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MULTIVARIATE SOBEL–UPPULURI–GALAMBOS-TYPE BOUNDS

ПРО БАГАТОВИМІРНІ ГРАНИЦІ SOBEL–UPPULURI–GALAMBOS

We improve the known upper and lower bounds for the probability of the fact that exactly k_i events should occur in a group consisting of n_i events simultaneously for all $i = 1, 2, \ldots, d$.

Уточнюються відомі верхні та нижні оцінки для ймовірностей того, що відбудеться рівно k_i подій в групі з n_i подій одночасно при всіх $i = 1, 2, \ldots, d$.

The upper and lower bounds in a recent multivariate generalization (Galambos and Xu) [1] of the univariate Sobel–Uppuluri–Galambos inequalities [2, 3] are shown in the following to be weighted averages of individual multivariate bounds, and hence can be sharpened by optimizing over these individual bounds. Examples are used to illustrate the improvement and to illustrate the difference between bounds of this kind, and the kind of multivariate bounds appearing in Chen and Seneta [4] and Galambos and Lee [5]. We consider only bounds for the probability of the intersection of exactly k_i events occuring, from the ith event group of n_i events, $i = 1, \ldots, d$.

1. Introduction and main results. Suppose $A_1 \ldots A_n$ are arbitrary events in an arbitrary probability space (Ω, B, P) and υ_1 be the number of events which occur at a given sample point. For any positive integers u and r, Sobel–Uppuluri bounds refer to

$$\sum_{k=0}^{2u+1} (-1)^k \binom{k+r}{r} S_{k+r} + \frac{2u+2}{n-r} \binom{2u+r}{r} S_{2u+r+2} \le \mathrm{P}(\upsilon_1 = r) \le$$

$$\le \sum_{k=0}^{2u} (-1)^k \binom{k+r}{r} S_{k+r} - \frac{2u+1}{n-r} \binom{2u+r+1}{r} S_{2u+r+1}, \tag{1}$$

where

$$S_i = E\left(\binom{\upsilon_1}{i}\right) \quad \text{for} \quad 1 \le i \le n,$$

(see, for example, Galambos [6] or Recsei and Seneta [7]).

A bound (upper or lower) is called "a degree s bound" if $\max_{i \in I} i = s$, where I is the index set of all S_i used in the bound. A higher degree bound usually provides more accurate evaluation at the cost of more complicated computation in calculating S_i's, since higher dimensional joint probabilities are involved. The application of (1) is extensive; details can be found in sources such as the book by Galambos and Simonelli [8].

In multivariate setting, d sets of events are considered, A_{ij}, $i = 1, \ldots, n_j$ for $j = 1, \ldots, d$ respectively. Let $\mathbf{v} = (\upsilon_1, \ldots, \upsilon_d)'$ and $\mathbf{r} = (r_1, \ldots, r_d)'$. Bounds extending (1) are sought for $\mathrm{P}(\mathbf{v} = \mathbf{r})$ using

$$S_{\mathbf{k}} = E\left(\prod_{i=1}^{d} \binom{\upsilon_i}{k_i}\right) = E\left(\binom{\mathbf{v}}{\mathbf{k}}\right),$$

where $\mathbf{k} = (k_1, \ldots, k_d)'$ is a vector of non-negative integers.

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Galambos and Lee [5] proposed bounds extending (1) to a bivariate setting ($d = 2$), which were then uniformly improved by Chen and Seneta [4] as follows, where u_1 and u_2 are non-negative integers:

$$\sum_{k_1=0}^{2u_1+1} \sum_{k_2=0}^{2u_2+1} (-1)^{k_1+k_2} \binom{k_1+r_1}{r_1} \binom{k_2+r_2}{r_2} S_{k_1+r_1,\, k_2+r_2} + \Delta_1 + \Delta_2 - \Delta_3 \le$$

$$\le \mathrm{P}(\nu_1 = r_1, \nu_2 = r_2) \le$$

$$\le \sum_{k_1=0}^{2u_1} \sum_{k_2=0}^{2u_2} (-1)^{k_1+k_2} \binom{k_1+r_1}{r_1} \binom{k_2+r_2}{r_2} S_{k_1+r_1,\, k_2+r_2} - H_1 - H_2 + H_3, \tag{2}$$

where

$$\Delta_1 = \begin{cases} 0 & \text{for } n_1 = r_1; \\ \binom{r_1+2u_1+1}{r_1} \dfrac{r_1+2u_1+2}{n_1-r_1} \displaystyle\sum_{y=0}^{n_2-r_2} (-1)^y S_{r_1+2u_1+2,\, r_2+y} \binom{r_2+y}{r_2} & \text{for } n_1 > r_1, \end{cases}$$

$$\Delta_2 = \begin{cases} 0 & \text{for } n_2 = r_2; \\ \binom{r_2+2u_2+1}{r_2} \dfrac{r_2+2u_2+2}{n_2-r_2} \displaystyle\sum_{x=0}^{n_1-r_1} (-1)^x S_{r_1+x,\, r_2+2u_2+2} \binom{r_1+x}{r_1} & \text{for } n_2 > r_2, \end{cases}$$

$$\Delta_3 = \begin{cases} 0 & \text{for } n_1 = r_1 \quad \text{or} \quad n_2 = r_2; \\ \binom{r_1+2u_1+1}{r_1} \binom{r_2+2u_2+1}{r_2} \dfrac{r_1+2u_1+2}{2u_1+2} \dfrac{r_2+2u_2+2}{2u_2+2} S_{r_1+2u_1+2,\, r_2+2u_2+2} & \\ & \text{for } n_1 > r_1 \quad \text{and} \quad n_2 > r_2, \end{cases}$$

$$H_1 = \begin{cases} 0 & \text{for } n_1 = r_1; \\ \binom{r_1+2u_1}{r_1} \dfrac{r_1+2u_1+1}{n_1-r_1} \displaystyle\sum_{y=0}^{2u_2} (-1)^y S_{r_1+2u_1+1,\, r_2+y} \binom{2u_2+y}{2u_2} & \text{for } n_1 > r_1, \end{cases}$$

$$H_2 = \begin{cases} 0 & \text{for } n_2 = r_2; \\ \binom{r_2+2u_2}{r_2} \dfrac{r_2+2u_2+1}{n_2-r_2} \displaystyle\sum_{x=0}^{2u_1} (-1)^x S_{r_1+x,\, r_2+2u_2+1} \binom{2u_1+x}{2u_1} & \text{for } n_2 > r_2, \end{cases}$$

$$H_3 = \begin{cases} 0 & \text{for } n_1 = r_1 \quad \text{or} \quad n_2 = r_2; \\ \binom{r_1+2u_1}{r_1} \binom{r_2+2u_2}{r_2} \dfrac{r_1+2u_1+1}{n_1-r_1} \dfrac{r_2+2u_2+1}{n_2-r_2} S_{r_1+2u_1+1,\, r_2+2u_2+1} & \\ & \text{for } n_1 > r_1 \quad \text{and} \quad n_2 > r_2, \end{cases}$$

The nature of this extension is to use the quantity

$$\sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{i+r_1}{r_1} \binom{j+r_2}{r_2} S_{i+r_1,\, j+r_2}$$

as the generic term for inclusion and exclusion, with upper bounds (lower bounds) being obtained by setting a and b both odd (even), and then to bound the difference from $\mathrm{P}(\nu_1 = r_1, \nu_2 = r_2)$. The central idea is the application of Meyer's identity

$$S_{kt} = E\left(\binom{\nu_1}{k} \binom{\nu_2}{t} \right) = \sum_{i=k}^{n_1} \sum_{j=t}^{n_2} \binom{\nu_1}{k} \binom{\nu_2}{t} \mathrm{P}(\nu_1 = i, \nu_2 = j),$$

in conjunction with lemmas on combinatorial coefficients via the method of indicators. This direction of generalization in theory separately involves all the degrees for event sets $\{A_{11}, \ldots, A_{n_1 1}\}$ and $\{A_{12}, \ldots, A_{n_2 2}\}$ from 0 to $a + r_1$ and 0 to $b + r_2$ respectively.

This involvement of individual bivariate degrees differs from that in multivariate inequalities where degree p is that figuring in the quantity

$$d_p = \sum_{|\mathbf{k}|=p} \binom{\mathbf{k}}{\mathbf{r}} S_{\mathbf{k}},$$

with $|\mathbf{k}| = \sum_{i=1}^{d} k_i$, $\mathbf{k} = (k_1, \ldots, k_d)'$ and

$$\binom{\mathbf{k}}{\mathbf{r}} = \prod_{i=1}^{d} \binom{k_i}{r_i}, \quad S_{\mathbf{k}} = S_{k_1, \ldots, k_d},$$

in the sense that p-dimensional joint probabilities are calculated. For example, the multivariate identity stated in Meyer [9] reads

$$\mathrm{P}(\mathbf{v} = \mathbf{r}) = \sum_{i=|\mathbf{r}|}^{|\mathbf{n}|} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}}{\mathbf{r}} S_{\mathbf{k}} \tag{3}$$

and its inverse form

$$S_{\mathbf{k}} = \sum_{p=|\mathbf{k}|}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} \binom{\mathbf{s}}{\mathbf{k}} \mathrm{P}(\mathbf{v} = \mathbf{s}), \tag{4}$$

and so (3) is of degree $|\mathbf{n}|$. Thus, bounds generalizing (1) by using multivariate degree p quantities d_p is a perhaps more natural way of extension, although each direction of generalization has its own advantage under different circumstances. We shall illustrate by constructing numerical examples in Section 3.

The manner of generalizing univariate inequalities using d_p's first appeared in [9] where univariate Bonferroni bounds were extended to multivariate. Recently, by using the method of polynomials, Galambos and Xu [1] put forward the bounds:

$$\sum_{i=0}^{2t+1} (-1)^i \sum_{|k|=i} \binom{k+r}{r} S_{k+r} +$$

$$+ \frac{2t+2}{|n|-|r|} \sum_{|k|=2t} \binom{k+r}{r} S_{k+r} \le \mathrm{P}(\mathbf{v} = \mathbf{r}) \le$$

$$\le \sum_{i=0}^{2t} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} - \frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} \sum_{|\mathbf{k}|=2t+1} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}}, \tag{5}$$

aiming at extension of univariate Sobel–Uppuluri–Galambos bounds. They assert that their result is analogous to the way Galambos [6] extended Bonferroni's classical univariate bounds; but that univariate extension in fact did not come from the method of polynomials. The dissonance between the method used by Galambos and Xu [1] and the result achieved prompted our investigation, and we obtained by extending

Galambos' [6] original methodology, the following two theorems.

Theorem 1. *For any integer* $t \geq 0$ *with* $2t+1 \leq |\mathbf{n}| - |\mathbf{r}|$,

$$\sum_{i=0}^{2t+1} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} + \max_i A_i(2t+1) \leq \mathrm{P}(\mathbf{v}=\mathbf{r}) \leq$$

$$\leq \sum_{i=0}^{2t} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} - \max_i A_i(2t), \qquad (6)$$

where

$$A_i(x) = \sum_{|\mathbf{k}|=x} S_{k_1+r_1,\ldots,k_i+r_i+1\ldots k_d+r_d} \binom{k_1+r_1}{r_1} \cdots \binom{k_i+r_i+1}{r_i} \cdots \binom{k_d+r_d}{r_d} \frac{k_i+1}{n_i-r_i},$$

$i = 1, \ldots, d$. *Note that* $A_i(x)$ *has degree* $|\mathbf{r}| + x + 1$.

Theorem 2. *Inequality (5) can be written*

$$\sum_{i=0}^{2t+1} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} + \sum_{i=1}^{d} \frac{n_i - r_i}{|\mathbf{n}| - |\mathbf{r}|} A_i(2t+1) \leq \mathrm{P}(\mathbf{v}=\mathbf{r}) \leq$$

$$\leq \sum_{i=0}^{2t} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} - \sum_{i=1}^{d} \frac{n_i - r_i}{|\mathbf{n}| - |\mathbf{r}|} A_i(2t),$$

so (6) *is sharper than* (5).

Remark. Theorem 2 shows that Galambos and Xu's ([1], Theorem 1) can be improved uniformly. The nature of improvement, from averaging to optimization, is similar in sense to the way in which Hunter's inequality improves certain univariate inequalities [10].

2. Proof of theorems.

Lemma 1. *For any integer* $T \geq 0$ *and non-negative integers* $x_1, \ldots, x_d$,

$$\sum_{i=0}^{T} \sum_{|\mathbf{k}|=i} (-1)^i \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} = (-1)^T \sum_{|\mathbf{k}|=T} \binom{x_1-1}{k_1} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d}. \qquad (7)$$

Since all (x_i, k_i), $i = 1, \ldots, d$, are interchangeable on the left, the right hand side can be rewritten in d ways.

Proof of Lemma 1. We use proof by induction.

When $T = 0$, left $= 1 =$ right.

When $T = 1$, the left becomes $1 - x_1 - x_2 - \ldots - x_d$ while the right equals $(-1)(x_1 - 1 + x_2 + \ldots + x_d)$ which is the same as the left.

Supposing that (5) is true for any integer $T \geq 0$, in the case of $T+1$ we have

$$\sum_{i=0}^{T+1} \sum_{|\mathbf{k}|=i} (-1)^i \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} =$$

$$= \sum_{i=0}^{T} \sum_{|\mathbf{k}|=i} (-1)^i \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} + (-1)^{T+1} \sum_{|\mathbf{k}|=T+1} \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} =$$

$$= (-1)^T \sum_{|\mathbf{k}|=T} \binom{x_1-1}{k_1} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d} + (-1)^{T+1} \sum_{|\mathbf{k}|=T+1} \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} =$$

$$= (-1)^{T+1} \left\{ - \sum_{|\mathbf{k}|=T} \binom{x_1-1}{k_1} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d} + \sum_{|\mathbf{k}|=T+1} \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} \right\} =$$

$$= (-1)^{T+1}\left\{\sum_{|\mathbf{k}|=T}\binom{x_1}{k_1+1}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d} - \sum_{|\mathbf{k}|=T}\binom{x_1-1}{k_1}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d} + \right.$$

$$\left. + \sum_{|\mathbf{k}|=T+1}\binom{x_1}{0}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d}\right\} =$$

$$= (-1)^{T+1}\left\{\sum_{|\mathbf{k}|=T}\binom{x_1-1}{k_1+1}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d} + \sum_{|\mathbf{k}|=T+1}\binom{x_1}{0}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d}\right\},$$

by putting the first two summation together and using

$$\binom{x_1}{k_1+1} - \binom{x_1-1}{k_1} = \binom{x_1-1}{k_1+1}.$$

Thus

$$\sum_{i=0}^{T+1}\sum_{|\mathbf{k}|=i}(-1)^i\binom{x_1}{k_1}\cdots\binom{x_d}{k_d} = (-1)^{T+1}\sum_{|\mathbf{k}|=T+1}\binom{x_1-1}{k_1}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d},$$

since

$$\sum_{|\mathbf{k}|=T}\binom{x_1-1}{k_1+1}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d} + \sum_{|\mathbf{k}|=T+1}\binom{x_1}{0}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d} =$$

$$= \sum_{|\mathbf{k}|=T+1}\binom{x_1-1}{k_1}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d}.$$

***Proof of Theorem* 1.** Let a be any non-negative integer and write $\mathrm{P}(\mathbf{s}) = \mathrm{P}(\mathbf{v} = \mathbf{s}) = \mathrm{P}(v_1 = s_1, \ldots, v_d = s_d)$. Then

$$\sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}S_{\mathbf{k}+\mathbf{r}} = \sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}\sum_{p=i+|\mathbf{r}|}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}\binom{\mathbf{s}}{\mathbf{k}+\mathbf{r}}P(\mathbf{s}),$$

by (4), nothing that $|k| = i$. Thus

$$\sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}S_{\mathbf{k}+\mathbf{r}} = \sum_{p=|\mathbf{r}|}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}\mathrm{P}(\mathbf{s})\sum_{i=0}^{J}\sum_{|\mathbf{k}|=i}(-1)^i\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}\binom{\mathbf{s}}{\mathbf{k}+\mathbf{r}},$$

where

$$J = \min(a, |\mathbf{n}|-|\mathbf{r}|) = \sum_{p=|\mathbf{r}|}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}\mathrm{P}(\mathbf{s})\binom{\mathbf{s}}{\mathbf{r}}\sum_{i=0}^{J}\sum_{|\mathbf{k}|=i}(-1)^i\binom{\mathbf{s}-\mathbf{r}}{\mathbf{k}},$$

this is because

$$\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}\binom{\mathbf{s}}{\mathbf{k}+\mathbf{r}} = \binom{\mathbf{s}}{\mathbf{r}}\binom{\mathbf{s}-\mathbf{r}}{\mathbf{k}}, \tag{8}$$

so

$$\sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}S_{\mathbf{k}+\mathbf{r}} = \mathrm{P}(\mathbf{r}) + \sum_{p=|\mathbf{r}|+1}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}\mathrm{P}(\mathbf{s})\binom{\mathbf{s}}{\mathbf{r}}\sum_{i=0}^{J}\sum_{|\mathbf{k}|=i}(-1)^i\binom{\mathbf{s}-\mathbf{r}}{\mathbf{k}} =$$

$$= \mathrm{P}(\mathbf{r}) + \sum_{p=|\mathbf{r}|+1}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}\mathrm{P}(\mathbf{s})\binom{\mathbf{s}}{\mathbf{r}}(-1)^J\sum_{|\mathbf{k}|=J}\binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\cdots\binom{s_d-r_d}{k_d},$$

by Lemma 1. Then

$$\sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}S_{\mathbf{k}+\mathbf{r}} =$$

$$= \begin{cases} \mathrm{P}(\mathbf{r}) + \sum\limits_{p=|\mathbf{r}|+1}^{|\mathbf{n}|}\sum\limits_{|\mathbf{s}|=p}\mathrm{P}(\mathbf{s})\binom{\mathbf{s}}{\mathbf{r}}(-1)^a \times \\ \quad \times\sum\limits_{|\mathbf{k}|=a}\binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\cdots\binom{s_d-r_d}{k_d} \quad \text{for} \quad a\le|\mathbf{n}|-|\mathbf{r}|; \\ \mathrm{P}(\mathbf{r}) + (-1)^a\sum\limits_{|\mathbf{k}|=a}\sum\limits_{p=|\mathbf{r}|+1}^{|\mathbf{n}|} \times \\ \quad \times\sum\limits_{|\mathbf{s}|=p}\mathrm{P}(\mathbf{s})\binom{\mathbf{s}}{\mathbf{r}}\binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\cdots\binom{s_d-r_d}{k_d} \quad \text{otherwise.} \end{cases}$$

Thus we have

$$\sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}S_{\mathbf{k}+\mathbf{r}} = \mathrm{P}(\mathbf{r}) + (-1)^a B_1(a), \tag{9}$$

where

$$B_1(a) = \sum_{|\mathbf{k}|=a}\sum_{p=|\mathbf{r}|+a+1}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}\mathrm{P}(\mathbf{s})\binom{\mathbf{s}}{\mathbf{r}}\binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\cdots\binom{s_d-r_d}{k_d},$$

since only when $p\ge|\mathbf{r}|+a+1$ can we obtain

$$\binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\cdots\binom{s_d-r_d}{k_d} \neq 0.$$

With (9), we are ready to prove Theorem 1. To show the upper bounds of the theorem, we set $a=2t$, which makes (9) become

$$\sum_{i=0}^{2t}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}S_{\mathbf{k}+\mathbf{r}} = \mathrm{P}(\mathbf{r}) + B_1(2t).$$

We shall first show $A_1(2t)\le B_1(2t)$, which can be seen by recalling that

$$A_1(2t) = \sum_{|\mathbf{k}|=2t}S_{k_1+r_1+1,\,k_2+r_2,\,\ldots,\,k_d+r_d}\binom{k_1+r_1+1}{r_1}\binom{k_2+r_2}{r_2}\cdots\binom{k_d+r_d}{r_d}\frac{k_1+1}{n_1-r_1} =$$

$$= \sum_{|\mathbf{k}|=2t}\sum_{p=|\mathbf{k}|+|\mathbf{r}|+1}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}\mathrm{P}(\mathbf{s})\binom{s_1}{k_1+r_1+1}\binom{s_2}{k_2+r_2}\cdots\binom{s_d}{k_d+r_d}\times$$

$$\times\binom{k_1+r_1+1}{r_1}\binom{k_2+r_2}{r_2}\cdots\binom{k_d+r_d}{r_d}\frac{k_1+1}{n_1-r_1}.$$

This is because by (4)

$$S_{k_1+r_1+1,\,k_2+r_2,\,\ldots,\,k_d+r_d} = \sum_{p=|\mathbf{k}|+|\mathbf{r}|+1}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} \mathrm{P}(\mathbf{s}) \binom{s_1}{k_1+r_1+1}\binom{s_2}{k_2+r_2}\ldots\binom{s_d}{k_d+r_d}.$$

On the other hand,

$$B_1(2t) =$$

$$= \sum_{|\mathbf{k}|=2t} \sum_{p=|\mathbf{r}|+2t+1}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} \mathrm{P}(\mathbf{s}) \binom{s_1}{r_1}\ldots\binom{s_d}{r_d}\binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\ldots\binom{s_d-r_d}{k_d}.$$

Comparing

$$\binom{s_1}{k_1+r_1+1}\binom{k_1+r_1+1}{r_1} \quad \text{vs} \quad \binom{s_1}{r_1}\binom{s_1-r_1-1}{k_1}\frac{n_1-r_1}{k_1+1}$$

will find by expanding the left-hand expression that

$$\binom{s_1}{k_1+r_1+1}\binom{k_1+r_1+1}{r_1} \le \frac{n_1-r_1}{k_1+1}\binom{s_1}{r_1}\binom{s_1-r_1-1}{k_1}. \tag{10}$$

Further, as in (8):

$$\binom{s_i}{k_i+r_i}\binom{k_i+r_i}{r_i} = \binom{s_i}{r_i}\binom{s_i-r_i}{k_i} \quad \text{for} \quad i=2,\,\ldots,\,d. \tag{11}$$

Thus, for each $\mathrm{P}(\mathbf{s})$ in $A_1(2t)$ and $B_1(2t)$, we can compare pairs of associated coefficients in $A_1(2t)$ and $B_1(2t)$ individually, by (10) and (11), and the immediate conclusion is that $A_1(2t) \le B_1(2t)$. Noting that x_1 can be replaced by any one of $x_2, \ldots, x_d$ establishes the upper bounds in Theorem 1, for $2t \le |\mathbf{n}| - |\mathbf{r}|$.

Taking $a = 2t+1$ in (9) yields

$$\sum_{i=0}^{2t+1} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}} S_{\mathbf{k}+\mathbf{r}} = \mathrm{P}(\mathbf{r}) - B_1(2t+1).$$

To establish the lower bounds we need first to show

$$\sum_{i=0}^{2t+1} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}} S_{\mathbf{k}+\mathbf{r}} \le \mathrm{P}(\mathbf{r}) - A_1(2t+1).$$

This can be achieved if $A_1(2t+1) \le B_1(2t+1)$, which follows from applying (10) and (11) to the comparison between $A_1(2t+1)$ and $B_1(2t+1)$ term by pairwise term. Considering the fact that x_1 can be replaced by any one of $x_2, \ldots, x_d$ completes the proof of Theorem 1.

The following well-known result is necessary to establish Theorem 2.

Lemma 2. *For positive numbers $a_1, \ldots, a_d$ with positive weights $c_1, \ldots, c_n$ respectively, such that $\sum_{i=1}^{n} c_i = 1$,*

$$\max_{1 \le i \le n} a_i \le c_1 a_1 + \ldots + c_n a_n,$$

and equality can be attained only when $a_i = \mathrm{const}$ for all $i = 1, \ldots, n$.

Proof of Theorem 2. For notational convenience, we state the proof of Theorem 2 in the bivariate setting. The proof for the multivariate setting follows analogously.

In the bivariate case, Theorem 1 yields

$$\sum_{i=0}^{2t+1}(-1)^i\sum_{k_1+k_2=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}S_{\mathbf{k}+\mathbf{r}}+A_1(2t+1)\le \mathrm{P}(\mathbf{v}=\mathbf{r})\le$$
$$\le\sum_{i=0}^{2t}(-1)^i\sum_{k_1+k_2=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}S_{\mathbf{k}+\mathbf{r}}-A_1(2t), \tag{12}$$

and

$$\sum_{i=0}^{2t+1}(-1)^i\sum_{k_1+k_2=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}S_{\mathbf{k}+\mathbf{r}}+A_2(2t+1)\le \mathrm{P}(\mathbf{v}=\mathbf{r})\le$$
$$\le\sum_{i=0}^{2t}(-1)^i\sum_{k_1+k_2=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}S_{\mathbf{k}+\mathbf{r}}-A_2(2t), \tag{13}$$

where

$$A_1(2t)=\sum_{k_1+k_2=2t}S_{k_1+r_1+1,\,k_2+r_2}\binom{k_1+r_1+1}{r_1}\binom{k_2+r_2}{r_2}\frac{k_1+1}{n_1-r_1}$$

and

$$A_2(2t)=\sum_{k_1+k_2=2t}S_{k_1+r_1,\,k_2+r_2+1}\binom{k_1+r_1}{r_1}\binom{k_2+r_2+1}{r_2}\frac{k_2+1}{n_2-r_2}.$$

Multiplying (12) by $\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|}$, (13) by $\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|}$ respectively and adding them yields

$$\sum_{i=0}^{2t+1}(-1)^i\sum_{k_1+k_2=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}S_{\mathbf{k}+\mathbf{r}}+A_1(2t+1)\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|}+A_2(2t+1)\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|}\le$$
$$\le \mathrm{P}(\mathbf{v}=\mathbf{r})\le\sum_{i=0}^{2t}(-1)^i\sum_{k_1+k_2=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}S_{\mathbf{k}+\mathbf{r}}-A_1(2t)\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|}-A_2(2t)\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|}. \tag{14}$$

Comparing (14) with (5), if we prove

$$A_1(2t)\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|}+A_2(2t)\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|}=\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|}\sum_{|\mathbf{k}|=2t+1}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}S_{\mathbf{k}+\mathbf{r}}, \tag{15}$$

we can then claim that the upper bound given in (5) is a weighted average of d upper bounds in Theorem 1, by Lemma 2, we can have the upper bounds of Theorem 2. The proof for the sharperness of lower bounds in Theorem 2 follows similarily, thus the key here is to prove equation (15). To see (15), note that

$$A_1(2t)\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|}+A_2(2t)\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|}=$$
$$=\sum_{k_1+k_2=2t}S_{k_1+r_1+1,\,k_2+r_2}\binom{k_1+r_1+1}{r_1}\binom{k_2+r_2}{r_2}\frac{k_1+1}{|\mathbf{n}|-|\mathbf{r}|}+$$
$$+\sum_{k_1+k_2=2t}S_{k_1+r_1,\,k_2+r_2+1}\binom{k_1+r_1}{r_1}\binom{k_2+r_2+1}{r_2}\frac{k_2+1}{|\mathbf{n}|-|\mathbf{r}|}=$$
$$=\sum_{k_1=0}^{2t-1}S_{k_1+r_1+1,\,2t-k_2+r_2}\binom{k_1+r_1+1}{r_1}\binom{2t-k_1+r_2}{r_2}\frac{k_1+1}{|\mathbf{n}|-|\mathbf{r}|}+$$

$$+ S_{r_1+2t+1,\, r_2}\binom{2t+r_1+1}{r_1}\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} + S_{r_1,\, 2t+r_2+1}\binom{2t+r_2+1}{r_2}\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} +$$

$$+ \sum_{k_1=1}^{2t} S_{k_1+r_1,\, 2t-k_1+r_2+1}\binom{k_1+r_1}{r_1}\binom{2t-k_1+r_2+1}{r_2}\frac{2t-k_1+1}{|\mathbf{n}|-|\mathbf{r}|}.$$

Setting $k_1+1=j$ in the first summations yields

$$A_1(2t)\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|} + A_2(2t)\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|} =$$

$$= \sum_{j=1}^{2t} S_{j+r_1,\, 2t-j+r_2+1}\binom{j+r_1}{r_1}\binom{2t-j+r_2+1}{r_2}\frac{j}{|\mathbf{n}|-|\mathbf{r}|} +$$

$$+ S_{r_1+2t+1,\, r_2}\binom{2t+r_1+1}{r_1}\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} + S_{r_1,\, 2t+r_2+1}\binom{2t+r_2+1}{r_2}\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} +$$

$$+ \sum_{k_1=1}^{2t} S_{k_1+r_1,\, 2t-k_1+r_2+1}\binom{k_1+r_1}{r_1}\binom{2t-k_1+r_2+1}{r_2}\frac{2t-k_1+1}{|\mathbf{n}|-|\mathbf{r}|} =$$

$$= S_{r_1+2t+1,\, r_2}\binom{2t+r_1+1}{r_1}\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} + S_{r_1,\, 2t+r_2+1}\binom{2t+r_2+1}{r_2}\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} +$$

$$+ \sum_{k_1=1}^{2t} S_{k_1+r_1,\, 2t-k_1+r_2+1}\binom{k_1+r_1}{r_1}\binom{2t-k_1+r_2+1}{r_2}\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} =$$

$$= \frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|}\sum_{|\mathbf{k}|=2t+1}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}}.$$

3. Numerical examples. In this Section, we shall present two examples. Example 1 shows that in some cases (5) does not give a useful result while (6) can still be very effective. Example 2, in conjunction with Example 1, clarities the relationship between the two directions of generalization as mentioned in Section 1. We see from the examples that in some cases, bounds (2) based on the direction of Galambos and Lee [5] and then Chen and Seneta [4] work better, while in some other cases, the bounds from (6) are sharper.

***Example* 1.** Consider A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 with non-zero probabilities of elementary conjunctions assigned as follows

$$\mathrm{P}\left(A_1 A_2 A_3^c A_4 B_1^c B_2^c B_3^c B_4\right) = 0{,}18; \qquad \mathrm{P}\left(A_1 A_2 A_3 A_4^c B_1^c B_2 B_3^c B_4^c\right) = 0{,}12;$$

$$\mathrm{P}\left(A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4^c\right) = 0{,}31; \qquad \mathrm{P}\left(A_1 A_2 A_3 A_4 B_1 B_2^c B_3 B_4\right) = 0{,}19;$$

$$\mathrm{P}\left(A_1 A_2 A_3 A_4^c B_1 B_2 B_3 B_4\right) = 0{,}01; \qquad \mathrm{P}\left(A_1^c A_2^c A_3^c A_4^c B_1^c B_2^c B_3^c B_4^c\right) = 0{,}19.$$

Hence we have the corresponding non-zero probabilities of the number of exactly $A's$ $B's$ which occur as

$$\mathrm{P}(\upsilon_A = 3, \upsilon_B = 1) = 0{,}3; \qquad \mathrm{P}(\upsilon_A = 3, \upsilon_B = 4) = 0{,}01;$$

$$\mathrm{P}(\upsilon_A = 4, \upsilon_B = 3) = 0{,}50; \qquad \mathrm{P}(\upsilon_A = 0, \upsilon_B = 0) = 0{,}19,$$

and then the associated Bonferroni summations

$$S_{31} = 6{,}34; \quad S_{32} = 6{,}06; \quad S_{33} = 2{,}04; \quad S_{34} = 0{,}01;$$

$$S_{41} = 1{,}5; \quad S_{42} = 1{,}5; \quad S_{43} = 0{,}5; \quad S_{44} = 0.$$

Example 2. Consider A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 with non-zero probabilities of elementary conjunctions specified as follows

$$\mathrm{P}\left(A_1 A_2 A_3 A_4^c B_1 B_1 B_3^c B_4^c\right) = 0{,}1; \quad \mathrm{P}\left(A_1^c A_2 A_3 A_4 B_1^c B_2^c B_3 B_4\right) = 0{,}1;$$

$$\mathrm{P}\left(A_1 A_2^c A_3 A_4 B_1^c B_2 B_3 B_4^c\right) = 0{,}1; \quad \mathrm{P}\left(A_1 A_2 A_3^c A_4 B_1 B_2 B_3^c B_4^c\right) = 0{,}1;$$

$$\mathrm{P}\left(A_1 A_2 A_3^c A_4 B_1 B_2^c B_3^c B_4^c\right) = 0{,}2; \quad \mathrm{P}\left(A_1^c A_2^c A_3^c A_4^c B_1^c B_2^c B_3^c B_4^c\right) = 0{,}1;$$

$$\mathrm{P}\left(A_1 A_2 A_3 A_4^c B_1^c B_2^c B_3^c B_4\right) = 0{,}3.$$

Such an assignment gives the corresponding non-zero probabilities of the number of exactly $A's$ $B's$ which occur as

$$\mathrm{P}(\nu_A = 3, \nu_B = 2) = 0{,}4; \quad \mathrm{P}(\nu_A = 3, \nu_B = 1) = 0{,}5; \quad \mathrm{P}(\nu_A = 0, \nu_B = 0) = 0{,}1;$$

and then corresponding Bonferroni summations can be calculated as

$$S_{31} = 1{,}3; \quad S_{32} = 0{,}4; \quad S_{33} = 0; \quad S_{34} = 0;$$

$$S_{41} = 0; \quad S_{42} = 0; \quad S_{43} = 0; \quad S_{44} = 0.$$

We first compare the upper bounds in (5) and (6).

In Example 1, $d = 2$, $n_1 = n_2 = 4$ and setting $r_1 = 3$, $r_2 = 1$ and $t = 1$ in Theorem 1 yields as the upper bounds entering into (6), using $A_i(\cdot)$ at each of $i = 1$ and 2 respectively:

$$\mathrm{P}(\nu_A = 3, \nu_B = 1) \le S_{31} - (2S_{32} + 4S_{41}) + (3S_{33} + 8S_{42}) - 12S_{43} = 0{,}34$$

and

$$\mathrm{P}(\nu_A = 3, \nu_B = 1) \le S_{31} - (2S_{32} + 4S_{41}) + (3S_{33} + 8S_{42}) - (4S_{34} + 8S_{43}) = 2{,}3.$$

Thus, the upper bound from Theorem 1 reads

$$\mathrm{P}(\nu_A = 3, \nu_B = 1) \le \min(0{,}34; 2{,}3) = 0{,}34$$

while the upper bound of (5) reads

$$\mathrm{P}(\nu_A = 3, \nu_B = 1) \le S_{31} - (2S_{32} + 4S_{41}) + (3S_{33} + 8S_{42}) - (3S_{34} + 9S_{43}) = 1{,}81.$$

Thus (5) gives a trivial upper bound (greater than 1). The actual value is $\mathrm{P}(\nu_A = 3, \nu_B = 1) = 0{,}30$. (15) in this example reads:

$$0{,}34 \times \frac{4-3}{4+4-3-1} + 2{,}3 \times \frac{4-1}{4+4-3-1} = 1{,}81.$$

Both (5) and (6) give the lower bound 0,3.

Next we shall use Example 1 to thow that (2) can be sharper than (6), and then use Example 2 to show that (6) can be sharper than (2).

In Example 1, we know that (6) gives a upper bound of 0,34 which is very effective in estimating the exact value of 0,3. However, (2) does even better. Letting $u_1 = 1$, $u_2 = 0$, $n_1 = n_2 = 4$, $r_1 = 3$, $r_2 = 1$ the upper bound in (2) yields

$$\mathrm{P}(\nu_A = 3, \nu_B = 1) \le \sum_{k=0}^{2} \sum_{t=0}^{0} (-1)^{k+t} \binom{k+3}{3} \binom{t+1}{1} S_{k+3,\, t+1} - H_2 \le$$

$$\le S_{31} - 4S_{41} - \frac{2}{3}(S_{32} - 4_{42}).$$

Applying the values of S_{ij}'s in Example 1 yields

$$P(\nu_A = 3, \nu_B = 1) \leq 0{,}3$$

which exactly hits the value to be evaluated. However, the upper bound in (2) is not always better than (6) as demonstrated by Example 2, where (2) gives the bound

$$P(\nu_A = 3, \nu_B = 1) \leq 1{,}3 - \frac{2}{3}\times 0{,}4 = 1{,}03$$

which is a trivial upper bound. On the other hand, (6) provides an upper bound

$$P(\nu_A = 3, \nu_B = 1) \leq 1{,}3 - (2\times 0{,}4 + 4\times 0) = 0{,}5$$

which is the exact value to be estimated. Both (2) and (6) give the lower bound 0,5.

Commemorative note. As a graduate student at the Australian National University, Canberra, in 1966 I (E.S.) received as a gift from David Vere–Jones a copy of A.V. Skorokhod's (1964) *Случайные процессы с независимыми приращениями,* and soon after received from my wife's parents as a gift for my 25 th birthday Gikhman and Skorokhod's (1965) *Введение в теорию случайных процессов.* I was pleased and proud to see in these books references in my first language, Ukrainian, since A. V.'s name was becoming very well known through the Skorokhod topology. Eventually I was able to acquire his Ukrainian-language textbooks of 1975 and 1990 and was able to use the former: *Елементи теорії ймовірностей та випадкових процесів* to help in the construction of my own lectures. I hope the preceding paper in a small sub-area of probability reflects my admiration, and that of my former student John Chen, for A. V's contributions to this field and to Ukrainian mathematics.

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