinity.

This property is compatible with the time-reversal assumption and consequently it must allow the material to be classically elastic, i.e., $\mathbf{G}(s)=\mathbf{G}_0=\mathbf{G}_\infty$ $\forall s \in [0, +\infty)$, because in this case the memory effects are negligible. Thus Fichera's question can be answered by appropriate regularity assumptions on the relaxation and Boltzmann functions so that the convolution integral of the constitutive functional of linear viscoelasticity theory is well defined and materials of linear elastic type can be considered as classical linear viscoelastic materials.

The difficulty to answer this question lies in assuming general and physically admissible hypotheses. We assume that $\mathbf{G}(s)-\mathbf{G}_\infty$ and $\mathbf{G}'(s)$ are inverse Fourier transforms and that the integral

$$\int_{-at}^{+\infty} a\left[\mathbf{G}\left(t+\frac{y}{a}\right)-\mathbf{G}_\infty\right]\left[\left(\frac{\sin y - y\cos y}{y^2}\right)\right]dy$$

approaches zero when the parameter a, which has the dimension of frequency, approaches infinity, if both the functions $\mathbf{G}(s)-\mathbf{G}_\infty$ and $\mathbf{G}(s)-\mathbf{G}_0$ are summable in R^+. This last condition is very interesting from the physical viewpoint, because it reveals a relation between the macroscopic body behaviour and microscopic quantities.

In our context, the function $\mathbf{G}(s)-\mathbf{G}_\infty$ has evolutionary character in the sense that the Boltzmann function and $\mathbf{G}_0$ may rightly be used to describe the effective value of the Cauchy stress tensor at instant t and that this tensor may be expressed simultaneously by $\mathbf{G}(s)-\mathbf{G}_\infty$ and by the elastic equilibrium module $\mathbf{G}_\infty$, i.e., by imagining to affect the initial elastic properties of the material by the final properties. This is effectively possible in the elastic case where we have $\mathbf{G}_0=\mathbf{G}_\infty$ neglecting microstructural oscillations of the crystal lattice.

With these ideas we are able not only to explain how to define linear and strongly viscoelastic material and to resolve the open question on the major symmetry property of the relaxation function, but also to establish necessary and sufficient uniformly elliptic conditions, which are closely connected with the constructed definition of a linear and strongly viscoelastic material.

By using the last conditions and the Fourier anti-transform method [5, 6], we will establish an existence and uniqueness theorem for bounded and rapidly solutions of the quasi-static problem, with assigned boundary data, relating to a linear strongly viscoelastic material in a particular subclass of functions of $H^{1,1}(R; H^{1,2}(\Omega)) \cap$ $\cap\ H^{1,2}(R; H^{1,2}(\Omega))$, where Ω is an open bounded domain of R^3.

We conclude by remarking that the considered space of the solutions is a direct consequence of the assumed hypotheses together with the boundedness of the work of internal stresses on $(-\infty, +\infty)$, the cause-effect principle and the conditions of thermodynamic compability.

2. Let β be a linear viscoelastic and homogeneous material system described by the following constitutive functional:

$$\mathbf{T}(\mathbf{x}, t) = \mathbf{G}_0(x)\mathbf{E}(\mathbf{x}, t) + \int_0^{+\infty} \mathbf{G}'(\mathbf{x}, s)\mathbf{E}^t(\mathbf{x}, s)\,ds =$$

$$= \mathbf{G}_\infty(\mathbf{x})\mathbf{E}(\mathbf{x}, t) + \int_0^{+\infty} [\mathbf{G}(\mathbf{x}, s) - \mathbf{G}_\infty(\mathbf{x})]\dot{\mathbf{E}}^t(\mathbf{x}, s)\,ds, \tag{1}$$

$$\mathbf{T}(\mathbf{x}, t) = \mathbf{T}^T(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in \Omega \times [0, +\infty) = Q,$$

where $\mathbf{T}(\mathbf{x}, t)$ is the Cauchy stress tensor, $\mathbf{G}(\mathbf{x}, s)$ and $\mathbf{G}'(\mathbf{x}, s)$ are respectively the relaxation and Boltzmann fourth-order Cartesian tensors, $\mathbf{G}_0(\mathbf{x})$ and $\mathbf{G}_\infty(\mathbf{x})$ denote respectively the instantaneous and equilibrium elastic moduli defined by:

$$\mathbf{G}_0(\mathbf{x}) = \lim_{s \to \infty} \mathbf{G}(\mathbf{x}, s) = \mathbf{G}(\mathbf{x}, s) - \int_0^s \mathbf{G}'(\mathbf{x}, \tau)\,d\tau,$$

$$\mathbf{G}_\infty(\mathbf{x}) = \lim_{s \to +\infty} \mathbf{G}(\mathbf{x}, s) = \mathbf{G}_0(\mathbf{x}) - \int_0^{+\infty} \mathbf{G}'(\mathbf{x}, \tau)\,d\tau,$$

$\mathbf{E}(\mathbf{x}, t) = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]$ is the second-order infinitesimal strain tensor, where $\mathbf{u}(\mathbf{x}, t)$ denotes the displacement vector, $\mathbf{E}^t(\mathbf{x}, s) = \mathbf{E}(\mathbf{x}, t - s)$, $s \in [0, +\infty)$, for every fixed $t \in [0, +\infty)$, denotes the history of the infinitesimal strain tensor at instant t; finally Ω is an open and bounded domain of R^3 with sufficiently regular boundary $\partial\Omega$.

For convenience, we extend $\mathbf{G}(\mathbf{x}, \cdot)$ to $(-\infty, +\infty)$ as an odd function, $\mathbf{G}(\mathbf{x}, s) =$ $= -\mathbf{G}(\mathbf{x}, -s)$ and $\mathbf{G}'(\mathbf{x}, s) = -\mathbf{G}'(\mathbf{x}, -s)$ $\forall\, s \in [0, +\infty)$, and make the following hypotheses $\forall\, x \in \Omega$:

$$s\,\mathbf{G}'(\mathbf{x}, \cdot) \in L^1(0, +\infty),$$

$$\mathbf{G}'(\mathbf{x}, \cdot) - \mathbf{G}_\infty(\mathbf{x}) = -\int_s^{+\infty} \mathbf{G}'(\mathbf{x}, \tau)\,d\tau \in H^{1,1}(0, +\infty) \cap H^{1,2}(0, +\infty), \tag{2}$$

$$\lim_{s \to +\infty} s^2\left[\mathbf{G}(\mathbf{x}, \cdot) - \mathbf{G}_\infty(\mathbf{x})\right] = 0.$$

If and only if $\mathbf{G}_0(\mathbf{x}) = \mathbf{G}_\infty(\mathbf{x})$, then for all $t \geq 0$.

$$\lim_{a \to +\infty} \int_{-at}^{+\infty} a\left[\mathbf{G}\left(\mathbf{x}, t + \frac{y}{a}\right) - \mathbf{G}_\infty(\mathbf{x})\right]\left[\left(\frac{\sin y - y\cos y}{y^2}\right)\right]dy = 0,$$

where $y = a(s - t)$ and $a > 0$.

The following hypotheses will be used:

1. It is assumed that $\mathbf{G}'(\mathbf{x}, \cdot)$ is continuous $\forall\, \mathbf{x} \in \Omega$ while $\mathbf{G}''(\mathbf{x}, \cdot)$ is piecewise continuous; furthermore $\mathbf{G}'(\mathbf{x}, \cdot)$ verifies Dini condition at every point of discontinuity and in a neighbourhood of such points $\mathbf{G}''(\mathbf{x}, \cdot)$ is bounded.

2. The fourth-order symmetric tensors $\mathbf{G}_0(\mathbf{x})$ and $\mathbf{G}_\infty(\mathbf{x})$ are positive definite and continuous in $\overline{\Omega}$; furthermore $\mathbf{G}(\mathbf{x}, \cdot)$ and $\mathbf{G}'(\mathbf{x}, \cdot)$ are continuous in $\overline{\Omega}$ with respect to every fixed s.

We can formulate the following definitions.

Definition 1. *A continuous material system defined by the constitutive functional* (1) *is called to be strictly viscoelastic if and only if in hypotheses* (2) *the following conditions hold:*

I) $\mathbf{G}(\mathbf{x}, \cdot) - \mathbf{G}_0(\mathbf{x}) \notin L^1(0, +\infty)\ \forall\, \mathbf{x} \in \Omega$,
$\mathbf{G}(\mathbf{x}, s) = \mathbf{G}^T(\mathbf{x}, s)\ \forall\ (\mathbf{x}, s) \in \Omega \times [\,0, +\infty)$;

II) *there exist two constants* $\mu_1 > \mu_2 > 0$, *such that:*

$$\mu_1 \mathbf{A} : \mathbf{A} > \mathbf{A} : [\mathbf{G}_0(\mathbf{x}) - \mathbf{G}_\infty(\mathbf{x})]\mathbf{A} > \mu_2 \mathbf{A} : \mathbf{A}$$
$$\forall\, \mathbf{A} \in \mathrm{Sym}\,(V) / \{0\} \quad \text{and} \quad \forall\, \mathbf{x} \in \Omega,$$

where $\mathrm{Sym}\,(V)$ *is the space of second-order Cartesian symmetric tensor of* R^3*; and symbol* $:$ *denotes a scalar product of tensors;*

III) *the dynamic viscosity tensor*

$$\hat{\mathbf{G}}_c(\mathbf{x}, w) = \int_0^{+\infty} [\mathbf{G}(\mathbf{x}, s) - \mathbf{G}_\infty(\mathbf{x})] \cos ws\, ds$$

is positive definite and bounded, i.e., there exist two constants $\beta_1 > \beta_2 > 0$, *independent of* w, *such that:*

$$\beta_1 \mathbf{A} : \mathbf{A} > \mathbf{A} : \hat{\mathbf{G}}_c(\mathbf{x}, w)\mathbf{A} > \beta_2 \mathbf{A} : \mathbf{A} \quad \forall\, \mathbf{A} \in \mathrm{Sym}\,(V) / \{0\},$$
$$\forall\, w \in (-\infty, +\infty) \quad \text{and} \quad \forall\, \mathbf{x} \in \Omega,$$

in particular, $\forall\, \mathbf{x} \in \Omega$, *we have:*

$$\beta_1 \mathbf{A} : \mathbf{A} > \mathbf{A} : \lim_{w \to 0} \hat{\mathbf{G}}_c(\mathbf{x}, w)\mathbf{A} =$$

$$= -\mathbf{A} : \int_0^{+\infty} s\, \mathbf{G}'(\mathbf{x}, s)\, ds\, \mathbf{A} = \mathbf{A} : \int_0^{+\infty} [\mathbf{G}(\mathbf{x}, s) - \mathbf{G}_\infty(\mathbf{x})]\, ds\, \mathbf{A} > \beta_2 \mathbf{A} : \mathbf{A}$$

and $\lim_{w \to \pm\infty} \hat{\mathbf{G}}_c(\mathbf{x}, w) = 0$;

IV) $\forall\, \mathbf{x} \in \Omega$, $\forall\, \mathbf{A} \in \mathrm{Sym}\,(V) / \{0\}$, $\exists\, \nu_1, \nu_2 > 0$ *such that:*

$$\mathbf{A} : \left[\mathbf{G}_0(\mathbf{x}) + \hat{\mathbf{G}}'_c(\mathbf{x}, w)\right]\mathbf{A} = \mathbf{A} : \left[\mathbf{G}_\infty(\mathbf{x}) + w\, \hat{\mathbf{G}}_s(\mathbf{x}, w)\right]\mathbf{A} \ge \nu_1 \mathbf{A} : \mathbf{A} \quad \forall\, w \in R,$$

$$w^2 \mathbf{A} : \hat{\mathbf{G}}_c(\mathbf{x}, w)\mathbf{A} = -w\, \hat{\mathbf{G}}'_s(\mathbf{x}, w) \ge \nu_2 \mathbf{A} : \mathbf{A} \quad \forall\, w \neq 0, \quad \text{where}$$

$$\hat{\mathbf{G}}_c(\mathbf{x}, w) = \int_0^{+\infty} \mathbf{G}'(s) \cos ws\, ds, \qquad \hat{\mathbf{G}}_s(\mathbf{x}, w) = \int_0^{+\infty} [\mathbf{G}(\mathbf{x}, s) - \mathbf{G}_\infty(\mathbf{x})] \sin ws\, ds,$$

$$\hat{\mathbf{G}}'_s(\mathbf{x}, w) = \int_0^{+\infty} \mathbf{G}'(s) \sin ws\, ds,$$

and ν_1, ν_2 *do not depend on* w. *In particular for all* $\mathbf{x} \in \Omega$ *we have:*

$$\mathbf{A} : \lim_{w \to 0} \left[\mathbf{G}_0(\mathbf{x}) + \hat{\mathbf{G}}'_s(\mathbf{x}, w)\right]\mathbf{A} =$$

$$= \mathbf{A} : \lim_{w \to 0} \left[\mathbf{G}_\infty(\mathbf{x}) + w\, \hat{\mathbf{G}}_s(\mathbf{x}, w)\right]\mathbf{A} = \mathbf{A} : \mathbf{G}_\infty(\mathbf{x})\mathbf{A} \ge \nu_1 \mathbf{A} : \mathbf{A},$$

$$\mathbf{A}: \lim_{w\to\pm\infty}\left[\mathbf{G}_0(\mathbf{x})+\hat{\mathbf{G}}'_s(\mathbf{x},w)\right]\mathbf{A} =$$

$$= \mathbf{A}: \lim_{w\to\pm\infty}\left[\mathbf{G}_\infty(\mathbf{x})+w\,\hat{\mathbf{G}}_s(\mathbf{x},w)\right]\mathbf{A} = \mathbf{A}: \mathbf{G}_0(\mathbf{x})\mathbf{A} > \nu_1\mathbf{A}: \mathbf{A}.$$

Definition 2. *The body* β *is linear elastic if and only if* $\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})$, $\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_0(\mathbf{x})\in L^1(0,+\infty)$ $\forall\,\mathbf{x}\in\Omega$ *in hypotheses* (2), *i.e., if*

$$\mathbf{T}(\mathbf{x},t) = \mathbf{G}_0(\mathbf{x})\mathbf{E}(\mathbf{x},t) = \mathbf{G}_\infty(\mathbf{x})\mathbf{E}(\mathbf{x},t)\ \ \forall\, t\in[\,0,T_c) \quad \textit{where} \quad T_c<+\infty.$$

In particular the body β *is classically linear elastic if and only if*

$$\mathbf{T}(\mathbf{x}) = \mathbf{G}_0(\mathbf{x})\mathbf{E}(\mathbf{x}) = \mathbf{G}_\infty(\mathbf{x})\mathbf{E}(\mathbf{x}).$$

By the Young inequality and the conditions (2), setting $\mathbf{E}(\mathbf{x},t)=\mathbf{E}(\mathbf{x},-s)$, $\forall\,(\mathbf{x},t)\in\Omega\times(\,0,-\infty)$ we have:

$$\begin{aligned}
\|\mathbf{T}(\mathbf{x},\cdot)\|_{L^p(R)} &\le |\mathbf{G}_0(\mathbf{x})|\,\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^p(R)} + \|\mathbf{G}'(\mathbf{x},\cdot)\mathbf{E}(\mathbf{x},\cdot)\|_{L^p(R)} \le \\
&\le |\mathbf{G}_0(x)|\,\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^p(R)} + \|\mathbf{G}'(\mathbf{x},\cdot)\|_{L^1(R)}\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^p(R)} \quad \forall\,\mathbf{x}\in\Omega, \\
\|\mathbf{T}(\mathbf{x},\cdot)\|_{L^p(R)} &\le |\mathbf{G}_\infty(\mathbf{x})|\,\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^p(R)} + \left\|[\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})]\dot{\mathbf{E}}(\mathbf{x},\cdot)\right\|_{L^p(R)} \le \\
&\le |\mathbf{G}_\infty(\mathbf{x})|\,\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^p(R)} + \|\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})\|_{L^1(R)}\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^p(R)},
\end{aligned} \tag{3}$$

provided that $\mathbf{E}(\mathbf{x},\cdot)\in H^{1,p}(-\infty,+\infty)$ and $\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})\in H^{1,1}(0,+\infty)$ $\forall\,\mathbf{x}\in\Omega$ and $p\ge1$.

In order to explain that the value of the convolution integral of the functional (1) in $(t,+\infty)$ does not contradict the cause effect principle, we observe that by setting $\tau=t-s$ and $\mathbf{E}(\mathbf{x},-\tau)$ $\forall\,(\mathbf{x},\tau)\in\Omega\times(-\infty\,,+\infty)$ we can rewrite (1) in the following manner:

$$\mathbf{T}(\mathbf{x},t) = \mathbf{G}_0(\mathbf{x})\mathbf{E}(\mathbf{x},t) + \int_{-\infty}^{0}\dot{\mathbf{G}}(\mathbf{x},\tau-t)\mathbf{E}(\mathbf{x},\tau)\,d\tau + \int_0^t\dot{\mathbf{G}}(\mathbf{x},t-\tau)\mathbf{E}(\mathbf{x},\tau)\,d\tau =$$

$$= \mathbf{G}_\infty(\mathbf{x})E(\mathbf{x},t) + \int_{-\infty}^{0}\left[\mathbf{G}_{-\infty}(\mathbf{x})-\mathbf{G}(\mathbf{x},\tau-t)\right]\mathbf{E}'(\mathbf{x},\tau)\,d\tau +$$

$$+\int_0^t[\mathbf{G}(\mathbf{x},t-\tau)-\mathbf{G}_\infty(\mathbf{x})]\mathbf{E}'(\tau)\,d\tau. \tag{4}$$

By Young's inequality it yields:

$$\begin{aligned}
\|\mathbf{T}(\mathbf{x},\cdot)\|_{L^1(R)\cap L^p(R)} &\le |\mathbf{G}_0(\mathbf{x})|\,\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^1(R)\cap L^p(R)} + \|\mathbf{G}'(\mathbf{x},\cdot)\mathbf{E}(\mathbf{x},\cdot)\|_{L^1(R)\cap L^p(R)} \le \\
&\le |\mathbf{G}_0(\mathbf{x})|\,\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^1(R)\cap L^p(R)} + \|\mathbf{G}'(\mathbf{x},\cdot)\|_{L^p(R)}\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^1(R)\cap L^p(R)} \quad \forall\,\mathbf{x}\in\Omega, \\
\|\mathbf{T}(\mathbf{x},\cdot)\|_{L^1(R)\cap L^p(R)} &\le |\mathbf{G}_\infty(\mathbf{x})|\,\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^1(R)\cap L^p(R)} + \\
&\quad + \left\|[\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})]\dot{\mathbf{E}}(\mathbf{x},\cdot)\right\|_{L^1(R)\cap L^p(R)} \le \\
&\le |\mathbf{G}_\infty(\mathbf{x})|\,\|\mathbf{E}(\mathbf{x},\cdot)\|_{L^1(R)\cap L^p(R)} + \|\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}(\mathbf{x})\|_{L^p(R)}\left\|\dot{\mathbf{E}}(\mathbf{x},\cdot)\right\|_{L^1(R)\cap L^p(R)},
\end{aligned} \tag{5}$$

provided that $\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})\in H^{1,p}(0,+\infty)$ and $\mathbf{E}(\mathbf{x},\cdot)\in H^{1,1}(-\infty,+\infty)$ $\forall\,\mathbf{x}\in\Omega$ and $p\ge1$.

Equating (3) with (5) we obtain:

$$\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})\in H^{1,1}(0,+\infty)\cap H^{1,p}(0,+\infty),$$
$$\mathbf{E}(\mathbf{x},\cdot)\in H^{1,1}(-\infty,+\infty)\cap H^{1,p}(0,+\infty)\quad\forall\,\mathbf{x}\in\Omega\quad\text{and}\quad p\ge 1. \tag{6}$$

Moreover the work done by the Cauchy stress

$$L(-\infty,+\infty)=\int_{-\infty}^{+\infty}\mathbf{T}(\mathbf{x},t):\dot{\mathbf{E}}(\mathbf{x},t)\,dt$$

is bounded because the power $W(\mathbf{x},t)$ of the stress tensor $\mathbf{T}(\mathbf{x},t)$ verifies, by virtue of (3) and (5), the following inequality:

$$\|W(\mathbf{x},\cdot)\|_{L^1(-\infty,+\infty)}\le\|\mathbf{T}(\mathbf{x},\cdot)\|_{L^1(-\infty,+\infty)\cap L^p(-\infty,+\infty)}\times$$
$$\times\left\|\dot{\mathbf{E}}(\mathbf{x},\cdot)\right\|_{L^\infty(-\infty,+\infty)\cap L^{p/(p-1)}(-\infty,+\infty)}. \tag{7}$$

This inequality and the conditions (6) in turn imply that

$$\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})\in H^{1,1}(-\infty,+\infty)\cap H^{1,p}(-\infty,+\infty),\quad p\ge 1,\quad\text{and}\quad\forall\,\mathbf{x}\in\Omega,$$
$$\mathbf{E}(\mathbf{x},\cdot)\in H^{1,1}(-\infty,+\infty)\cap H^{1,p}(-\infty,+\infty), \tag{8}$$
$$\dot{\mathbf{E}}(\mathbf{x},\cdot)\in L^\infty(-\infty,+\infty)\cap L^{p/(p-1)}(-\infty,+\infty).$$

Interesting cases from the physical viewpoint are those in which in (8) $p=2$ and $p=\infty$, i.e., respectively

$$\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})\in H^{1,1}(-\infty,+\infty)\cap H^{1,2}(-\infty,+\infty)\quad\forall\,\mathbf{x}\in\Omega,$$
$$\mathbf{E}(\mathbf{x},\cdot)\in H^{1,1}(-\infty,+\infty)\cap H^{1,2}(-\infty,+\infty),\quad\dot{\mathbf{E}}(\mathbf{x},\cdot)\in L^\infty(-\infty,+\infty), \tag{9}$$

$$\mathbf{G}(\mathbf{x},\cdot)-\mathbf{G}_\infty(\mathbf{x})\in H^{1,1}(-\infty,+\infty)\cap H^{1,\infty}(-\infty,+\infty)\quad\forall\,\mathbf{x}\in\Omega,$$
$$\mathbf{E}(\mathbf{x},\cdot)\in H^{1,1}(-\infty,+\infty)\cap H^{1,\infty}(-\infty,+\infty)^*. \tag{10}$$

3. The quasi-static problem for a viscoelastic body expressed by the functional (1) is formulated by the following Dirichlet problem:

$$\operatorname{div}\left\{\mathbf{G}_\infty(\mathbf{x})\operatorname{grad}u(\mathbf{x},t)+\int_0^{+\infty}[\mathbf{G}(\mathbf{x},s)-\mathbf{G}_\infty(\mathbf{x})]\operatorname{grad}\dot{\mathbf{u}}^t(\mathbf{x},s)\,ds\right\}+b(\mathbf{x},t)=$$

$$=\operatorname{div}\left\{\mathbf{G}_0(\mathbf{x})\operatorname{grad}u(\mathbf{x},t)+\int_0^{+\infty}\mathbf{G}'(\mathbf{x},s)\operatorname{grad}u^t(\mathbf{x},s)\,ds\right\}+b(\mathbf{x},t)=0,\qquad(\mathbf{x},t)\in Q, \tag{11}$$

$$u(\mathbf{x},t)\big|_{\partial\Omega}=0,\quad\text{where}\quad u(\mathbf{x},t)=\mathbf{u}(\mathbf{x},t)-\mathbf{u}_\infty(\mathbf{x}),\qquad\lim_{t\to+\infty}\mathbf{u}(\mathbf{x},t)=\mathbf{u}_\infty(x),$$

$$b(x,t)=\mathbf{b}(\mathbf{x},t)-\mathbf{b}_\infty(\mathbf{x}),\qquad\lim_{t\to+\infty}\mathbf{b}(\mathbf{x},t)=\mathbf{b}_\infty(\mathbf{x})^{**}.$$

We can extend this problem on all of R if we consider the symmetry properties of the relaxation and Boltzmann function and if we introduce this history of the body forces $\mathbf{b}^t(\mathbf{x},s)=\mathbf{b}(\mathbf{x},t-s)$, $s\in[0,+\infty)$, respect with every fixed $t\in[0,+\infty)$, setting $\mathbf{b}(\mathbf{x},t)=\mathbf{b}(\mathbf{x},-s)$ and $\mathbf{u}(\mathbf{x},t)=\mathbf{u}(\mathbf{x},-s)$ $\forall\,t,s\in\{0,+\infty)$ and $\forall\,\mathbf{x}\in\Omega$.

* This case is interesting because of thermodynamic compability conditions.

** We introduce the function $u(x,t)$ in order to discuss also the case $\operatorname{grad}u(x,-\infty)\ne 0$.

The notation $\mathbf{b}(\mathbf{x}, t) = L^p\big(R; H^{1,2}(\Omega)\big)$, $1 \le p < \infty$, means that in further function $b(\mathbf{x}, t)$ as a function of $t \in R$ maps R into $H^{1,2}(\Omega)$ for all $\mathbf{x} \in \Omega$, is measurable and its norm is calculated according to the formula

$$\| b(\mathbf{x}, t) \|_{L^p\left(R; H^{1,2}(\Omega)\right)} = \left(\int_R \| b(\mathbf{x}, t) \|^p_{H^{1,2}(\Omega)} dt \right)^{p^{-1}} < \infty, \quad 1 \le p < \infty.$$

If hypotheses 2, (9) and (10) hold, it is possible to prove the following theorem.

Theorem 1. *Assume that the body* β *is strictly viscoelastic according to definition* 1, *that*

$$b(\mathbf{x}, t) \in L^1\big(R; H^{1,2}(\Omega)\big) \cap L^2\big(R; H^{1,2}(\Omega)\big), \quad b(\mathbf{x}, \cdot) \in S_\infty(R),$$

and it has compact support in R. *Then there exists at least one solution with compact support*

$$u(\mathbf{x}, t) \in H^{1,1}\big(R; H^{1,2}(\Omega)\big) \cap H^{1,2}\big(R; H^{1,2}(\Omega)\big), \quad u(\mathbf{x}, \cdot) \in S_\infty(R),$$

such that

$$\int_{\Omega'} \left\{ \mathbf{G}_\infty(\mathbf{x}) \nabla u(\mathbf{x}, t) + \int_0^{+\infty} [\mathbf{G}(\mathbf{x}, s) - \mathbf{G}_\infty(\mathbf{x})] \nabla \dot{u}^t(\mathbf{x}, s) ds \right\} : \nabla \mathbf{H}(\mathbf{x}, \mathbf{x}', t) d\mathbf{x}' =$$

$$= \int_{\Omega'} \left\{ \mathbf{G}_0(\mathbf{x}) \nabla u(\mathbf{x}, t) + \int_0^{+\infty} \mathbf{G}'(\mathbf{x}, s) \nabla \dot{u}^t(\mathbf{x}, s) ds \right\} : \nabla \mathbf{H}(\mathbf{x}, \mathbf{x}', t) d\mathbf{x}' =$$

$$= \int_{\Omega'} b(\mathbf{x}, t) \mathbf{H}(\mathbf{x}, \mathbf{x}', t) d\mathbf{x}' \qquad (12)$$

$$\forall\, \mathbf{H}(\mathbf{x}, \mathbf{x}', t) \in L^\infty\big(-\infty, +\infty; H^{1,2}(\Omega) \times H^{1,2}(\Omega')\big) : \mathbf{H}(\mathbf{x}, \mathbf{x}', t)\Big|_{\partial\Omega} = 0,$$

where $\mathbf{H}(\mathbf{x}, \mathbf{x}', t)$ *is strongly measurable, if* $\mathbf{x}' \ne \mathbf{x}$, *and* $S_\infty(R)$ *denotes the class of* C^∞ *functions* $u(\mathbf{x}, t)$ *with respect to* t, *for which there exist constants* C_{pq}, *dependent on* $u(\mathbf{x}, t)$ *and on numbers* p *and* q, *such that:*

$$\int_\Omega \left| t^p \partial_t^{(q)} u(\mathbf{x}, t) \right|^2 dx < C_{pq}^2, \quad \int_\Omega \left| t^p \partial_t^{(q)} \nabla u(\mathbf{x}, t) \right|^2 dx < C_{pq}^2.$$

Proof. We consider the Fourier transformed problem of (11) in $L^1 \cap L^2$:

$$\nabla \cdot \left\{ \left[\mathbf{G}_\infty(\mathbf{x}) + iw \hat{\mathbf{G}}(\mathbf{x}, w) \right] \nabla \hat{u}(\mathbf{x}, w) \right\} + \hat{b}(\mathbf{x}, w) =$$

$$= \nabla \cdot \left\{ \left[\mathbf{G}_0(\mathbf{x}) + \hat{\mathbf{G}}'(x, w) \right] \nabla \hat{u}(\mathbf{x}, w) \right\} + \hat{b}(\mathbf{x}, w) = 0, \qquad (13)$$

$$(\mathbf{x}, w) \in \Omega \times (-\infty, +\infty), \quad \hat{u}(\mathbf{x}, w)\Big|_{\partial\Omega} = 0,$$

where

$$\hat{\mathbf{G}}(\mathbf{x}, w) = \int_0^{+\infty} [\mathbf{G}(\mathbf{x}, s) - \mathbf{G}_\infty(\mathbf{x})] \exp(-iws)\, ds,$$

$$\hat{u}(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(\mathbf{x}, t) \exp(-iwt)\, dt, \quad \hat{b}(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} b(\mathbf{x}, t) \exp(-iwt)\, dt$$

and

$$\hat{G}'(\mathbf{x}, w) = \int_0^{+\infty} \mathbf{G}'(\mathbf{x}, t) \exp(-iws)\, ds.$$

The first equation in (13) is strongly elliptic [7], as we can verify by part IV of definition 1; consequently, when $b(\mathbf{x}, t) = 0$, $u(\mathbf{x}, t) = 0$ is the solution of the problem (11) [7]. In order to prove the existence of nonzero solutions of problem (11) one must show that equation (13) is not only strongly elliptic but in fact also uniformly elliptic [7], that at least one of the following conditions holds: $\forall\, \mathbf{x} \in \Omega$, $\forall\, \mathbf{A} \in \mathrm{Sym}(V)/\{0\}$, $\exists \nu_1, \nu_2 > 0$, independent on w such that:

$$\mathbf{A} : \left[\mathbf{G}_0(\mathbf{x}) + \hat{\mathbf{G}}'_c(\mathbf{x}, w)\right]\mathbf{A} = \mathbf{A} : \left[\mathbf{G}_\infty(\mathbf{x}) + w\hat{\mathbf{G}}_s(\mathbf{x}, w)\right]\mathbf{A} \geq \nu_1 \mathbf{A} : \mathbf{A} \quad \forall w \in R,$$
$$w^2 \mathbf{A} : \hat{\mathbf{G}}_c(\mathbf{x}, w)\mathbf{A} = -w\hat{\mathbf{G}}'_s(\mathbf{x}, w) \geq \nu_2 \mathbf{A} : \mathbf{A} \quad \forall w \in R. \tag{14}$$

As a consequence of the part IV of definition 1 the first condition of (14) holds while the second condition is verified if $w \neq 0$. Furthermore we have

$$\hat{u}^{(h)}(\mathbf{x}, w) = (iw)^h \hat{u}(\mathbf{x}, w) = 0 \quad \forall\, (\mathbf{x}, w) \in \Omega \times R \quad \text{and} \quad h \geq 2\ ^3, \tag{14$'$}$$

where h denotes the derivative of order h of $u(x, t)$ with respect to time. This condition restricts strongly the function class in which solutions of problem (13) may exist: one must seek the solutions of problem (13) in the class of functions $\hat{u}(\mathbf{x}, \cdot)$ such that $\hat{u}(\mathbf{x}, \cdot)$ and $\hat{\hat{u}}(\mathbf{x}, \cdot)$ are null outside a suitable compact interval and rapidly decreasing.

We consider the following dual problem of (13):

$$\nabla \cdot \left\{\left[\mathbf{G}_\infty(\mathbf{x}) + iw\hat{\mathbf{G}}(\mathbf{x}, w)\right]\nabla \mathbf{H}(\mathbf{x}, \mathbf{x}', t)\right\} + \delta(\mathbf{x}' - \mathbf{x}) =$$
$$= \nabla \cdot \left\{\left[\mathbf{G}_0(\mathbf{x}) + \hat{\mathbf{G}}'(\mathbf{x}, w)\right]\nabla \mathbf{H}(\mathbf{x}, \mathbf{x}', t)\right\} + \delta(\mathbf{x}' - \mathbf{x}) = 0 \tag{15}$$
$$\forall\, (\mathbf{x}, \mathbf{x}', w) \in \Omega \times \Omega' \times (-\infty, +\infty), \qquad \mathbf{H}(\mathbf{x}, \mathbf{x}', t)\big|_{\partial\Omega} = 0,$$

where $\mathbf{H}(\mathbf{x}, \mathbf{x}', t)$ is the Green function of problem the (15), $\delta(\mathbf{x}' - \mathbf{x}) = \delta(x^{ii} - x^i)\mathbf{e}_i$ and $(0, e_i)$ is a fixed reference system of R^3.

Due to strong and uniform ellipticity conditions of equation in (14) and hypotheses assumed on the Boltzmann and relaxation function there exists at least one solution $\mathbf{H}(\mathbf{x}, \mathbf{x}', t) \in L^\infty\left(-\infty, +\infty; H^{1,2}(\Omega) \times H^{1,2}(\Omega)\right)$ of the problem (15) [5, 6], which is strongly measurable if $\mathbf{x}' \neq \mathbf{x}$.

Taking the scalar product of (13) and (5) by $\mathbf{H}(\mathbf{x}, \mathbf{x}', t)$ and $\hat{u}(\mathbf{x}, w)$ respectively and integrating over Ω' we obtain

$$\int_{\Omega'} \left\{\left[\mathbf{G}_\infty(\mathbf{x}) + iw\hat{\mathbf{G}}(\mathbf{x}, w)\right]\nabla u(\mathbf{x}, t)\right\} : \nabla \mathbf{H}(\mathbf{x}, \mathbf{x}', t)\, d\mathbf{x}' =$$
$$= \int_{\Omega'} \left\{\left[\mathbf{G}_0(\mathbf{x}) + \hat{\mathbf{G}}'(\mathbf{x}, w)\right]\nabla \hat{u}(\mathbf{x}, t)\right\} : \nabla \mathbf{H}(\mathbf{x}, \mathbf{x}', t)\, d\mathbf{x}' = \int_{\Omega'} b(\mathbf{x}, t)\mathbf{H}(\mathbf{x}, \mathbf{x}', t)\, d\mathbf{x}';$$
$$\int_{\Omega'} \left\{\left[\mathbf{G}_\infty(\mathbf{x}) + iw\hat{\mathbf{G}}(\mathbf{x}, w)\right]\nabla \mathbf{H}(\mathbf{x}, \mathbf{x}', t)\right\} : \nabla \hat{u}(\mathbf{x}, t)\, d\mathbf{x}' = \tag{16}$$
$$= \int_{\Omega'} \left\{\left[\mathbf{G}_0(\mathbf{x}) + \hat{\mathbf{G}}'(\mathbf{x}, w)\right]\nabla \mathbf{H}(\mathbf{x}, \mathbf{x}', t)\right\} : \nabla \hat{u}(\mathbf{x}, t)\, d\mathbf{x}' = \sum_1^3 i\hat{u}^i(\mathbf{x}, w).$$

Because of the major symmetry property of the relaxation and Boltzmann function, from (16) we find:

$$\hat{u}(\mathbf{x}, w) = \int_{\Omega'} \hat{b}^i(\mathbf{x}, w) H_i(\mathbf{x}, x' w)\, \delta_i^j e_j \, d\mathbf{x}'. \tag{17}$$

The properties of boundedness and continuity of $H(\mathbf{x}, \mathbf{x}' w)$, $\mathbf{x}' \neq \mathbf{x}$, and of the coefficients of (13)) even as $w \to \pm\infty$, of compactness of the interval on which $\hat{\mathbf{b}}(\mathbf{x}, \cdot)$ is defined, and of the continuity of $\hat{\mathbf{b}}(\mathbf{x}, \cdot)$, are sufficient conditions so that a solution of the problem (13)

$$\hat{u}(\mathbf{x}, t) \in H^{1,1}\big(R; H^{1,2}(\Omega)\big) \cap H^{1,2}\big(R; H^{1,2}(\Omega)\big)$$

exists such that $\hat{u}(\mathbf{x}, \cdot) \in S_\infty(R)$.

As it is easy to verify,

$$u(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}(\mathbf{x}, w) \exp(iwt)\, dw \in H^{1,1}\big(R; H^{1,2}(\Omega)\big) \cap H^{1,2}\big(R; H^{1,2}(\Omega)\big)$$

and $\hat{u}(\mathbf{x}, \cdot) \in S_\infty(R)$ is a solution of problem (11) verifying (12).

The uniqueness of the solution of problem (11) is stated by the following theorem.

Theorem 2. *Under the assumed hypotheses of Theorem* 1 *there exists one and only one solution of the problem* (11) *that verifies* (12).

Proof. If we consider two solutions $u_1(\mathbf{x}, t)$ and $u_2(\mathbf{x}, t)$ of the problem (11) verifying (12), by (13) we have that

$$\nabla \cdot \Big\{ \big[\mathbf{G}_\infty(\mathbf{x}) + iw\hat{\mathbf{G}}(\mathbf{x}, w) \big] \nabla \big[\hat{u}_1(\mathbf{x}, w) - \hat{u}_2(\mathbf{x}, w) \big] \Big\} =$$

$$= \nabla \cdot \Big\{ \big[\mathbf{G}_0(\mathbf{x}) + \hat{\mathbf{G}}'(\mathbf{x}, w) \big] \nabla \big[\hat{u}_1(\mathbf{x}, w) - \hat{u}_2(\mathbf{x}, w) \big] \Big\} = 0, \tag{18}$$

$$(\mathbf{x}, w) \in \Omega \times (-\infty, +\infty), \qquad \hat{u}(\mathbf{x}, w)\big|_{\partial\Omega} = 0.$$

As we have proved before the only solution of the problem (18) is $\hat{u}_1(\mathbf{x}, w) - \hat{u}_2(\mathbf{x}, w) = 0$, from which it follows that $u_1(\mathbf{x}, t) = u_2(\mathbf{x}, t)$.

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Received 17.12.98,
after revision — 09.03.2000