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THE STOCHASTIC FLOW AND THE NOISE ASSOCIATED TO TANAKA'S STOCHASTIC DIFFERENTIAL EQUATION

СТОХАСТИЧНИЙ ПОТІК ТА ШУМ, АСОЦІЙОВАНИЙ ІЗ СТОХАСТИЧНИМ ДИФЕРЕНЦІАЛЬНИМ РІВНЯННЯМ ТАНАКИ

We study properties of a noise (in the Tsirelson sense) which is generated by solutions of the well-known Tanaka equation.

Досліджуються властивості шуму (в розумінні Б. Цірельсона), який породжується розв'язками відомого рівняння Танаки.

1. Introduction. Consider the following one-dimensional stochastic differential equation (SDE) for given $s, x \in R$:

$$dX_t = \operatorname{sgn}(X_t) dw_t, \qquad t \geq s, \quad X_s = x, \tag{1}$$

where $\operatorname{sgn}(y) = 1_{[0,\infty)}(y) - 1_{(-\infty,0)}(y)$, $w = (w_t)$ is a one-dimensional Wiener process and dw_t is the stochastic differential in the Itô sense. This SDE was first introduced by H. Tanaka [1] as a simplest example of SDE's having law unique solutions which, however, *can not possess any strong solutions.* Indeed, any solution $X = (X_t)$ of (1) is a Wiener process starting from x at time s so that it is unique in law. On the other hand, we have by Itô – Tanaka formula

$$|X_t| - |x| - \lim_{\varepsilon \to 0} (2\varepsilon)^{-1} \int_s^t 1_{[0,\varepsilon)}(|X_u|)\, du = \int_s^t \operatorname{sgn}(X_u)\, dX_u = w_t - w_s, \tag{2}$$

and we have also, by the uniqueness of solutions for the Skorokhod equation (cf. [2], also [3, p. 122]),

$$|X_t| = |x| + w_t - w_s - \min_{s \leq u \leq t} [(|x| + w_u - w_s) \wedge 0]. \tag{3}$$

Then it holds that $\sigma\{|X_u|; s \leq u \leq t\} = \sigma\{w_u - w_s; s \leq u \leq t\}$ which is strictly smaller than $\sigma\{X_u; s \leq u \leq t\}$. This clearly implies that X can not be a strong solution to SDE (1).

In Section 2, we show the existence and the uniqueness in law of a family $X = \{X_{s,t}(x)\}$ of solutions to SDE (1) which forms a *coalescing stoshastic flow.* This stoshastic flow X naturally generates a *noise.* The noise is a notion in continuous time which corresponds to the notion of i.i.d. random sequences in discrete time. This notion has been introduced and studied deeply by Tsirelson [4–6]. The noise generated by the flow X of solutions to SDE (1) may be a simplest example of predictable, *non-Gaussian* or *non-white* noises.

Given a noise $\{\mathcal{F}_{s,t}\}$ and $f \in L^2(\mathcal{F}_{0,1})$ with the L^2-norm 1, the notion of the *spectral measure* for the noise has been introduced by Tsirelson as a probability measure μ_f on the space $C_{[0,1]}$ formed of all closed subsets in $[0, 1]$. In Section 3, we will determine μ_f for the noise generated by the flow $X = \{X_{s,t}(x)\}$ of solutions to SDE (1) when $f = \operatorname{sgn}(X_{0,1}(0))$. This problem has been already studied by Warren [7]. Our approach is somewhat different; we compute it directly without relating it to a random walk approximation.

2. A stoshastic flow made from solutions to SDE (1). For a fixed $s \in R$, we consider a family $X^{(s)} = \{X_{s,t}(x);\ t \in [t, \infty)\}_{x \in R}$ of continuous stochastic processes in time t defined on a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $F = = \{\mathcal{F}_t\}_{t \geq s}$ and a one-dimensional F-Wiener process $w = (w_t)_{t \geq s}$, such that, for each $x \in R$, $t \in [s, \infty) \mapsto X_{s,t}(x) \in R$ is an F-adapted solution to SDE (1). We call such a family $X^{(s)}$ *a family of solutions to SDE* (1) starting at time s. For such a family, we consider the following mutually equivalent conditions:

(A_1) $X^{(s)}$ is *monotone* in the sense that, for each $x, y \in R$ with $x \leq y$ and $t \geq s$,

$$P(X_{s,t}(x) \leq X_{s,t}(y)) = 1.$$

(A_2) $X^{(s)}$ is *monotone in the strict sense* that

$$P(X_{s,t}(x) \leq X_{s,t}(y) \text{ for each } x \leq y \text{ and } t \geq s) = 1.$$

(A_3) $X^{(s)}$ is *coalescing* in the sense that, for each $x, y \in R$ with $x \neq y$, if the paths $t \in [s, \infty) \mapsto X_{s,t}(x) \in R$ and $t \in [s, \infty) \mapsto X_{s,t}(y) \in R$ meet at time t_0 for the first time, then they coinside for all $t \geq t_0$.

The equivalence of (A_1), (A_2) and (A_3) is easy to see and its proof is omitted.

Proposition 1. *There exists a family* $X^{(s)} = \{X_{s,t}(x);\ t \in [s, \infty)\}_{x \in R}$ *of solutions to SDE* (1) *satisfying one of the above equivalent conditions* (A_1), (A_2) *and* (A_3). *Furthermore, its law is unique.*

Proof. First, we construct a family $X^{(s)} = \{X_{s,t}(x); t \in [s, \infty)\}_{x \in R}$ of solutions to SDE (1) satisfying the condition (A_3). Take a one-dimensional Wiener process $\{B(t)\}_{t \geq s}$ with the natural filtration $F = \{\mathcal{F}_t\}_{t \geq s}$ and define

$$X_{s,t}(0) = B(t) - B(s)$$

and

$$w_t = \int_s^t \operatorname{sgn}(X_{s,u}(0))\, dB(u) \quad \text{for} \quad t \geq s.$$

Then $(w_t)_{t \geq s}$ is an F-Wiener process with $w_s = 0$ and $\{X_{s,t}(0)\}_{t \geq s}$ is an F-adapted solution to SDE (1) for $x = 0$. For $x \neq 0$, let $\sigma = \min\{u > s \mid |x| + w_u = 0\}$. Since $|X_{s,t}(0)| = w_t - \min_{s \leq u \leq t} w_u$ by (3), we see easily that $X_{s,\sigma}(0) = 0$. Now we define $\{X_{s,t}(0)\}_{t \geq s}$ as follows; if $s \leq t \leq \sigma$, then $X_{s,t}(x) = x + w_t$ or $X_{s,t}(x) = x - w_t$ accordingly as $x > 0$ or $x < 0$, and if $t \geq \sigma$, then $X_{s,t}(x) = X_{s,t}(0)$. It is easy to verify that $X^{(s)} = \{X_{s,t}(x); t \in [s, \infty)\}_{x \in R}$ so defined is a family of solutions to SDE (1) satisfying the condition (A_3).

Next we prove the uniqueness in law. But this is simple; for any family $X^{(s)} = \{X_{s,t}(x);\ t \in [s, \infty)\}_{x \in R}$ of solutions to SDE (1) satisfying the conditions (A_3), $\{X_{s,t}(x)\}$ must be given from $\{X_{s,t}(0)\}$ and $\{w_t\}$ with $w_s = 0$ as in the above construction and the joint law of $\{X_{s,t}(0)\}$ and $\{w_t\}$ is obviously unique.

For $a, b \in R$ and $c \in \{-1, +1\}$ such that $b \geq 0$, $a + b \geq 0$, define a transformation $f^c_{a,b} : x \in R \mapsto f^c_{a,b}(x) \in R$, by

$$f_{a,b}^{\pm 1}(x) = \begin{cases} x+a, & b<x; \\ \pm(a+b), & -b \le x \le b; \\ x-a, & x<-b. \end{cases}$$

Then the above construction of $X^{(s)} = \{X_{s,t}(x); t \in [s, \infty)\}_{x \in R}$ can be rephrased as follows:

$$X_{s,t}(x) = f_{a,b}^{c}(x), \tag{4}$$

where $a = w_t$, $b = -\min_{s \le u \le t} w_u$, $c = \operatorname{sgn}\{X_{s,t}(0))$.

We briefly recall a general definition of stochastic flows [8, 9]; let $\mathcal{S}$ be a topological space and $\mathcal{T}$ be a class of trasformations on $\mathcal{S}$ containing the identity transformation and forming a semigroup under the composition of transformations. We assume that a suitable topology is given on $\mathcal{T}$ so that it is a Polish space as well as a topological semigroup. Now a $\mathcal{T}$-valued random variable is well understood with respect to the natural Borel structure on $\mathcal{T}$.

Definition 1. *By a $\mathcal{T}$-stochastic flow, we mean a family of $\mathcal{T}$-valued random variables $\Phi = \{\varphi_{s,t};\ s \le t\}$ having the following properties:*

1) (*the flow property*), $\varphi_{s,u} = \varphi_{t,u} \circ \varphi_{s,t}$ *and* $\varphi_{t,t} = \mathrm{id}$, *for all* $s \le t \le u$,

2) (*the independent increment property*), *for any sequence* $t_0 \le t_1 \le \ldots \le t_n$, *$\mathcal{T}$-valued random variables* φ_{t_{k-1},t_k}, $k = 1, \ldots, n$, *are independent,*

3) (*the stationarity*), *for any* $h > 0$, $\varphi_{s,t} \overset{\mathrm{d}}{=} \varphi_{s+h,t+h}$.

In the following, we consider a particular case of

$$\mathcal{S} = \boldsymbol{R}, \quad \text{and} \quad \mathcal{T} = \{f_{a,b}^{c};\ b \ge 0,\ a+b \ge 0,\ c \in \{-1, +1\}\}$$

exclusively; $\mathcal{T}$ forms a semigroup of transformations on $\boldsymbol{R}$ and the composition rule is given by

$$f_{a',b'}^{c'} \circ f_{a,b}^{c} = f_{a'',b''}^{c''}, \quad a'' = a + a', \quad b'' = b \vee (b' - a),$$
$$c'' = \begin{cases} c, & b > b' - a; \\ c', & b \le b' - a. \end{cases} \tag{5}$$

The topology of $\mathcal{T}$ is defined by the Euclidean topology of the parameter (a, b, c).

Theorem 1. *There exists a $\mathcal{T}$-stoshastic flow $X = \{X_{s,t};\ s \le t\}$ such that, for each $s \in \boldsymbol{R}$, $X^{(s)} = \{X_{s,t}(x); t \in [s, \infty)\}_{x \in R}$ defines a family of solutions to SDE* (1) *satisfying the condition* (A_3). *Furthermore, the law of such a flow is unique.*

Proof. For a given s, let $X^{(s)} = \{X_{s,t}(x)\}$ be a family of solutions to SDE (1) satisfying the condition (A_3). We know that, for each $t \ge s$, $X_{s,t} := [x \mapsto X_{s,t}(x)]$ is a $\mathcal{T}$-valued random variable as given by (4). Denote its probability law by $\mu_{s,t}$ which is a Borel probability on $\mathcal{T}$. It is easy to see that $\mu_{s,t} = \mu_{0,t-s}$ and we denote $\mu_{0,t}$ by μ_t. Then we have

$$\mu_{t+s} = \mu_s * \mu_t, \quad s \geq 0, \quad t \geq 0, \quad \mu_0 = \delta_{\mathrm{id}}, \tag{6}$$

that is $\{\mu_t\}$ constitutes a convolution semigroup of probabilities on $\mathcal{T}$. Here, the convolution $\mu * \nu$ of two Borel probabilities on $\mathcal{T}$ is defined, as usual, by

$$\int_{\mathcal{T}} f(\varphi)\, \mu * \nu(d\varphi) = \int_{\mathcal{T}} \int_{\mathcal{T}} f(\varphi \circ \psi)\, \mu(d\varphi)\, \nu(d\psi), \quad f \in C_b(\mathcal{T}).$$

The relation (6) follows from the fact (easily obtained by a standard argument) that, for $s_1 < s_2$, if $X^{(s_1)} = \{X_{s_1,t}(x)\}_{t \in [s_1, s_2]}$ and $Y^{(s_2)} = \{Y_{s_2,t}(x)\}_{t \in [s_2, \infty)}$ are mutually independent families of solutions to SDE (1) at time s_1 and at time s_2, respectively, both satisfying the condition (A_3), then $Z^{(s_1)} = \{Z_{s_1,t}(x)\}_{t \in [s_1, \infty)}$ defined by

$$Z_{s_1,t}(x) = \begin{cases} X_{s_1,t}(x), & s_1 \leq t \leq s_2; \\ Y_{s_2,t}(X_{s_1,s_2}(x)), & t \geq s_2, \end{cases}$$

is a family of solutions to SDE (1) at time s_1 satisfying the condition (A_3). The proof of the theorem can be completed by applying the following general lemma.

Lemma 1. *For given convolution semigroup $\{\mu_t\}_{t \geq 0}$ on $\mathcal{T}$, there exists a $\mathcal{T}$-stochastic flow $\Phi = \{\varphi_{s,t}; s \leq t\}$ such that the law of $\varphi_{s,t}$ coinsides with μ_{t-s}. Furthermore, the law of such a flow is unique.*

This lemma can be proved by a standard application of the Kolmogorov extension theorem and may be well known. For the completeness, however, we present its proof.

Proof of Lemma **1.** Let $I = \{\lambda = (s, t); s \leq t\}$ and, for $\lambda_1, \ldots, \lambda_n \in I$, define a Borel probability $Q_{\lambda_1, \ldots, \lambda_n}$ on the n-fold product $\mathcal{T}^n$ of $\mathcal{T}$ as follows; let $\lambda_i = (s_i, t_i)$ and let $\{u_0 < u_1 < \ldots < u_l\}$ be the set $\bigcup_{i=1}^{n} \{s_i, t_i\}$ arranged in the order of elements. Take mutually independent $\mathcal{T}$-valued random variables $\xi_1, \ldots, \xi_l$ such that ξ_k is distributed by $\mu_{u_k - u_{k-1}}$, $k = 1, \ldots, l$. Define $\mathcal{T}$-valued random variable η_{λ_i}, $i = 1, \ldots, n$, by

$$\eta_{\lambda_i} = \begin{cases} \mathrm{id}, & \text{when } s_i = t_i; \\ \xi_{k+m} \circ \ldots \circ \xi_{k+1}, & \text{when } s_i = u_k < \ldots < u_{k+m} = t_i. \end{cases}$$

Finally, define $Q_{\lambda_1, \ldots, \lambda_n}$ to be the law on $\mathcal{T}^n$ of $\{\eta_{\lambda_1}, \ldots, \eta_{\lambda_n}\}$.

We can easily verify that the family $\{Q_{\lambda_1, \ldots, \lambda_n}\}$ satisfies the consistency condition so that, by the Kolmogorov extension theorem, we can construct a family $\{\varphi_\lambda;\ \lambda \in I\}$ of $\mathcal{T}$-valued random variables such that the law of $(\varphi_{\lambda_1}, \ldots, \varphi_{\lambda_n})$ coincides with $Q_{\lambda_1, \ldots, \lambda_n}$. Then, $\varphi_{s,t} = \varphi_\lambda$, $\lambda = (s, t)$, is what we want.

The uniqueness in law of $\{\varphi_{s,t}\}_{s \leq t}$ is obvious.

Let $X = \{X_{s,t}; s \leq t\}$ be the $\mathcal{T}$-stoshastic flow of Theorem 1. We know by (4) that $X_{s,t}(x)$ has a representation

$$X_{s,t}(x) = f_{a,b}^{c}(x)$$

with $a = w_t^{(s)}$, $b = -\min_{s \leq u \leq t} w_u^{(s)}$, $c = \mathrm{sgn}(X_{s,t}(0))$, where $w^{(s)} = (w_t^{(s)})_{t \geq s}$ is a Wiener process with $w_s^{(s)} = 0$. By the composition rule (5) for $f_{a,b}^{c}$, we have

$$w_t^{(s)} + w_u^{(t)} = w_u^{(s)} \quad \text{for} \quad s \le t \le u.$$

Then, by setting

$$w(t) = \begin{cases} w_t^{(0)}, & t \ge 0, \\ -w_0^{(t)}, & t < 0, \end{cases}$$

we have $w_t^{(s)} = w(t) - w(s)$ for every $s \le t$. It is easy to verify that $w = \{w(t)\}_{-\infty<t<\infty}$ is a Wiener process. Thus, $X_{s,t}(x)$ has a representation

$$X_{s,t}(x) = f_{a,b}^c(x),$$
$$a = w(t) - w(s), \quad b = -\min_{s \le u \le t} [w(u) - w(s)], \quad c = \operatorname{sgn}(X_{s,t}(0)), \tag{7}$$

where $w = \{w(t)\}_{-\infty<t<\infty}$ is a Wiener process and $X_t = X_{s,t}(0)$ is a solution to SDE (1) for $x = 0$ with resprect to the Wiener process w.

For a fixed s, the process $t \in [s, \infty) \mapsto X_{s,t}(0)$ is continuous as a solution to SDE (1). On the other hand, the process $s \in (-\infty, t] \mapsto X_{s,t}(0)$, for a fixed t, is highly discontinuous. We state, without proof, the following description of this process which can be obtained from the composition rule (5) of $\{f_{a,b}^c\}$ combined with the representation (7).

Theorem 2. *For a fixed t, set*

$$Y_s = X_{t-s,t}(0), \quad B_s = w(t) - w(t-s), \quad s \ge 0,$$

where $w = (w(t))$ is the Wiener process (7). Then $|Y_s| = \max_{0 \le u \le s} B_u$ and the process $(\Xi_s)_{s\ge0}$ defined by

$$\Xi_s = \operatorname{sgn}(Y_s)\left[\max_{0 \le u \le s} B_u - B_s\right]$$

is a one-dimensional Wiener process.

In other words, take a one-dimensional Wiener process $(b_s)_{s\ge0}$ with $b_0 = 0$ and set $r_s = |b_s|$, $s \ge 0$.

Let

$$r_s = -\beta_s + L_s, \quad L_s = \lim_{\varepsilon \to 0} (2\varepsilon)^{-1} \int_0^s 1_{[0,\varepsilon)}(r_u)\, du = \max_{0 \le u \le s} \beta_u, \quad s \ge 0,$$

be the Lévy–Skorokhod decomposition of the reflecting Brownian motion $(r_s)_{s\ge0}$, in which $(\beta_s)_{s\ge0}$ is a one-dimensional Wiener process with $\beta_0 = 0$. Set, finally

$$y_s = \operatorname{sgn}(b_s) L_s, \quad s \ge 0.$$

Then,

$$(Y_s, B_s)_{s\ge0} \overset{\mathrm{d}}{=} (y_s, \beta_s)_{s\ge0}.$$

3. The noise generated by the flow associated to SDE (1). The notion of noises has been introduced and studied by Tsirelson [4–6]. Before giving a formal definition, we prepare some general notions and notations. In the following, a probability space $(\Omega, \mathcal{F}, P)$ is always assumed to be complete and, when we speak of a sub-σ-field of $\mathcal{F}$, it is assumed to contain all P-null sets, unless otherwise stated. The trivial σ-field, which consists of events with probability 0 or 1, is denoted simply by $\{\Omega, \varnothing\}$. For a sub-σ-field $\mathcal{G}$ of $\mathcal{F}$, we denote by $L^0(\Omega\,;\mathcal{G})$, or simply by $L^0(\mathcal{G})$ when Ω is well understood, the space of all $\mathcal{G}$-measurable real random variables (more precisely, the space of all the equivalence classes of $\mathcal{G}$-measurable real random variables coinciding each other P-almost surely). $L^p(\Omega\,;\mathcal{G})$, $1 \le p < \infty$, is the subspace of $L^0(\Omega\,;\mathcal{G})$ formed of all p-th integrable random variables.

Definition 2. *Let $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$ be two probability spaces and let $\mathcal{G}$ and $\mathcal{G}'$ be sub-σ-fields of $\mathcal{F}$ and $\mathcal{F}'$, respectively. By a morphism π from $(\Omega, \mathcal{G})$ to $(\Omega', \mathcal{G}')$, denoted by $\pi\colon(\Omega, \mathcal{G}) \to (\Omega', \mathcal{G}')$, we mean a mapping*

$$\pi_*\colon L^0(\Omega', \mathcal{G}') \to L^0(\Omega, \mathcal{G}),$$

with the following properties:

(i) *for any $X_1, \ldots, X_n \in L^0(\Omega', \mathcal{G}')$,*

$$[(X_1, \ldots, X_n), P'] \overset{d}{=} [(\pi_*(X_1), \ldots, \pi_*(X_n)), P];$$

(ii) *for any $X_1, \ldots, X_n \in L^0(\Omega', \mathcal{G}')$ and any Borel function $f\colon \mathbf{R}^n \to \mathbf{R}$,*

$$\pi_*[f(X_1, \ldots, X_n)] = f(\pi_*(X_1), \ldots, \pi_*(X_n)).$$

Remark **1.** Of course, what we have in mind in the above definition of morphism is a point transformation $\pi\colon \Omega \to \Omega'$ which is $\mathcal{G}/\mathcal{G}'$-measurable and satisfies $P' = P \circ \pi^{-1}$ on $\mathcal{G}'$, so that it induces π_* by $\pi_*(X) = X \circ \pi$, $X \in L^0(\Omega', \mathcal{G}')$. However, we would not like to mention of a point transformation explicitly.

Definition 3. *By a noise, we mean a family $\{\mathcal{F}_{s,t}\,;\, -\infty < s \le t < \infty\}$ of sub-σ-fields of $\mathcal{F}$ on a probability space $(\Omega, \mathcal{F}, P)$ with the following properties:*

(i) *$\mathcal{F}_{t,t} = \{\Omega, \varnothing\}$ for every $t \in \mathbf{R}$;*

(ii) *$\mathcal{F}_{s,u} = \mathcal{F}_{s,t} \vee \mathcal{F}_{t,u}$, and $\mathcal{F}_{s,t}$ and $\mathcal{F}_{t,u}$ are independent for every $s \le t \le u$;*

(iii) *denoting $\mathcal{F}_{-\infty,\infty} = \bigvee_{s\le t} \mathcal{F}_{s,t}$, there exists a one-parameter group $\{T_h\}_{h\in \mathbf{R}}$ of morphisms $T_h\colon (\Omega, \mathcal{F}_{-\infty,\infty}) \to (\Omega, \mathcal{F}_{-\infty,\infty})$ such that $(T_h)_*[L^0(\mathcal{F}_{s,t})] = L^0(\mathcal{F}_{s+h,t+h})$ for every $h \in \mathbf{R}$ and $s \le t$.*

We denote the noise in this definition as $N = [(\mathcal{F}_{s,t})_{s\le t}, \{T_h\}_{h\in \mathbf{R}}]$ or simply as $N = \{\mathcal{F}_{s,t}\}$ when $\{T_h\}$ is well inderstood.

Remark **2.** If we define the notion of noise similarly in the case of discrete time $n \in \mathbf{Z}$, then $\{\mathcal{F}_{m,n};\, m \le n\}$ must be given as $\mathcal{F}_{m,n} = \sigma[\xi_{m+1}, \ldots, \xi_n]$ where $\{\xi_n\}$ is an i.i.d. random sequence. Hence, the notion of noises in the discrete time is essentially equivalent to that of i.i.d. random sequences.

Example **1.** *Let $w = (w_t)_{-\infty<t<\infty}$ be a d-dimensional Wiener process $(1 \le d \le \infty)$ and let $\mathcal{F}_{s,t}$, $s \le t$, be the σ-fields generated by $\{w_v - w_u;\, s \le u \le v \le t\}$. Let T_h, $h \in \mathbf{R}$, be a morphism $T_h\colon (\Omega, \mathcal{F}_{-\infty,\infty}) \to (\Omega, \mathcal{F}_{-\infty,\infty})$ uniquely determined by*

$$(T_h)_*[f(w_t - w_s)] = f(w_{t+h} - w_{s+h}),$$

for any Borel function $f: \boldsymbol{R}^d \to \boldsymbol{R}$.

Then the family $\{\mathcal{F}_{s,t}; s \le t\}$ *together with the one-parameter group of morphisms* $\{T_h\}$ *defines a noise* N_w. *This noise* N_w *is called a* d*-dimensional Gaussian noise or white noise.*

Example 2. *Let* S *be a Polish space and* $n(dx)$ *be a* σ*-finite Borel measure on* S. *Let* $\rho(dt, dx)$ *be a Poisson random measure on* $(-\infty, \infty) \times S$ *with the mean measure* $dt \cdot n(dx)$. *Let* $\mathcal{F}_{s,t}$, $s \le t$, *be the* σ*-field generated by* $\{p((u, v] \times E);\ s \le u \le v \le t,\ E \in \mathcal{B}(S)\}$ *and* T_h, $h \in \boldsymbol{R}$, *be a morphism* $T_h: (\Omega, \mathcal{F}_{-\infty,\infty}) \to (\Omega, \mathcal{F}_{-\infty,\infty})$ *uniquely determined by*

$$(T_h)_*[p((s, t] \times E)] = p((s+h, t+h] \times E), \quad s \le t, \quad E \in \mathcal{B}(S).$$

Then the family $\{\mathcal{F}_{s,t}; s \le t\}$ *together with the one-parameter group of morphisms* $\{T_h\}$ *defines a noise* N_p. *This noise* N_p *is called a Poissonian noise.*

Similarly, we can define a noise from a process with stationary independent increments which we call an *additive noise* or a *linearizable noise.* Gaussian noises and Poissonian noises are particular examples of additive noises.

Definition 4. *Let* $N = [\{\mathcal{F}_{s,t}\}_{s\le t}, \{T_h\}_{h\in R}]$ *and* $N' = [\{\mathcal{F}'_{s,t}\}_{s\le t}, \{T'_h\}_{h\in R}]$ *be two noises defined on probability spaces* Ω *and* Ω', *respectively. We say that* N' *is homomorphic to* N *if there exists a morphism* $\pi: (\Omega, \mathcal{F}_{-\infty,\infty}) \to (\Omega', \mathcal{F}'_{-\infty,\infty})$ *such that*

$$\pi_*[L^0(\Omega', \mathcal{F}'_{s,t})] \subset L^0(\Omega, \mathcal{F}_{s,t}) \quad \forall\, s \le t,$$

and

$$\pi_* \circ (T'_h)_* = (T_h)_* \circ \pi_* \quad \forall\, h \in \boldsymbol{R}.$$

If, furthermore, $\pi_*[L^0(\Omega', \mathcal{F}'_{s,t})] = L^0(\Omega, \mathcal{F}_{s,t})$ *for all* $s \le t$, *then we say that* N' *is isomorphic to* N.

When N' is isomorphic to N, noting that π_* is always injective, we see easily that $(\pi_*)^{-1}$ defines a morphism $\pi^{-1}: (\Omega', \mathcal{F}'_{-\infty,\infty}) \to (\Omega, \mathcal{F}_{-\infty,\infty})$ so that N is isomorphic to N'. Thus, in this case, we may well say that *N and N' are isomorphic.*

Definition 5. *Let* $N = [(\mathcal{F}_{s,t})_{s\le t}, \{T_h\}_{h\in R}]$ *and* $N' = [\{\mathcal{F}'_{s,t}\}_{s\le t}, \{T'_h\}_{h\in R}]$ *be two noises defined on a same probability space. We say that* N' *is a subnoise of* N *if* $\mathcal{F}'_{s,t} \subset \mathcal{F}_{s,t}$, *for all* $s \le t$, *and* $\{T'_h\}_*$ *is the restriction of* $(T_h)_*$ *for every* $h \in \boldsymbol{R}$.

The following two propositions are easy to prove.

Proposition 2. *A noise* N' *is homomorphic to a noise* N *if and only if* N' *is isomorphic to a subnoise of* N.

Proposition 3. 1. *A subnoise of Gaussian noise is Gaussian. More generally, any noise homomorphic to a Gaussian noise is also Gaussian.*

2. *Let* N *and* N' *be two Gaussian noises with dimension* d *and* d', *respectively. Then* N' *is homomorphic to* N *if and only if* $d' \le d$. N' *is isomorphic to* N *if and only if* $d' = d$.

Definition 6. *For a noise* $N = [(\mathcal{F}_{s,t})_{s\le t}, \{T_h\}_{h\in R}]$, *define*

$$\hat{\mathcal{F}}_{s,t} = \mathcal{F}_{-t,-s}, \quad s\le t, \quad and \quad \hat{T}_h = T_{-h}, \quad h\in \boldsymbol{R}.$$

Then, obviously, $\hat{N} = [\{\hat{\mathcal{F}}_{s,t}\}_{s\le t}, \{\hat{T}_h\}_{*_{h\in R}}]$ *is a noise and we call it the reversed noise of* N.

The reversed noise of a Gaussian noise is a Gaussian noise isomorphic to it, more generally, the reversed noise of an additive noise is an additive noise isomorphic to it.

Given a noise $N = \{\mathcal{F}_{s,t}\}_{s\le t}$, the filtration $\boldsymbol{F} = \{\mathcal{F}_{0,t}\}_{t\ge 0}$ is called the *filtration associated to the noise* N.

Definition 7. *A noise* $N = \{\mathcal{F}_{s,t}\}_{s\le t}$ *is called predictable if, for the filtration* $\boldsymbol{F}$ *associated to* N, *it holds that* $\mathcal{M}(\boldsymbol{F}) = \mathcal{M}^c(\boldsymbol{F})$, *where* $\mathcal{M}(\boldsymbol{F})$ *is the space of locally square-integrable* $\boldsymbol{F}$*-martingales* $M = (M_t)$ *with* $M_0 = 0$ *and* $\mathcal{M}^c(\boldsymbol{F})$ *its subspace formed of all continuous martingales.*

Gaussian noises are predictable. Poissonian noises are not predictable and, more generally, an additive noise is predictable if and only if it is Gaussian. In the following, we consider predictable noises only, unless otherwise stated.

Definition 8. *Let* $N = [(\mathcal{F}_{s,t})_{s\le t}, \{T_h\}_{h\in R}]$ *be a predictable noise. By a linear representation of a noise, we mean a two-parameter family* $R = \{R_{s,t};\ s\le t\}$ *of real* $\mathcal{F}_{-\infty,\infty}$*-measurable random variables* $R_{s,t}$ *such that*

(i) $R_{s,t}$ *is* $\mathcal{F}_{s,t}$*-measurable for any* $s\le t$,

(ii) $R_{s,t}+R_{t,u} = R_{s,u}$ *for any* $s\le t\le u$,

(iii) $(T_h)_*[R_{s,t}] = R_{s+h,t+h}$ *for any* $s\le t$ *and* $h\in \boldsymbol{R}$.

The totality $\mathcal{R}$ *of linear representation of a noise* N *forms a Gaussian system and each* $R = \{R_{s,t}\}\in \mathcal{R}$ *is given by* $R_{s,t} = w_t - w_s$ *from a Wiener process* $(w_t)_{-\infty<t<\infty}$.

Then, setting

$$\mathcal{F}^{\text{lin}}_{s,t} = \sigma\{R_{u,v};\ s\le u\le v\le t,\ R\in\mathcal{R}\},$$

a subnoise $N^{\text{lin}} = [\{\mathcal{F}^{\text{lin}}_{s,t}\}_{s\le t}, \{T_h\}_{h\in R}]$ *of* N *is defined, which is obviously Gaussian.*

It is a maximal Gaussian subnoise of N; N is Gaussian if and only if $N^{\text{lin}} = N$ so that it is non-Gaussian or non-white if $N^{\text{lin}} \ne N$. Tsirelson called a nontrivial predictable noise for which N^{lin} is trivial, i.e. $\mathcal{F}^{\text{lin}}_{s,t} = \{\Omega, \varnothing\}$ for any (= for some) $s<t$, a *black noise.*

Let $X = \{X_{s,t};\ s\le t\}$ be the $\mathcal{T}$-stoshastic flow of Theorem 1 defined on a probability space Ω. Let $\mathcal{F}_{s,t} = \sigma[X_{u,v};\ s\le u\le v\le t]$ for $s\le t$. For $h\in \boldsymbol{R}$, a morphism $T_h\colon (\Omega, \mathcal{F}_{-\infty,\infty}) \to (\Omega, \mathcal{F}_{-\infty,\infty})$ is uniquely determined by $(T_h)_*[X_{s,t}(x)] = X_{s+h,t+h}(x)$ for every $s\le t$ and $h\in \boldsymbol{R}$, and $\{T_h;\ h\in \boldsymbol{R}\}$ forms a one-parameter group of morphisms. It is easy to deduce properties (i), (ii) and (iii) in Definition 3 from the properties 1), 2, and 3) in Definition 1 so that $N = [(\mathcal{F}_{s,t})_{s\le t}, \{T_h\}_{h\in R}]$ defines a noise. Let $w = (w(t))_{-\infty<t<\infty}$ be the Wiener process in the representation (7) of the flow X. It is easy to see that $(T_h)_*[w(t) - w(s)] = w(t +$

$+h)-w(s+h)$ so that $\{w(t)-w(s)\}$ defines a linear representation of the noise N. Set $\mathcal{F}^w_{s,t} = \sigma[w(v)-w(u);\ s \le u \le v \le t]$, $s \le t$. Then $N^w := [\{\mathcal{F}^w_{s,t}\}_{s\le t}, \{T_h\}_{h\in R}]$ is, obviously, a Gaussian subnoise of N.

Theorem 3. *$N = [\{\mathcal{F}_{s,t}\}_{s\le t}, \{T_h\}_{h\in R}]$ is a predictable non-Gaussian noise and $N^{\text{lin}} = N^w$.*

Proof. First, we establish a martingale representation theorem for the filtration $F = \{\mathcal{F}_{0,t})_{t\ge 0}$ associated to the noise N.

Lemma 2. *Every $M = (M_t) \in \mathcal{M}(F)$ can be represented as a stoshastic integral of a F-predictable process $\Phi = (\Phi_s)$ as*

$$M_t = \int_0^t \Phi_s \, dw(s), \qquad t \ge 0,$$

where $w = (w(t))$ is the Wiener process in (7).

Proof. We have

$$X_{s,t}(x) - x = \int_s^t \operatorname{sgn}(X_{s,u}(x))\, dw(s), \qquad t \in [s,\infty),$$

for fixed $s \ge 0$ and $x \in R$ and hence, for $s_0 = 0 < s_1 < \ldots < s_n < \infty$, $x_1, \ldots, x_n \in R$ and $\xi_1, \ldots, \xi_n \in R$,

$$\sum_{i=1}^n \xi_i \cdot (X_{s_{i-1}, s_i \wedge t}(x_i) - x_i) \cdot 1_{[s_{i-1}<t]} = \int_0^t \Psi(s)\, dw(s)$$

for some bounded F-predictable process $\Psi = (\Psi(s))$ such that $|\Psi(s)|^2 = |\xi_i|^2$ for $s \in [s_{i-1}, s_i)$, $i = 1, \ldots, n$. Set

$$N(t) = \exp\left[\int_0^t \Psi(s)\, dw(s) - \frac{1}{2}\int_0^t |\Psi(s)|^2\, ds\right].$$

Since $|\Psi(s)|^2$ is deterministic, we obtain easily by the Itô formula,

$$Y := \exp\left[\sum_{i=1}^n \xi_i X_{s_{i-1}, s_i}(x_i)\right] = CN(s_n) = C\left[1 + \int_0^{s_n} N(s)\Psi(s)\, dw(s)\right],$$

where $C = \exp\left(\sum_{i=1}^n \xi_i x_i + \frac{1}{2}\int_0^{s_n} |\Psi(s)|^2\, ds\right)$ is a positive constant. Then

$$Y = E(Y) + \int_0^{s_n} \Psi'(s)\, dw(s)$$

with $\Psi'(s) = CN(s)\Psi(s)$.

Linear combinations of random variables in the form of Y are dense in $L^2(\mathcal{F}_{0,\infty})$ and, by a standard approximation argument, we see that, for every $F \in L^2(\mathcal{F}_{0,\infty})$, $E(F \mid \mathcal{F}_{0,t}) - E(F)$ is a stoshastic integral by dw.

We return to the proof of Theorem. From the lemma, N is obviously predictable. To prove that N is not Gaussian, suppose on the contrary, that N is a d-dimensional Gaussian noise. Then the filtration $F = \{\mathcal{F}_{0,t}\}_{t\ge 0}$ associated to the noise N is a

d-dimensional Brownian filtration. By the lemma, we must have $d = 1$. This means that N is a one-dimensional Gaussian noise. As we saw above, for the Wiener process $w = (w(t))$ in (7), N^w is a Gaussian subnoise of N and, by the comparison of the dimension, we must have $N^w = N$. But this contradicts to the fact that $\mathcal{F}^w_{s,t}$ is strictly smaller than $\mathcal{F}_{s,t}$ which was already explained in the introduction.

In [5], Tsirelson discussed the spectral decomposition for a family of projection operators on L^2-spaces associated to a noise; in particular, he defined the notion of *spectral measures* associated to a noise. To introduce this, we need the following notions and notations: Let $C_{[0,1]}$ be the space of all closed subsets in $[0, 1]$ endowed with the Hausdorff distance. Let $C^{\text{finite}}_{[0,1]} = \{E \in C_{[0,1]} \mid \#(E) < \infty\}$. Each $E \in C^{\text{finite}}_{[0,1]}$ can be expressed as $E = \{t_1, \ldots, t_n\}$, $0 \le t_1 < \ldots < t_n \le 1$, for some $n = 0, 1, \ldots$; $n = 0$ means that $E = \varnothing$.

By an elementary set E, we mean a finite disjoint union $E = \bigcup_{i=1}^n [u_i, v_i]$, $0 \le u_1 < v_1 < u_2 < v_2 < \ldots < u_n < v_n \le 1$, of closed intervals. Set $\mathcal{F}_E = \mathcal{F}_{u_1,v_1} \vee \mathcal{F}_{u_2,v_2} \vee \ldots \vee \mathcal{F}_{u_n,v_n}$.

Definition 9. *Given a noise* $N = [(\mathcal{F}_{s,t})_{s\le t}, \{T_h\}_{h\in R}]$ *and* $f \in L^2(\mathcal{F}_{0,1})$ *such that* $\|f\|^2_{L^2} = E(f^2) = 1$, *there exists a unique Borel probability* μ_f *on* $C_{[0,1]}$ *with the property that, for every elementary set* E,

$$\mu_f(\{F \in C_{[0,1]};\ F \subset E\}) = E(E(f \mid \mathcal{F}_E)^2). \tag{8}$$

μ_f *is called the spectral measure of the noise* N *with respect to* $f \in L^2(\mathcal{F}_{0,1})$.

***Example* 3.** *Let* $N_w = \{\mathcal{F}^w_{s,t}\}_{s\le t}$ *be a* d*-dimensional Gaussian noise generated by a* d*-dimensional Wiener process* $w = (w(t))_{-\infty<t<\infty}$. *By the Wiener–Itô expansion theorem for Wiener functionals, we have, for every* $f \in L^2(\mathcal{F}^w_{0,1})$ *with* $E(f^2) = 1$, *the following expansion in the sense of iterated Itô stoshastic integrals:*

$$f = f_0 + \sum_{n=1}^{\infty} \int_0^1 \left\{ \int_0^{t_n} \cdots \left\{ \int_0^{t_2} f_n(t_1, t_2, \ldots, t_{n-1}, t_n)\, dw(t_1) \right\} \ldots dw(t_{n-1}) \right\} dw(t_n).$$

where $f_0 = E(f)$ *and* f_n, $n = 1, 2, \ldots$, *are* L^2*-functions defined on the* $\Delta_n = \{(t_1, \ldots, t_n) \mid 0 < t_1 < \ldots < t_n < 1\}$ *with values in the* n*-fold tensc* $\bigotimes_{i=1}^n R^d$ *of* R^d. *The condition* $E(f^2) = 1$ *implies that*

$$f_0^2 + \sum_{n=1}^{\infty} \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} |f_n(t_1, \ldots, t_{n-1}, t_n)|^2\, dt_1 \ldots dt_{n-1}\, dt_n = 1.$$

In this case, the spectral measure μ_f *is given as follows:* $\mu_f(C^{\text{finite}}_{[0,1]}) = 1$ *under* μ_f, $E = \{t_1, \ldots, t_n\}$, $0 \le t_1 < \ldots < t_n \le 1$, *is distributed by*

$$|f_n(t_1, \ldots, t_n)|^2\, dt_1 \ldots dt_n, \quad n = 0, 1, \ldots.$$

In particular, $\mu_f(\{\varnothing\}) = f_0^2 = E(f)^2$.

Tsirelson [5] showed that a predictable noise is Gaussian if and only if $\mu_f(C^{\text{finite}}_{[0,1]}) = 1$ for every $f \in L^2(\mathcal{F}_{0,1})$ with $E(f^2) = 1$.

Now we return to the noise N associated to the flow X of Theorem 1. In the following, we compute the spectral measure μ_f in the case $f = \operatorname{sgn}(X_{0,1}(0))$.

Let $w = (w(t))_{-\infty < t < \infty}$ be the 1-dimensional Wiener process in the representation (7) for X and N^w be the Gaussian noise generated by w. We know that N^w is a proper subnoise of N.

For fixed $0 < t < 1$ and $x > 0$, let $\sigma_t^x = \min\{u > t \mid x + w(u) - w(t) = 0\}$. Then

$$1_{[\sigma_t^x > 1]} = 1_{\left[x + \min\limits_{t \le u \le 1}(w(u) - w(t)) > 0\right]}$$

is $\mathcal{F}^w_{t,1}$-measurable and let

$$1_{[\sigma_t^x > 1]} = f_0[x, t] + \sum_{n=1}^{\infty} \int_t^1 \left\{ \ldots \left\{ \int_t^{t_2} f_n[x, t](t_1, \ldots, t_n)\, dw(t_1) \right\} \ldots \right\} dw(t_n) \qquad (9)$$

be its Wiener–Itô expansion by itereted stochastic integrals. By the method of Veretennikov–Krylov [10], $f_0[x, t]$ and $f_n[x, t](t_1, \ldots, t_n)$, $t < t_1 < \ldots < t_n < 1$, can be explicitly given as follows. First, introduce some notations: let

$$p^{\pm}(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left[\exp\left\{ -\frac{(x-y)^2}{2t} \right\} \pm \exp\left\{ -\frac{(x+y)^2}{2t} \right\} \right],$$

$$x \ge 0, \quad y \ge 0, \quad t > 0,$$

and defines the Markovian semigroups $\{T_t^+\}$ and $\{T_t^-\}$ on $C_b([0, \infty))$ by

$$T_t^{\pm} f(x) = \int_0^{\infty} p^{\pm}(t, x, y) f(y)\, dy, \qquad t > 0, \quad f \in C_b([0, \infty)).$$

$\{T_t^+\}$ and $\{T_t^-\}$ are semigroups corresponding to reflecting Brownian motion and absorbing Brownian motion on $[0, \infty)$, respectively. Introduce, further, operators ∂T_t^+ and ∂T_t^- by

$$[(\partial T_t^{\pm}) f](x) = \frac{\partial}{\partial x} T_t^{\pm} f(x) = \int_0^{\infty} \frac{\partial p^{\pm}}{\partial x}(t, x, y) f(y)\, dy, \qquad t > 0, \quad x \in [0, \infty).$$

Then,

$$f_0[x, t] = T_{1-t}^- 1(x) = \frac{2}{\sqrt{2\pi(1-t)}} \int_0^x e^{-y^2/\{2(1-t)\}}\, dy \qquad (10)$$

and

$$f_n[x, t](t_1, \ldots, t_n) = T_{t_1 - t}^- (\partial T_{t_2 - t_1}^- (\ldots (\partial T_{1 - t_n}^- 1)))(x),$$
$$t \le t_1 < \ldots < t_n < 1. \qquad (11)$$

Let

$$\mu_t(x) = \frac{x}{t^{3/2}} e^{-x^2/\{2t\}}, \qquad x > 0, \quad t > 0. \qquad (12)$$

Then

$$\int_0^\infty \mu_t(y)\, p^-(s,y,x)\,dy = \mu_{t+s}(x), \qquad t,s>0, \quad x>0,$$
$$\int_0^\infty \mu_t(y)\,dy = \frac{1}{\sqrt{t}}, \tag{13}$$

that is, $\mu_t(y)\,dy$ is an entrance law for the absorbing semigroup T_t^-.

From (9), we have

$$f_0[x,t]^2 + \sum_{n=1}^\infty \int_t^1 dt_n \int_t^{t_n} dt_{n-1} \ldots \int_t^{t_2} dt_1 \cdot f_n[x,t](t_1,\ldots,t_n)^2 =$$
$$= E[1_{[\sigma_t^x>1]}] = T_{1-t}^- 1(x)$$

and so, for $0<r<t<1$,

$$\sqrt{1-r}\int_0^\infty \mu_{t-r}(x)\left[f_0[x,t]^2 + \sum_{n=1}^\infty \int_t^1 dt_n \int_t^{t_n} dt_{n-1}\ldots\int_t^{t_2} dt_1\cdot f_n[x,t](t_1,\ldots,t_n)^2\right]dx =$$
$$= \sqrt{1-r}\int_0^\infty \mu_{t-r}(x)\,T_{1-t}^- 1(x)\,dx = \sqrt{1-r}\int_0^\infty \mu_{1-r}(x)\,dx = 1. \tag{14}$$

We denote by $C_{[a,b]}$, $0\le a<b\le 1$, the subspace of $C_{[0,1]}$ formed of all closed subsets in $[a,b]$. By $C_{[a,b]}^{\text{finite}}$, we denote the subclass formed of all finite subsets in $[a,b]$. For each $0\le r<t<1$, we define a Borel probability $m_{r,t}$ on $C_{[t,1]}$ as follows: $m_{r,t}(C_{[t,1]}^{\text{finite}}) = 1$ and, under $m_{r,t}$, $E=\{t_1,\ldots,t_n\}$, $t<t_1<\ldots<t_n<1$, is distributed by

$$\sqrt{1-r}\left[\int_0^\infty \mu_{t-r}(x)\,f_n[x,t](t_1,\ldots,t_n)^2\,dx\right]dt_1\ldots dt_n, \qquad n=0,1,\ldots. \tag{15}$$

By (14), this is well defined. In particular,

$$\mu_{r,t}(\{\varnothing\}) = \sqrt{1-r}\int_0^\infty \mu_{t-r}(x)\,f_0[x,t]^2\,dx =$$
$$= \sqrt{1-r}\int_0^\infty \mu_{t-r}(x)\,(T_{1-t}^- 1(x))^2\,dx.$$

For $0\le s<t<1$, define the projection operator $\pi_{s,t}\colon C_{[s,1]}\to C_{[t,1]}$ by

$$\pi_{s,t}(E) = E\cap[t,1]\in C_{[t,1]}, \qquad E\in C_{[s,1]}.$$

Lemma 3 [7]. *For a fixed* $0\le r<1$, *the family of Borel probabilities* $m_{r,t}$ *on* $C_{[t,1]}$, $1>t>r$, *is consistent in the sense that*

$$m_{r,s}\circ\pi_{s,t}^{-1} = m_{r,t} \quad \textit{for every} \quad r<s<t.$$

Then, by a standard application of the Kolmogorov extension theorem, we obtain the next corollary.

Corollary **1** [7]. *For a fixed* $0 \le r < 1$, *there exists a unique Borel probability* m_r *on* $C_{[r,1]}$ *such that*

$$m_{r,t} = m_r \circ \pi_{r,t}^{-1} \quad \text{for every} \quad r < t < 1.$$

Proof of Lemma **3.** It is enough to show that, for every $0 \le r < s < t < 1$ and $t < t_1 < \ldots < t_n < 1$, $n = 1, 2, \ldots$,

$$\sum_{k=0}^{\infty} \int_0^{\infty} \mu_{s-r}(x) \left[\int \cdots \int_{s<s_1<\ldots<s_k<t} f_{n+k}^2[x,s](s_1,\ldots,s_k,t_1,\ldots,t_n)\, ds_1 \ldots ds_k \right] dx =$$

$$= \int_0^{\infty} \mu_{t-r}(x) \cdot f_n^2[x,t](t_1,\ldots,t_n)\, dx. \tag{16}$$

By comparing the Wiener – Itô expansion of both sides in the following equation:

$$1_{\left[x + \min\limits_{s\le u\le 1}(w(u)-w(s)) > 0\right]} =$$

$$= 1_{\left[x + \min\limits_{s\le u\le t}(w(u)-w(s)) > 0\right]} 1_{\left[x + (w(t)-w(s)) + \min\limits_{t\le u\le 1}(w(u)-w(t)) > 0\right]},$$

we obtain, for $s < s_1 < \ldots < s_k < t < t_1 < \ldots < t_n < 1$,

$$f_{n+k}[x,s](s_1,\ldots,s_k,t_1,\ldots,t_n) = g_{k,n}[x,s,t,t_1,\ldots,t_n](s_1,\ldots,s_k),$$

where the functions $g_{k,n}$ are determined as kernels of the following Wiener – Itô expansion:

$$f_n[x + (w(t) - w(s)), t](t_1,\ldots,t_n) \cdot 1_{\left[x + \min\limits_{s\le u\le t}(w(u)-w(s)) > 0\right]} =$$

$$= g_{0,n}[x,s,t,t_1,\ldots,t_n] +$$

$$+ \sum_{k=1}^{\infty} \int \cdots \int_{s<s_1<\ldots<s_k<t} g_{k,n}[x,s,t,t_1,\ldots,t_n](s_1,\ldots,s_k)\, dw(s_1) \ldots dw(s_k).$$

Then,

$$\sum_{k=0}^{\infty} \int \cdots \int_{s<s_1<\ldots<s_k<t} f_{n+k}^2[x,s](s_1,\ldots,s_k,t_1,\ldots,t_n)\, ds_1 \ldots ds_k =$$

$$= \sum_{k=0}^{\infty} \int \cdots \int_{s<s_1<\ldots<s_k<t} g_{k,n}^2[x,s,t,t_1,\ldots,t_n](s_1,\ldots,s_k)\, ds_1 \ldots ds_k =$$

$$= E\left[f_n^2[x + (w(t) - w(s)), t](t_1,\ldots,t_n) \cdot 1_{\left[x + \min\limits_{s\le u\le t}(w(u)-w(s)) > 0\right]} \right] = (T_{t-s}^{-} h)(x),$$

where $h(y) = f_n^2[y,t,](t_1,\ldots,t_n)$. From this, we have

$$\sum_{k=0}^{\infty} \int_0^{\infty} \mu_{s-r}(x) \left[\int \cdots \int_{s<s_1<\ldots<s_k<t} f_{n+k}^2[x,s](s_1,\ldots,s_k,t_1,\ldots,t_n)\, ds_1 \ldots ds_k \right] dx =$$

$$= \int_0^{\infty} \mu_{s-r}(x)(T_{t-s}^{-} h)(x)\, dx = \int_0^{\infty} \mu_{t-r}(x)\, h(x)\, dx,$$

which is exactly (16).

It is obvious from the definition of m_r that,

$$m_r\{E \in C_{[r,1]} \mid \#(E \cap [t,1]) < \infty \quad \text{for each} \quad t > r\} = 1.$$

However, we have the following.

Theorem 4.

$$m_r\{E \in C_{[r,1]} \mid \#(E) = \infty\} = 1.$$

Proof. For a fixed $0 < t < 1$, we denote the path $\{w(u) - w(t);\ t \le u \le 1\}$ by $w_{[t,1]}$. We may consider $w_{[t,1]}$ as a generic element of a Wiener space: Every $\mathcal{F}^w_{t,1}$-measurable random variable is a Wiener functional of $w_{[t,1]}$. We denote, in particular, $1_{[\sigma^t_x > 1]} = F_{[t,x]}(w_{[t,1]})$. If $(T_\lambda)_{\lambda > 0}$ is the Ornstein–Uhlenbeck semigroup on the Wiener space, then, from (9), we have

$$(T_\lambda F_{[t,x]})(w_{[t,1]}) =$$

$$= f_0[x,t] + \sum_{n=1}^{\infty} e^{-\lambda n} \int \ldots \int_{t < t_1 < \ldots < t_n < 1} f_n[x,t](t_1, \ldots, t_n)\, dw(t_1) \ldots dw(t_n).$$

Hence, for a fixed λ, and every $t > r$,

$$\int_{C_{[r,1]}} e^{-2\lambda \#(E \cap [t,1])}\, m_r(dE) = \sqrt{1-r} \int_0^\infty \mu_{t-r}(x) \| T_\lambda F_{[t,x]} \|_2^2\, dx,$$

where $\| \ \|_p$ is the L_p-norm on the Wiener space. By the hypercontractivity of T_λ [3, p. 367],

$$\| T_\lambda F_{[t,x]} \|_2^2 \le \| F_{[t,x]} \|_p^2 \quad \text{with} \quad p = 1 + e^{-2\lambda} < 2$$

and $\| F_{[t,x]} \|_p = E(1_{[\sigma^x_t > 1]})^{1/p} = [T^-_{1-t} 1(x)]^{1/p}$. Hence, by (10),

$$\| T_\lambda F_{[t,x]} \|_2^2 \le [T^-_{1-t} 1(x)]^{2/p} = O(x^{2/p})$$

as $x \downarrow 0$ and $t \downarrow r$. Then,

$$\int_{C_{[r,1]}} e^{-2\lambda \#(E \cap [t,1])}\, m_r(dE) \le \text{const.} \int_0^\infty \mu_{t-r}(x)\, x^{2/p}\, dx =$$

$$= \text{const.} \int_0^\infty \frac{x^{1+2/p}}{(t-r)^{3/2}} e^{-x^2/\{2(t-r)\}}\, dx =$$

$$= \text{const.}\ (t-r)^{(2-p)/(2p)} \to 0 \quad \text{as} \quad t \downarrow r.$$

Thus,

$$\int_{C_{[r,1]}} e^{-2\lambda \#(E)}\, m_r(dE) = 0,$$

that is, $\#(E) = \infty$ for m_r-a.a. $E \in C_{[r,1]}$.

It is easy to deduce the following from the scaling property of Brownian motion:

Proposition 4. *Let S be a $C_{[0,1]}$-valued random set with the law m_0. Then, for every $0 \le r < 1$, the $C_{[r,1]}$-valued random set defined by*

$$r + (1-r)\,\mathcal{S} = \{\, r + (1-r)\,t \mid t \in \mathcal{S} \,\},$$

has the law m_r.

Theorem 5 [7]. *For* $f = \operatorname{sgn}(X_{0,1}(0))$, *the spectral measure* μ_f *of the noise* N *is the law of a* $C_{[0,1]}$*-valued random set given by*

$$\{\, g + (1-g)\,\mathcal{S}_1 \} \cup \{\, g - g\,\mathcal{S}_2 \} =$$
$$= \{\, g + (1-g)\,t \mid t \in \mathcal{S}_1 \} \cup \{\, g\,(1-t) \mid t \in \mathcal{S}_2 \}.$$

Here, $\mathcal{S}_1$, $\mathcal{S}_2$ *and* g *are mutually independent random elements such that* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are copies of the random set* $\mathcal{S}$ *in the above proposition and* g *is an arc-sine distributed random variable with values in* $[0,1]$: *that is,*

$$P[g \in dt] = \frac{1}{\pi}\frac{1}{\sqrt{t(1-t)}}, \qquad 0 < t < 1.$$

Proof. First, we introduce, for a fixed $0 < r < 1$, a Borel probability $\breve{m}_r$ on $C_{[0,r]}$ as follows.

Lemma 4. *For* $0 < u < 1$ *and* $x > 0$, *define the kernel* $g_n[x,u](u_1,\ldots,u_n)$, $0 < u_1 < \ldots < u_n < u$, $n = 0, 1, \ldots$, *by the following Wiener – Itô expansion:*

$$1_{[r(u) < x]} = g_0[x,u] + \sum_{n=1}^{\infty} \int_0^u \left\{ \ldots \left\{ \int_0^{u_2} g_n[x,u](u_1,\ldots,u_n)\, dw(u_1) \right\} \ldots \right\} dw(u_n), \tag{17}$$

where $r(u) = [w(u) - w(0)] - \min_{0 \le v \le u} [w(v) - w(0)]$. *Then, there exists a unique Borel probability* $\breve{m}_r$ *on* $C_{[0,r]}$ *such that, for every* $0 < u < r$,

$$\breve{m}_r \{ E \in C_{[0,r]} \mid \#(E \cap [0,u]) < \infty \} = 1$$

and, under $\breve{m}_r$, $E \cap [0,u] = \{u_1, \ldots, u_n\}$, $0 < u_1 < \ldots < u_n < u$, *is distributed by*

$$\sqrt{r} \left[\int_0^{\infty} \mu_{r-u}(x)\, g_n^2[x,u](u_1,\ldots,u_n)\, dx \right] du_1 \ldots du_n. \tag{18}$$

Furthermore, $\breve{m}_r$ *is the law of the random set* $r - r\,\mathcal{S} = \{\, r(1-t);\ t \in \mathcal{S} \}$ *where* $\mathcal{S}$ *is the random set in the above proposition.*

Proof. The probability $\breve{m}_r$ can be obtained from the probability m_{1-r} on $C_{[1-r,1]}$ by the time reversal: Indeed, by considering the time-reversed Wiener process $\breve{w}(s) = w(1-s) - w(1)$, we easily see for $0 < u_1 < \ldots < u_n < u < 1$ that

$$g_n[x,u](u_1,\ldots,u_n) = f_n[x, 1-u](1-u_n, \ldots, 1-u_1). \tag{19}$$

Thus, $\breve{m}_r$ is obtained as the image mesure of m_{1-r} on $C_{[1-r,1]}$ by the map

$$E \in C_{[1-r,1]} \mapsto \breve{E} = \{\, 1 - t \mid t \in E \,\} \in C_{[0,r]}.$$

On the other hand, kernels $g_n[x,u]$ are given explicitly by the Veretennikov – Krylov formula in terms of the reflecting semigroup T_t^+ and the duality relation (19) is analytically equivalent to the following relation: for $t > 0$, $t_i > 0$ and $x > 0$,

$$T_t^- 1(x) = T_t^+ 1_{[0,x]}(0),$$

$$T_{t_1}^-(\partial T_{t_2}^-(\ldots(\partial T_{t_n}^- 1)))(x) = T_{t_n}^+(\partial T_{t_{n-1}}^+(\ldots(\partial T_{t_1}^+ 1_{[0,x]})))(0).$$

Now returning to the proof of the theorem, let $\mathscr{E}_m$, $m = 1, 2, \ldots$, be the family of elementary sets expressible as a finite union of dyadic intervals $[(k-1)2^{-m}, k2^{-m}]$, $k = 1, \ldots, 2^m$. It is enough to establish (8) for $E \in \mathscr{E}_m$. Let $w = (\dot{w}(t))_{-\infty<t<\infty}$ be the Wiener process in the expression (7) of the flow X for which we may assume $w(0) = 0$ without loss of generality. Recall that

$$|X_{s,t}(x)| = |x| + w(t) - w(s) -$$

$$- \min_{s \le u \le t} [(|x| + w(u) - w(s)) \wedge 0], \qquad s \le t, \quad x \in R.$$

Denoting by l^1 the last exit time from the origin of the Brownian path

$$t \in [0, 1) \mapsto X_{0,t}(0), \tag{20}$$

we have

$$f = \operatorname{sgn}(X_{0,1}(0)) =$$

$$= \sum_{k=1}^{2^m} 1_{[(k-1)2^{-m} < l^1 < k2^{-m}]} \operatorname{sgn}(X_{(k-1)2^{-m}, k2^{-m}}(0)) =$$

$$= \sum_{k=1}^{2^m} 1_{G_{m,k}} 1_{F_{m,k}} \operatorname{sgn}(X_{(k-1)2^{-m}, k2^{-m}}(0)), \tag{21}$$

where, by setting

$$r(t) = w(t) - \min_{0 \le s \le t} w(u),$$

$$\xi = - \min_{(k-1)2^{-m} \le u \le k2^{-m}} [w(u) - w((k-1)2^{-m})]$$

and

$$\eta = w(k2^{-m}) - w((k-1)2^{-m}) - \min_{(k-1)2^{-m} \le u \le k2^{-m}} [w(u) - w((k-1)2^{-m})],$$

events $G_{m,k}$ and $F_{m,k}$ are defined by

$$G_{m,k} = [r((k-1)2^{-m}) < \xi]$$

and

$$F_{m,k} = \left[\eta + \min_{k2^{-m} \le u \le 1} [w(u) - w(k2^{-m})] > 0\right].$$

The event $G_{m,k}$ indicates that the Brownian path (20) has at least one zero during the interval $[(k-1)2^{-m}, k2^{-m}]$ while, $F_{m,k}$ that the Brownian path (20) does not have any zero during the interval $[k2^{-m}, 1]$. On $G_{m,k}$, the path (20) and the path $t \mapsto$ $\mapsto X_{(k-1)2^{-m},t}(0)$ coincides for $t \ge k2^{-m}$ because of the coalescence of paths. Set, for $x > 0$ and $y > 0$,

$$G_{m,k}[x] = [r((k-1)2^{-m}) < x]$$

and

$$F_{m,k}[y] = \left[y + \min_{k2^{-m} \le u \le 1} [w(u) - w(2^{-m})] > 0 \right].$$

Note that $G_{m,k}[x] \in \mathcal{F}_{0,(k-1)2^{-m}}$, $F_{m,k}[y] \in \mathcal{F}_{k2^{-m},1}$ and $\{ \xi, \eta, \operatorname{sgn}(X_{(k-1)2^{-m},k2^{-m}}(0)) \}$ is $\mathcal{F}_{(k-1)2^{-m},k2^{-m}}$-measurable so that they are mutually independent. We fix m_0 and $E \in \mathcal{E}_{m_0}$. In computing the conditional expectation $E(f | \mathcal{F}_E)$ by the expression (21) for $m \ge m_0$, it is easy to see that the k-th summand in (21) has the zero conditional expectation unless $[(k-1)2^{-m}, k2^{-m}] \subset E$. Hence, setting

$$A(E) = \{ k \mid 1 \le k \le 2^{-m},\ [(k-1)2^{-m}, k2^{-m}] \subset E \},$$

we have, by (9) and (17),

$$E(f | \mathcal{F}_E) = \sum_{k \in A(E)} E(1_{G_{m,k}} | \mathcal{F}^w_{E \cap [0,(k-1)2^{-m}]}) \operatorname{sgn}(X_{(k-1)2^{-m},k2^{-m}}(0)) \times$$

$$\times E(1_{F_{m,k}} | \mathcal{F}^w_{E \cap [k2^{-m},1]}) =$$

$$= \sum_{k \in A(E)} \left(\sum_{n=0}^{\infty} \int_0^{(k-1)2^{-m}} \ldots \int_0^{u_2} g_n^E[\xi, (k-1)2^{-m}](u_1, \ldots, u_n)\, dw(u_1) \ldots dw(u_n) \right) \times$$

$$\times \left(\sum_{l=0}^{\infty} \int_{k2^{-m}}^{1} \ldots \int_{k2^{-m}}^{t_2} f_l^E[\eta, k2^{-m}](t_1, \ldots, t_l)\, dw(t_1) \ldots dw(t_l) \right) \times$$

$$\times \operatorname{sgn}(X_{(k-1)2^{-m},k2^{-m}}(0)),$$

where

$$g_n^E[x, (k-1)2^{-m}](u_1, \ldots, u_n) =$$

$$= g_n[x, (k-1)2^{-m}](u_1, \ldots, u_n) \cdot 1_E \times \ldots \times 1_E(u_1, \ldots, u_n)$$

and

$$f_l^E[y, k2^{-m}](t_1, \ldots, t_n) = f_l[y, k2^{-m}](t_1, \ldots, t_l) \cdot 1_E \times \ldots \times 1_E(t_1, \ldots, t_l).$$

We have

$$E\left(\operatorname{sgn}(X_{(k-1)2^{-m},k2^{-m}}(0)) \operatorname{sgn}(X_{(j-1)2^{-m},j2^{-m}}(0)) \mid \mathcal{F}^w_{-\infty,\infty} \right) = \delta_{k,j},$$

$$k, j = 1, \ldots, 2^m,$$

and hence,

$$E(E(f | \mathcal{F}_E)^2) = \sum_{k \in A(E)} \int_0^{\infty} \int_0^{\infty} P(\xi \in dx, \eta \in dy) \times$$

$$\times \left(\sum_{n=0}^{\infty} \int_0^{(k-1)2^{-m}} \ldots \int_0^{u_2} \{ g_n^E[x, (k-1)2^{-m}](u_1, \ldots, u_n) \}^2 du_1 \ldots du_n \right) \times$$

$$\times \left(\sum_{l=0}^{\infty} \int_{k2^{-m}}^{1} \ldots \int_{k2^{-m}}^{t_2} \{ f_l^E[y, k2^{-m}](t_1, \ldots, t_l) \}^2 dt_1 \ldots dt_l \right).$$

The joint distribution of ξ and η is well-known (cf. [11], Section 1.7): Also, by the well-known convolution property for one-sided stable laws with exponent $1/2$, the joint density $p(x, y)$ of ξ and η is given by

$$p(x, y) = \sqrt{\frac{2}{\pi}}\, \mu_{2^{-m}}(x+y) = \frac{1}{\pi}\int_{(k-1)2^{-m}}^{k2^{-m}} \mu_{k2^{-m}-u}(y)\, \mu_{u-(k-1)2^{-m}}(x)\, du.$$

Then, by recalling the definitions of $\check{m}_u$ and m_u (cf. (18) and (15)),

$$E(E(f|\mathcal{F}_E)^2) = \frac{1}{\pi}\sum_{k\in A(E)}\int_{(k-1)2^{-m}}^{k2^{-m}} \frac{1}{\sqrt{1-u}}\frac{1}{\sqrt{u}} \times$$

$$\times\, \check{m}_u(F\in C_{[0,u]};\, F\cap[0,(k-1)2^{-m}]\subset E)\, m_u(F\in C_{[u,1]};\, F\cap[k2^{-m},1]\subset E)\, du,$$

which clearly converges to

$$\frac{1}{\pi}\int_E \frac{1}{\sqrt{u(1-u)}}\, \check{m}_u(F\in C_{[0,u]};\, F\subset E)\, m_u(F\in C_{[u,1]};\, F\subset E)\, du,$$

as $m\to\infty$. This established the relation (8) for μ_f given in the theorem.

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