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ON RANDOMLY PERTURBED LINEAR OSCILLATING MECHANICAL SYSTEMS

ПРО ВИПАДКОВО ЗБУРЕНІ ЛИНІЙНІ КОЛИВНІ МЕХАНІЧНІ СИСТЕМИ

We prove that the amplitudes and the phases of eigen oscillations of a linear oscillating system perturbed by either a fast Markov process or a small Wiener process can be described asymptotically as a diffusion process whose generator is calculated.

Доведено, що амплітуди і фази власних коливань лінійної коливної системи, збуреної або „швидким" процесом Маркова, або ж малим вінеровим процесом, можна асимптотично описати дифузійним процесом, генератор якого обчислюється.

1. Introduction. A linear oscillating system in R^m is a system with the potential energy of the form

$$U(x) = \frac{1}{2}(\Lambda x, x), \qquad x \in R^m, \tag{1}$$

where Λ is a non-negative symmetric matrix. The kinetic energy of the system is

$$T(v) = \frac{1}{2}(v, v), \qquad v \in R^m. \tag{2}$$

The motion of the system is determined by the system of differential equations

$$\frac{d}{dt}x = v, \qquad \frac{d}{dt}v = -\Lambda x. \tag{3}$$

Let $\{e_1, \ldots, e_m\}$ be the basis formed by eigenvectors of the matrix Λ. Set

$$x_k = (x, e_k), \qquad \lambda_k^2 = (\Lambda e_k, e_k), \qquad k = 1, \ldots, m.$$

Then the system (2) can be rewritten in the form

$$\frac{d^2}{dt^2}x_k(t) + \lambda_k^2 x_k(t) = 0, \qquad k = 1, \ldots, m. \tag{4}$$

So

$$x_k(t) = a_k \sin \lambda_k (t + \varphi_k), \qquad v_k(t) = \lambda_k a_k \cos \lambda_k (t + \varphi_k), \qquad k = 1, \ldots, m, \tag{5}$$

where a_k, φ_k, $k = 1, \ldots, m$, are determined by initial values $x(0)$, $v(0)$. The functions represented by formulas (5) are called the eigen oscillations of the system.

A randomly perturbed linear oscillating system is defined as the solution to the system of differential equations

$$\frac{d}{dt}x_\varepsilon(t) = v_\varepsilon(t), \qquad \frac{d}{dt}v_\varepsilon(t) = -\Lambda x_\varepsilon(t) + F(\varepsilon, t, x_\varepsilon(t), v_\varepsilon(t), \omega), \tag{6}$$

where

$$F\colon R_+ \times R_+ \times R^m \times R^m \times \Omega \to R^m.$$

We assume that random perturbations are defined on a probability space $\{\Omega, \mathcal{F}, P\}$ and that

$$\lim_{\varepsilon \to 0} \int_0^t F(\varepsilon, s, x, v, \omega)\, dt = 0$$

and probability for all $t > 0$. We consider two particular cases.

A. ***Fast Markov perturbation.*** We assume that

$$F(\varepsilon, t, x, v, \omega) = f\left(x, v, y\left(\frac{t}{\varepsilon}\right), \omega\right),$$

where $f\colon R^m \times R^m \times Y \to R^m$ and (Y, C) is a measurable space, $y(t, \omega)$ is a homogeneous Markov process in (Y, C), this process is ergodic with an ergodic distribution $\rho(dy)$, satisfying the following strong mixing condition:

$$\sup_y \int_0^\infty \operatorname{var}(P(t, y, \cdot) - \rho(\cdot))\, dt < \infty,$$

where $P(t, y, \cdot)$ is the transition probability of the Markov process, and $\operatorname{var}(\cdot)$ is a variation of the signed measure under consideration. We suppose that the function $f(x, v, y)$ is bounded, measurable in y, twice differentiable in x, v with bounded derivatives, and the relation

$$\int f(x, v, y)\, \rho(dy) = 0, \quad x \in R^m, \quad v \in R^m, \tag{7}$$

is fulfilled.

B. ***Small Wiener perturbation.*** We assume that

$$F(\varepsilon, t, x, v, \omega) = \sqrt{\varepsilon}\, F(x, v) \frac{d}{dt} w(t),$$

here $F(x, v)$ is a twice differentiable $L(R^m)$-valued function which is bounded with its derivatives, and $w(t)$ is the Wiener process in R^m. In this case the second equation of system (6) should be rewritten as a stochastic differential equation.

Differential equations with random functions containing a small parameter were studied first by R. Z. Khasminskii [1 – 3]. The problems considered in the article are related to diffusion approximation for randomly perturbed differential equations. Under various conditions the problems of such a kind were studied by R. Z. Khasminskii [3], G. C. Papanicolaou, D. Stroock, and S. R. S. Varadhan [4], A. V. Skorokhod [5], M. I. Freidlin and A. D. Wentzell [6].

2. Asymptotic properties of unperturbed systems. We need results concerning the behaviour of averaged values of functions of phase variables along the trajectories of the system. Let $x(t)$, $v(t)$ be a solution to system (3). For a function $\Phi \in C(R^m \times R^m)$ denote:

$$A_T(\Phi; x(0), v(0)) = \frac{1}{T} \int_0^T \Phi(x(t), v(t))\, dt. \tag{8}$$

Theorem 1. *A limit exists*

$$\lim_{T \to \infty} A_T(\Phi; x(0), v(0)) = A(\Phi; x(0), v(0)), \tag{9}$$

where the function $A\colon C(R^m \times R^m) \times R^m \times R^m \to R$ *is a non-negative linear function in* Φ, *and it is determined by the relation*

$$A(\Phi; (a_1\cos\theta_1; \ldots; a_m\cos\theta_m), (-\lambda_1 a_1\sin\theta_1; \ldots; -\lambda_m a_m \sin\theta_m)) =$$

$$= \frac{\delta_1\ldots\delta_r}{(2\pi)^r}\int_0^{2\pi\delta_1^{-1}}\ldots\int_0^{2\pi\delta_r^{-1}} \Phi(X(\delta_1, s_1, \ldots, \delta_r, s_r), V(\delta_1, s_1, \ldots, \delta_r, s_r))\, ds_1\ldots ds_r, \tag{10}$$

where the vectors X, Y *are determined by their coordinates:*

$$X_k(\delta_1, s_1, \ldots, \delta_r, s_r) = a_k\cos\left(\sum_{j=1}^{r} n_{kj}\delta_j s_j + \theta_k\right),$$

$$V_k(\delta_1, s_1, \ldots, \delta_r, s_r) = -\lambda_k a_k \sin\left(\sum_{j=1}^{r} n_{kj}\delta_j s_j + \theta_k\right),$$

here r *is the dimension of the linear* span $\mathcal{L}(\lambda_1, \ldots, \lambda_m)$ *of* λ_k, $k = 1, \ldots, m$, *over the ring* $\mathcal{Z}$, *the positive numbers* δ_j, $j = 1, \ldots, r$, *are formed a basis in* $\mathcal{L}(\lambda_1, \ldots, \lambda_m)$, *and*

$$\lambda_k = \sum_{j=1}^{r} n_{kj}\delta_j, \quad n_{kj}\in\mathcal{Z}, \quad k = 1, \ldots, m, \quad j = 1, \ldots, r.$$

The proof of the theorem can be obtained from formula (5).

***Remark* 1.** It is easy to see that $\theta_k = \lambda_k\varphi_k$. Formula (10) implies that

$$A(\Phi; x(0), v(0)) = \hat{A}(\hat{\Phi}; a_1, \ldots, a_m, \varphi_2-\varphi_1, \ldots, \varphi_m-\varphi_1), \tag{11}$$

where $x(0)$, $v(0)$ are determined by formula (5) with $t = 0$, the function $\hat{A}(\hat{\Phi}; \cdot)$ from $R_+^m\times[-2\pi, 2\pi]^{m-1}$ into R is expressed through $A(\Phi; \cdot)$ in a natural way.

***Remark* 2.** Let

$$\Phi(x, v) = \Phi_1(r_1, \ldots, r_m)\Phi_2(\psi_1, \ldots, \psi_m),$$

where

$$x_k = r_k\cos\psi_k, \quad v_k = -\lambda_k r_k\sin\psi_k, \quad k = 1, \ldots, m,$$

and

$$r_k\in R_+, \quad \psi_k\in[0, 2\pi), \quad k = 1, \ldots, m.$$

Then

$$\hat{A}(\Phi; a_1, \ldots, a_m, \varphi_2-\varphi_1, \ldots, \varphi_m-\varphi_1) =$$

$$= \Phi(r_1, \ldots, r_m)\hat{A}(1, \ldots, 1, \varphi_2-\varphi_1, \ldots, \varphi_m-\varphi_1).$$

3. Fast Markov perturbations. We consider the stochastic process $(x_\varepsilon(t); v_\varepsilon(t))$ for which $x_\varepsilon(t)$ and $v_\varepsilon(t)$ satisfy the system of differential equations

$$\frac{d}{dt}x_\varepsilon(t) = v_\varepsilon(t),$$

$$\frac{d}{dt}v_\varepsilon(t) = -\Lambda x_\varepsilon(t) + f(x_\varepsilon(t), v_\varepsilon(t), y_\varepsilon(t)), \tag{12}$$

and $y_\varepsilon(t) = y\left(\frac{t}{\varepsilon}\right)$, where the stochastic process $y(t)$ satisfies condition A of Section 1. We assume that $x_\varepsilon(0) = x(0)$, $v_\varepsilon(0) = v(0)$ are non-random. We will use some results related to the Markov process $y(t)$ and the solutions to system (12). Denote

$$R(y, C) = \int_0^\infty (P(t, y, C) - \rho(C))\, dt \tag{13}$$

and set

$$R g(y) = \int g(y') R(y, dy') \tag{14}$$

for any measurable bounded function $g : Y \to R$.

Lemma 1. *Let A be the generator of the process $y(t)$:*

$$A g(y) = \lim_{h \downarrow 0} \frac{1}{h} (E_y g(y(h)) - g(y)) \tag{15}$$

which is defined on all measurable bounded function $g(y)$ for which

$$\frac{1}{h} (E_y g(y(h)) - g(y))$$

is bounded and the limit in the right-hand side of relation (15) *exists, E_y is the conditional expectation under the condition $y(0) = y$.*

Then for any measurable bounded function $g(y)$ satisfying the condition

$$\int g(y) \rho(dy) = 0$$

we have

$$A R g(y) = -g(y). \tag{16}$$

The proof is obtained by calculation.

Lemma 2. *Let a measurable bounded function $\varphi(y)$ satisfies the condition*

$$\int \varphi(y) \rho(dy) = 0.$$

Then the stochastic process

$$\xi_T(t) = \frac{1}{\sqrt{T}} \int_0^{tT} \varphi(y(t))\, dt \tag{17}$$

converges weakly to the Wiener process $\xi(t)$ for which

$$E\xi(t) = 0, \quad E\xi^2(t) = 2t \int \int \varphi(y) R\varphi(y) \rho(dy).$$

The proof can be derived from the general theorem on convergence to a diffusion process ([4, p. 78], theorem 1).

***Corollary* 1.** *The stochastic process $y(t)$ satisfying condition A of Section* 1 *is uniformly ergodic, i.e. for any measurable bounded function $g(y)$ the following relation is fulfilled*

$$\lim_{T \to \infty} \sup_y E_y \left(\frac{1}{T} \int_0^T g(y(t))\, dt - \int g(y) \rho(dy) \right)^2 = 0. \tag{18}$$

In the next theorem the results on averaging and normal deviations which can be derived from [1, 3, 4], Sec. 2.5, are formulated for system (12).

Theorem 2. *Let $(x_\varepsilon(t); v_\varepsilon(t))$ be the solution to system* (12), *the function $f(x, v, y)$ is bounded continuous in x, v and had bounded continuous in x, v derivatives*

$$f_x(x, v, y), \quad f_v(x, v, y),$$

$$f_{xx}(x, v, y), \quad f_{xv}(x, v, y), \quad f_{vv}(x, v, y),$$

and let $(x(t); v(t))$ be the solution to system (3) *satisfying the same initial conditions. Then*

(i) *for any* $T > 0$ *with probability* 1 *the relation*

$$\lim_{\varepsilon \to 0} \sup_{t \le T} \left(|x_\varepsilon(t) - x(t)| + |v_\varepsilon(t) - v(t)| \right) = 0 \tag{19}$$

is fulfilled;

(ii) *set*

$$\hat{x}_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} (x_\varepsilon(t) - x(t)), \qquad \hat{v}_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} (v_\varepsilon(t) - v(t)), \tag{20}$$

as $\varepsilon \to 0$ *the stochastic process* $(\hat{x}_\varepsilon(t); \hat{v}_\varepsilon(t))$ *converges weakly to the stochastic process* $(\hat{x}(t); \hat{v}(t))$ *satisfying the system of stochastic differential equations*

$$d\hat{x}(t) = \hat{v}(t)\,dt, \qquad d\hat{v}(t) = -\Lambda \hat{x}(t)\,dt + dz(t) \tag{21}$$

with the initial condition $\hat{x}(0) = \hat{v}(0) = 0$, *where* $z(t)$ *is the Gaussian process with independent increments with* $Ez(t) = 0$, *and*

$$E(z(t), u)^2 = \int_0^t \int \int (f(x(s), v(s), y), u)(f(x(s), v(s), y'), u)\, R(y, dy')\, \rho(dy)\, ds$$

for all $u \in R^m$.

Now we consider the composite stochastic process

$$X_\varepsilon(t) = (x_\varepsilon(t); v_\varepsilon(t); y_\varepsilon(t))$$

in the space $(R^m)^2 \times Y$, here $(x_\varepsilon(t); v_\varepsilon(t))$ is the solution to system (12). It is easy to see that $X_\varepsilon(t)$ is a homogeneous Markov process, and its generator is of the form

$$H_\varepsilon g(x, v, y) = H^0 g(x, v, y) + \frac{1}{\varepsilon} A g(x, v, y), \tag{22}$$

where

$$H^0 g(x, v, y) = (v, g_x(x, v, y)) - (\Lambda x, g_v(x, v, y)) + (f(x, v, y), g_v(x, v, y)) \tag{23}$$

and A is the generator of the process $y(t)$ which is acting on g as a function of y. The operator H_ε is defined on the functions $g : (R^m)^2 \times Y \to R$ satisfying the condition $(\mathcal{H})$:

a) $g(x, v, y)$, $g_x(x, v, y)$, $g_v(x, v, y)$ are measurable bounded function continuous in x, v uniformly with respect to y,

b) the limit

$$\lim_{h \downarrow 0} \frac{1}{h} (E_y g(x, v, y(h)) - g(x, v, y))$$

exists locally uniformly in x, v.

Denote by $E_{x,v,y}$ the conditional expectation under the condition

$$x_\varepsilon(0) = x, \quad v_\varepsilon(0) = v, \quad y_\varepsilon(0) = y.$$

For any function g satisfying condition $(\mathcal{H})$ the following formula is valid

$$E_{x,v,y} g(x_\varepsilon(t), v_\varepsilon(t), y_\varepsilon(t)) - g(x, v, y) = E_{x,v,y} \int_0^t H_\varepsilon f(x_\varepsilon(s), v_\varepsilon(s), y_\varepsilon(s))\, ds. \tag{24}$$

Denote by $\mathcal{F}_t^\varepsilon$ the σ-algebra generated by $\{X_\varepsilon(s),\ s \le t\}$.

Lemma 3. *Let g satisfy condition $(\mathcal{H})$ and $\int g(x, v, y)\,\rho(dy) = 0$. Set*

$$G(x, v, y) = \int g(x, v, y')\,R(y, dy'). \tag{25}$$

Then for $t_1 \le t_2$ the relation

$$E\left(\int_{t_1}^{t_2} g(x_\varepsilon(s), v_\varepsilon(t), y_\varepsilon(s))\,ds \,\middle|\, \mathcal{F}_{t_1}^\varepsilon\right) =$$

$$= \varepsilon E\Bigg(G(x_\varepsilon(t_1), v_\varepsilon(t_1), y_\varepsilon(t_1)) - G(x_\varepsilon(t_2), v_\varepsilon(t_2), y_\varepsilon(t_2)) +$$

$$+ \int_{t_1}^{t_2} H^0 G(x_\varepsilon(s), v_\varepsilon(s), y_\varepsilon(s))\,ds \,\Bigg|\, \mathcal{F}_{t_1}^\varepsilon\Bigg) \tag{26}$$

is valid.

The proof follows from formulas (16), (22), (24).

Corollary 2. *Let g satisfy condition $(\mathcal{H})$. Then for $t_1 < t_2$ we have*

$$E\left(\int_{t_1}^{t_2} g(x_\varepsilon(s), v_\varepsilon(t), y_\varepsilon(s))\,ds \,\middle|\, \mathcal{F}_{t_1}^\varepsilon\right) =$$

$$= E\left(\int_{t_1}^{t_2} \int g(x_\varepsilon(s), v_\varepsilon(s), y)\,\rho(dy)\,ds \,\middle|\, \mathcal{F}_{t_1}^\varepsilon\right) + O(\varepsilon(1 + (t_2 - t_1))). \tag{27}$$

To prove this we apply Lemma 3 to the function

$$\hat{g}(x, v, y) = g(x, v, y) - \int g(x, v, y)\,\rho(dy).$$

Denote by

$$\{x_{\varepsilon k},\ k = 1, \ldots, m\}, \qquad \{v_{\varepsilon k},\ k = 1, \ldots, m\}$$

the coordinates of the vectors x_ε, v_ε. Set

$$z_k^\varepsilon(t) = \lambda_k^2 x_{\varepsilon k}^2(t) + v_{\varepsilon k}^2(t). \tag{28}$$

Let $\{\hat\theta_k^\varepsilon,\ k = 1, \ldots, m\}$ be determined by relation

$$x_{\varepsilon k}(t) = (\lambda_k)^{-1}\sqrt{z_k^\varepsilon(t)}\cos\lambda_k\,\hat\theta_k^\varepsilon(t),$$
$$v_{\varepsilon k}(t) = -\sqrt{z_k^\varepsilon(t)}\sin\lambda_k\,\hat\theta_k^\varepsilon(t). \tag{29}$$

Set

$$\theta_k^\varepsilon(t) = \hat\theta_k^\varepsilon(t) - \hat\theta_1^\varepsilon(t), \qquad k = 2, \ldots, m. \tag{30}$$

Lemma 4. *The stochastic process $z_k^\varepsilon(t)$, $k = 1, \ldots, m$, and $\theta_k^\varepsilon(t)$, $k = 2, \ldots, m$, satisfy the system of differential equations*

$$\frac{d}{dt} z_k^\varepsilon(t) = f_k(x_\varepsilon(t), v_\varepsilon(t), y_\varepsilon(t)), \qquad k = 1, \ldots, m, \tag{31}$$

$$\frac{d}{dt}\theta_k^\varepsilon(t) = \frac{x_{\varepsilon 1}(s)\, f_1(x_\varepsilon(t), v_\varepsilon(t), y_\varepsilon(t))}{2 z_1^\varepsilon(t)} -$$

$$- \frac{x_{\varepsilon k}(s)\, f_k(x_\varepsilon(t), v_\varepsilon(t), y_\varepsilon(t))}{2 z_k^\varepsilon(t)}, \qquad k = 2, \ldots, m. \tag{32}$$

The proof follows from formulas (28) – (30).

Consider the compound stochastic process

$$(\tilde{z}^{\varepsilon}(t); \tilde{\theta}^{\varepsilon}(t)) = \left(z^{\varepsilon}\left(\frac{t}{\varepsilon}\right); \theta^{\varepsilon}\left(\frac{t}{\varepsilon}\right)\right) \tag{33}$$

in the space $R^m \times R^{m-1}$, where

$$z^{\varepsilon}(t) = (z_1^{\varepsilon}(t), \ldots, z_m^{\varepsilon}(t)),$$

$$\theta^{\varepsilon}(t) = (\theta_2^{\varepsilon}(t), \ldots, \theta_m^{\varepsilon}(t)).$$

We will prove that the stochastic process given by formula (33) converges weakly in C to a diffusion process. For the description of this process and the proof of the statement we need some notation. Let

$$x \in R^m, \quad v \in R^m, \quad x = (x_1, \ldots, x_m), \quad v = (v_1, \ldots, v_m).$$

We introduce new variables

$$z_k = \lambda_k^2 x_k^2 + v_k^2, \quad k = 1, \ldots, m,$$

and

$$\theta_k = \hat{\theta}_k - \hat{\theta}_1, \quad k = 2, \ldots, m,$$

where

$$x_k = \lambda_k^{-1} \sqrt{z_k} \cos \lambda_k \hat{\theta}_k,$$

$$v_k = -\sqrt{z_k} \sin \lambda_k \hat{\theta}_k.$$

Denote by $B(x, v)$ a $(m-1) \times m$ matrix with elements which are determined by the relations:

$$b_{ij}(x, v) = \frac{1}{2}\left(\frac{x_1}{z_1} 1_{\{j=1\}} - \frac{x_i}{z_i} 1_{\{j=i\}}\right), \quad i = 1, \ldots, m, \quad j = 2, \ldots, m.$$

Let

$$\hat{f}_{ij}(x, v) = \int\int f_i(x, v, y) f_j(x, v, y') R(y, dy') \rho(dy). \tag{34}$$

Denote

$$a(x, v) = \int\int f_v(x, v, y') f(x, v, y) R(y, dy') \rho(dy'), \tag{35}$$

and let the vector $b(x, v)$ be determined by its coordinates

$$b_k(x, v) = \frac{x_k v_k}{z_k^2} \hat{f}_{kk}(x, v) - \frac{x_1 v_1}{z_1^2} \hat{f}_{11}(x, v), \quad k = 2, \ldots, m. \tag{36}$$

Note that the following formulas are valid for the Jacobians:

$$\frac{Dz}{Dv} = 2V, \quad \frac{D\theta}{Dv} = 2B(x, v),$$

where the elements of the matrix V are given by the relation $v_{ij} = v_i 1_{\{i=j\}}$.

Introduce the matrices $\hat{F}(x, v)$ with elements $\hat{f}_{ij}(x, v)$ and

$$C^{zz}(x, v) = 2V \hat{F}^*(x, v), \quad C^{z\theta}(x, v) = 2B(x, v) \hat{F}^*(x, v),$$

$$C^{\theta z}(x, v) = 2V \hat{F}^*(x, v) B^*(x, v), \quad C^{\theta\theta}(x, v) = B(x, v) \hat{F}^*(x, v) B^*(x, v).$$

Let the vectors $\hat{a}(z, \theta)$, $\hat{b}(z, \theta)$ and matrices

$$\hat{C}^{zz}(z, \theta), \quad \hat{C}^{z\theta}(z, \theta), \quad \hat{C}^{\theta z}(z, \theta), \quad \hat{C}^{\theta\theta}(z, \theta)$$

are $\hat{A}$-transformations of the vectors $a(x, v)$, $b(x, v)$ and matrices

$$C^{zz}(x, v), \quad C^{z\theta}(x, v), \quad C^{\theta z}(x, v), \quad C^{\theta\theta}(x, v),$$

for example

$$\hat{a}_i(z, \theta) = \hat{A}(a_i; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m),$$

where the function $\hat{A}$ was introduced in Remark 1. Denote by $L^{z\theta}$ the differential operator which is determined for $\Phi \in C^{(2)}(R^m \times R^{m-1})$ by the relation

$$L^{z\theta}\Phi(z, \theta) = (\Phi_z(z, \theta), \hat{a}(z, \theta)) + \Big(\Phi_\theta(z, \theta), \hat{b}(z, \theta)\Big) + \\ + \operatorname{Tr}\Phi_{zz}(z, \theta)\,\hat{C}^{zz}(z, \theta) + \operatorname{Tr}\Phi_{z\theta}(z, \theta)\,(\hat{C}^{\theta z}(z, \theta))^* + \\ + \operatorname{Tr}\Phi_{\theta z}(z, \theta)\,(\hat{C}^{z\theta}(z, \theta))^* + \operatorname{Tr}\Phi_{\theta\theta}(z, \theta)\,\hat{C}^{\theta\theta}(z, \theta)\Big)\Big). \tag{37}$$

Theorem 3. *The compound stochastic process* $(\tilde{z}^\varepsilon(t); \tilde{\theta}^\varepsilon(t))$ *converges weakly in* C *as* $\varepsilon \to 0$ *to the diffusion process* $(\tilde{z}(t); \tilde{\theta}(t))$ *in the same space with the initial value* $(z^0; \theta^0)$, *where*

$$z_k^0 = E_k(x_k(0), v_k(0)), \quad k = 1, \ldots, m,$$
$$\theta_k^0 = \varphi_k^0 - \varphi_1^0, \quad k = 2, \ldots, m,$$

and the generator $L^{z\theta}$ *which is determined by formula* (37).

Proof. We will use Theorem 1 on [4, p. 78]. We have to prove the relation

$$\lim_{\varepsilon \to 0} E\left| E\left(\Phi(z^\varepsilon(\varepsilon^{-1}t_2), \theta^\varepsilon(\varepsilon^{-1}t_2)) - \Phi(z^\varepsilon(\varepsilon^{-1}t_1), \theta^\varepsilon(\varepsilon^{-1}t_1)) - \right.\right. \\ \left.\left. - \int_{t_1}^{t_2} L^{z\theta}(\tilde{z}^\varepsilon(s), \tilde{\theta}^\varepsilon(s))\,ds \,\middle|\, \mathcal{F}^\varepsilon_{\varepsilon^{-1}t_1} \right)\right| = 0 \tag{38}$$

for $t_1 < t_2$. To prove this we use the following sequence of relations

$$E\left(\Phi(z^\varepsilon(t_2), \theta^\varepsilon(t_2)) - \Phi(z^\varepsilon(t_1), \theta^\varepsilon(t_1)) \,\middle|\, \mathcal{F}^\varepsilon_{t_1}\right) =$$

$$= E\left(\int_{t_1}^{t_2} \left[\left(\Phi_z(z^\varepsilon(s), \theta^\varepsilon(s)), \frac{dz^\varepsilon(s)}{ds} \right) + \left(\Phi_\theta(z^\varepsilon(s), \theta^\varepsilon(s)), \frac{dz^\varepsilon(s)}{ds} \right) \right] ds \,\middle|\, \mathcal{F}^\varepsilon_{t_1} \right) =$$

$$= E\left(\int_{t_1}^{t_2} (\Phi_z(z^\varepsilon(s), \theta^\varepsilon(s)) + \right. \\ \left. + B^*(x_\varepsilon(s), v_\varepsilon(s))\,\Phi_\theta(z^\varepsilon(s), \theta^\varepsilon(s)), f(x_\varepsilon(s), v_\varepsilon(s), y_\varepsilon(s)))\,ds \,\middle|\, \mathcal{F}^\varepsilon_{t_1} \right).$$

Applying to the last integral Lemmas 4, 3, and Corollary 2 we can obtain the realtion

$$E\left(\Phi(z^\varepsilon(t_2), \theta^\varepsilon(t_2)) - \Phi(z^\varepsilon(t_1), \theta^\varepsilon(t_1)) \,\middle|\, \mathcal{F}^\varepsilon_{t_1}\right) =$$

$$= O(\varepsilon^2(t_2 - t_1 + 1)) + \varepsilon E\left(\int_{t_1}^{t_2} \left[(\Phi_z, a(x_\varepsilon(s), v_\varepsilon(s))) + \right.\right.$$

$$+ \left(\Phi_\theta, b(x_\varepsilon(s), v_\varepsilon(s))\right) + \operatorname{Tr} \Phi_{zz} \left(C^{zz}(x_\varepsilon(s), v_\varepsilon(s))\right)^* +$$

$$+ \operatorname{Tr} \Phi_{\theta z} \left(C^{z\theta}(x_\varepsilon(s), v_\varepsilon(s))\right)^* + \operatorname{Tr} \Phi_{\theta z} \left(C^{\theta z}(x_\varepsilon(s), v_\varepsilon(s))\right)^* +$$

$$+ \operatorname{Tr} \Phi_{\theta\theta} \left(C^{\theta\theta}(x_\varepsilon(s), v_\varepsilon(s))\right)^* \Big] ds \,\Big|\, \mathcal{F}^\varepsilon_{t_1} \Bigg). \tag{39}$$

In this formula the derivatives of the function Φ have as their arguments the functions $z^\varepsilon(s)$, $\theta^\varepsilon(s)$. It follows from statement (i) of Theorem 2 that for any continuous bounded functions $G: R^m \times R^m \to R$ and $\Psi: R^m \times R^{m-1} \to R$ the formula is fulfilled:

$$\lim_{\varepsilon \to 0} \varepsilon E \left| E \left(\int\limits_{\varepsilon^{-1} t_1}^{\varepsilon^{-1} t_2} \Psi(z^\varepsilon(s), \theta^\varepsilon(s)) \times \right.\right.$$

$$\left.\left. \times \left[G(x_\varepsilon(s), v_\varepsilon(s)) - \hat{G}(z^\varepsilon(s), \theta^\varepsilon(s)) \right] ds \,\Big|\, \mathcal{F}^\varepsilon_{\varepsilon^{-1} t_1} \right) \right| = 0, \tag{40}$$

where

$$\hat{G}(z, \theta) = \hat{A}(G; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m).$$

Formulas (39) and (40) implies the relation

$$E \left(\Phi(\tilde{z}^\varepsilon(t_2), \tilde{\theta}^\varepsilon(t_2)) - \Phi(\tilde{z}^\varepsilon(t_1), \tilde{\theta}^\varepsilon(t_1)) - \int\limits_{t_1}^{t_2} L^{z\theta} \Phi(\tilde{z}^\varepsilon(s), \tilde{\theta}^\varepsilon(s)) \, ds \,\Big|\, \mathcal{F}^\varepsilon_{t_1} \right) =$$

$$= O(\varepsilon) + \alpha(\varepsilon), \tag{41}$$

where

$$\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0.$$

Formula (41) implies formula (38), so the theorem is proved.

4. Wiener perturbations. We consider the functions $x_\varepsilon(t)$, $v_\varepsilon(t)$ satisfying the system of stochastic differential equations:

$$dx_\varepsilon(t) = v_\varepsilon(t)\, dt,$$

$$dv_\varepsilon(t) = -\Lambda x_\varepsilon(t) + \sqrt{\varepsilon} F(x_\varepsilon(t), v_\varepsilon(t))\, dw(t), \tag{42}$$

where the function

$$F: R^m \times R^m \to L(R^m)$$

is bounded and smooth enough, and $w(t)$ is R^m-valed Wiener process. Let the stochastic processes $z^\varepsilon(s)$, $\theta^\varepsilon(s)$ are determined by formulas (28) – (30), where $x_\varepsilon(t)$, $v_\varepsilon(t)$ satisfy the system (42).

Lemma 5. *The functions*

$$z^\varepsilon_k(t), \quad k = 1, \ldots, m,$$

$$\theta^\varepsilon_i(t), \quad i = 2, \ldots, m,$$

satisfy the system of stochastic differential equations

$$\begin{aligned} dz_k^{\varepsilon}(t) &= \sqrt{\varepsilon}\sum_j \alpha_{kj}(x_{\varepsilon}(t), v_{\varepsilon}(t))\,dw_j(t) + \varepsilon\beta_k(x_{\varepsilon}(t), v_{\varepsilon}(t))\,dt, \\ d\theta_i^{\varepsilon}(t) &= \sqrt{\varepsilon}\sum_j \gamma_{ij}(x_{\varepsilon}(t), v_{\varepsilon}(t))\,dw_j(t) + \varepsilon\delta_i(x_{\varepsilon}(t), v_{\varepsilon}(t))\,dt, \end{aligned} \tag{43}$$

where

$$\alpha_{kj}(x, v) = 2v_k F_{kj}(x, v), \qquad \beta_k(x, v) = 2v_k \sum_j F_{kj}^2(x, v) \tag{44}$$

and

$$\begin{aligned} \gamma_{ij}(x, v) &= \frac{x_1}{z_1}F_{ij}(x, v) - \frac{x_i}{z_i}F_{ij}(x, v), \\ \delta_i(x, v) &= \sum_j\left(\frac{x_1 v_1}{z_1}F_{1j}^2(x, v) - \frac{x_i v_i}{z_i}F_{ij}^2(x, v)\right) \end{aligned} \tag{45}$$

and F_{ij} are the elements of the matrix F.

The proof is obtained by calculation.

Theorem 4. *The compound stochastic process*

$$(\tilde{z}^{\varepsilon}(t); \tilde{\theta}^{\varepsilon}(t)) = \left(z^{\varepsilon}\left(\frac{t}{\varepsilon}\right); \theta^{\varepsilon}\left(\frac{t}{\varepsilon}\right)\right)$$

converges weakly in C as $\varepsilon \to 0$ to the same diffusion process $(z(t); \theta(t))$ as in Theorem 3 for which

$$\begin{aligned} \hat{a}_k(z, \theta) &= \hat{A}(\beta_k; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m), \\ \hat{b}_i(z, \theta) &= \hat{A}(\delta_i; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m), \\ \hat{G}_{kl}^{zz}(z, \theta) &= \hat{A}\left(\sum_j \alpha_{kj}\alpha_{lj}; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m\right), \\ \hat{G}_{ki}^{z\theta}(z, \theta) = \hat{G}_{ik}^{\theta z}(z, \theta) &= \hat{A}\left(\sum_j \alpha_{kj}\gamma_{ij}; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m\right), \\ \hat{G}_{ij}^{\theta\theta}(z, \theta) &= \hat{A}\left(\sum_k \gamma_{ik}; z_1, \ldots, z_m, \theta_2, \ldots, \theta_m\right). \end{aligned}$$

The proof of the theorem follows from the Itô's formula and the Theorem 1.

1. *Khasminskii R. Z.* On random processes determined by differential equations with a small parameter // Teor. Veroyatnost. i Primenen. – 1966. – **11**. – P. 240–259.
2. *Khasminskii R. Z.* A limit theorem for solutions of differential equations with a random right-hand side // Ibid. – P. 444–462.
3. *Khasminskii R. Z.* On the averaging principle for Itô stochastic differential equations // Kybernetika (Prague). – 1968. – **4**. – P. 260–279 (in Russian).
4. *Papanicolau G. C., Strook D., Varadhan S. R. S.* Martingale approach to some limit theorem // Stat. Mech. Dynam. Syst. Duke Univ. Math. Ser. Durhan (N. C.). – 1977. – **3**. – 20 p.
5. *Skorokhod A. V.* Asymptotic methods in the theory of stochastic differential equations. – Providence, Rhod Island: AMS, 1989. – 339 p.
6. *Freidlin M. I., Wentzell A. D.* Random perturbations of Hamiltonian systems // Mem. AMS. – 1994. – **523**. – P. 1–82.

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