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ON SUMS OF OVERLAPPING PRODUCTS OF INDEPENDENT BERNOULLI RANDOM VARIABLES

ПРО СУМИ ДОБУТКІВ ПОСЛІДОВНИХ ПАР ВЕЛИЧИН, ВИБРАНИХ З ПОСЛІДОВНОСТІ НЕЗАЛЕЖНИХ БЕРНУЛЛІЄВИХ ВЕЛИЧИН

We find the exact distribution of an arbitrary remainder of infinite sum of overlapping products of a sequence of independent Bernoulli random variables.

Знайдено точний розподіл довільного залишку нескінченної суми добутків послідовних пар величин, вибраних з даної послідовності незалежних бернуллієвих випадкових величин.

Results and discussion. Let $X_1, X_2, \ldots$ be independent random variables with distribution

$$\mathrm{P}\{X_n = 1\} = \frac{1}{\mu + n - 1} = 1 - \mathrm{P}\{X_n = 0\}, \qquad n \in \mathbf{N} := \{1, 2, \ldots\}, \tag{1}$$

where $\mu \geq 1$ is a fixed real-valued parameter, and introduce the random variable $N := N_1 = \sum_{n=1}^{\infty} X_n X_{n+1}$ along with the remainders

$$N_l := \sum_{n=l}^{\infty} X_n X_{n+1}, \qquad l \in \mathbf{N},$$

of the infinite sum. The random non-negative integer N is well defined; in fact by the monotone convergence theorem

$$E(N_l) = \sum_{n=l}^{\infty} \frac{1}{(\mu + n - 1)(\mu + n)} < \infty$$

and so $E(N_l) = 1/l$ in the particular case $\mu = 1$, for every $l \in \mathbf{N}$. The aim of this note is to determine the distribution of N_l for all $l \in \mathbf{N}$.

The problem of computing the distribution of $N = N_1$ was originally posed for the case $\mu = 1$ to the second-named author by Y. S. Chow. Having obtained the solution by the method of generating functions, which states that if $\mu = 1$ then N is a Poisson random variable with mean 1, P. Diaconis [1] kindly informed him that the result was known: Diaconis's own proof for this result was included in unpublished notes of Michel Emery in Strasbourg and in an unpublished dissertation by Lars-Ola Hahlin in Uppsala, and it also follows as the special case $\lambda = 1$ for the first coordinate of an infinite-dimensional convergence theorem in Section 3 of a paper by Arratia, Barbour and Tavare [2]. Considering the distributions

$$\mathrm{P}\{X_n = 1\} = \frac{\lambda}{\lambda + n - 1} = 1 - \mathrm{P}\{X_n = 0\}, \qquad n \in \mathbf{N}, \tag{2}$$

for some constant $\lambda > 0$ instead of (1), the method in [2] is purely combinatorial, it identifies the Poisson distribution of N with mean λ as the limiting distribution of the number of cycles of size 1 in a random permutation under the Ewens sampling formula. This method does not appear to produce the distribution of N_l for $l > 1$, even for $\lambda = 1$. Our direct proof here does this for all $l \in \mathbf{N}$ for all $\mu \geq 1$ for the distributions in (1), and in this case it is of independent interest even for $N = N_1$ when $\mu = 1$.

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Throughout, all empty sums are understood as zero and all empty products are understood as one.

Theorem 1. *Let* $X_1, X_2, \ldots$ *be independent random variables with the distributions in* (1) *for some* $\mu \geq 1$. *Then for any* $l, n \in \mathbf{N}$ *such that* $n \geq 1$,

$$\mathrm{P}\{X_l X_{l+1} + \ldots + X_n X_{n+1} + X_{n+1} = k\} = \sum_{j=l+k-2}^{n} \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1}(\mu+r-1)} \binom{j+2-l}{k} \tag{3}$$

and hence for all $l \in \mathbf{N}$,

$$\mathrm{P}\{N_l = k\} = \sum_{j=l+k-2}^{\infty} \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1}(\mu+r-1)} \binom{j+2-l}{k} \tag{4}$$

for every non-negative integer k, *and the generating function of* N_l *is*

$$E\left(s^{N_l}\right) = \sum_{j=l-2}^{\infty} \frac{(s-1)^{j+2-l}}{\prod_{r=l}^{j+1}(\mu+r-1)} = 1 + \frac{s-1}{\mu+l-1} + \frac{(s-1)^2}{(\mu+l-1)(\mu+l)} + \ldots \tag{5}$$

for all $s \in [0, 1]$.

Note that the first statement in (3) and formula (6) in the proof below also give the exact distribution of any section $X_l X_{l+1} + \ldots + X_n X_{n+1}$ of the series defining N.

In the special case $\mu = 1$, the formulae (3), (4) and (5) take the forms

$$\mathrm{P}\{X_l X_{l+1} + \ldots + X_n X_{n+1} + X_{n+1} = k\} = (l-1)! \sum_{j=l+k-2}^{n} \frac{(-1)^{j+k+l}}{(j+1)!} \binom{j+2-l}{k},$$

$$\mathrm{P}\{N_l = k\} = (l-1)! \sum_{j=l+k-2}^{\infty} \frac{(-1)^{j+k+l}}{(j+1)!} \binom{j+2-l}{k} \tag{4_1}$$

for every non-negative integer k and

$$E\left(s^{N_l}\right) = 1 + \frac{s-1}{l} + \frac{(s-1)^2}{l(l+1)} + \frac{(s-1)^3}{l(l+1)(l+2)} + \ldots = \frac{(l-1)!}{(s-1)^{l-1}} \sum_{j=l-1}^{\infty} \frac{(s-1)^j}{j!} \tag{5_1}$$

for all $s \in [0, 1]$. For $l = 1$ it follows from (4_1) in this particular case that

$$\mathrm{P}\{N = k\} = \sum_{j=k-1}^{\infty} \frac{(-1)^{j+1+k}}{(j+1)!} \binom{j+1}{k} = \frac{1}{k!} \sum_{j+1-k=0}^{\infty} \frac{(-1)^{j+1-k}}{(j+1-k)!} = \frac{1}{k!} e^{-1}$$

for all $k = 1, 2, \ldots$, or, equivalently from (5_1), $E\left(s^N\right) = e^{s-1}$, $0 \leq s \leq 1$, the generating function of the Poisson distribution with mean 1. Notice the interesting fact in this connection that the multiplying factor $\sum_{j=l-1}^{\infty} (s-1)^j / j!$ of the second formula in (5_1) is the remainder of the polynomial approximation of degree $l-2$ of e^{s-1}. All in all, the distributions equivalently given by (4) or (5) may be looked upon as a parametric family ($\mu \geq 1$, $l \in \mathbf{N}$) extending the Poisson distribution with mean 1.

In the converse direction we conjecture the following: If $X_1, X_2, \ldots$ are independent Bernoulli random variables such that $\mathrm{P}\{X_1 X_2 = 1\} > 0$ and the distribution of $N = N_1 = \sum_{n=1}^{\infty} X_n X_{n+1}$ is given by (4) with $l = 1$, for some $\mu = 1$, then $E(X_n) = 1/(\mu + n - 1)$ for the same μ, for each $n \in \mathbf{N}$. As a special case for $\mu = 1$, this would give a joint characterization of the standard (mean 1) Poisson and the Bernoulli distributions in (1) with $\mu = 1$. The following result confirms the conjecture under the extra condition that an extended „scaled" version of the full conclusion of Theorem 1 holds.

Theorem 2. *Let* $X_1, X_2, \ldots$ *be independent random variables with distribution given by* $\mathrm{P}\{X_n = 1\} = p_n = 1 - \mathrm{P}\{X_n = 0\}$ *for some* $p_n \in (0, 1)$, $n \in \mathbf{N}$, *such that the generating functlon of* $N_l = \sum_{n=l}^{\infty} X_n X_{n+1}$ *is*

$$f_{l,\lambda}(s) := E\left(s^{N_l}\right) =$$

$$= 1 + \frac{\lambda(s-1)}{\mu+l-1} + \frac{[\lambda(s-1)]^2}{(\mu+l-1)(\mu+l)} + \frac{[\lambda(s-1)]^3}{(\mu+l-1)(\mu+l)(\mu+l+1)} + \ldots,$$

$0 \le s \le 1$, *for all* $l \in \mathbf{N}$ *for some* $\lambda > 0$ *and* $\mu \ge 1$. *Then, necessarily,* $\lambda = 1$ *and* $p_n = 1/(\mu+n-1)$ *for every* $n \in \mathbf{N}$.

The function $f_{l,\lambda}(\cdot)$ here is a seemingly natural generalization of the generating function in Theorem 1 since for the pair $(\lambda, \mu) = (1, 1)$ it reduces to $f_{1,\lambda}(s) = e^{\lambda(s-1)}$, $0 \le s \le 1$, the generating function of the Poisson distribution with mean λ. However, Theorem 2 excludes this parameterization by asserting that the only possible λ is 1. The result in [2], stated above, suggests that version of the conjecture above that if $X_1, X_2, \ldots$ are independent Bernoulli random variables such that $\mathrm{P}\{X_1X_2 = 1\} > 0$ and the distribution of $N = N_1 = \sum_{n=1}^{\infty} X_nX_{n+1}$ is Poisson with mean $\lambda > 0$, then $E(X_n) = \lambda/(\lambda + n - 1)$ for each $n \in \mathbf{N}$. To prove the corresponding weaker version, an analogue of Theorem 2, would require the presently unavailable knowledge of the generating functions of N_l for all $l \in \mathbf{N}$ under the distributions in (2), i.e. the corresponding version of Theorem 1. A remark on this and related problems is placed after the proof of Theorem I below.

Finally, we mention another problem that arises naturally and is open even for our present sequence of independent variables $X_1, X_2, \ldots$ satisfying (1) with $\mu = 1$. For a number $k \in \mathbf{N}$, what is the distribution of $S_k := \sum_{n=1}^{\infty} \prod_{j=n}^{n+k} X_j$? Here $S_1 = N$ of course, and so Theorem 1 answers the question for $k = 1$, but, while various systems of recursive equations may be derived as in the proof of Theorem 1 below, we were unable to identify in any explicit sense the distribution of even the next case, the distribution of $S_2 = \sum_{n=1}^{\infty} X_nX_{n+1}X_{n+2}$.

***Proof of Theorem* 1.** For all admissible values of the integers l, n and k, introduce

$$p_{l,n}(k) := \mathrm{P}\{X_lX_{l+1} + \ldots + X_nX_{n+1} + X_{n+1} = k\},$$

$$p^*_{l,n}(k) = \sum_{j=l+k-2}^{n} \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1}(\mu+r-1)} \binom{j+2-l}{k}$$

and

$$q_{l,n}(k) := \mathrm{P}\{X_lX_{l+1} + \ldots + X_nX_{n+1} = k\},$$

and let us agree to understand $p_{l,n}(k)$, $p^*_{l,n}(k)$ and $q_{l,n}(k)$ as zero if k is negative. Conditioning on X_{n+2}, we obtain

$$p_{l,n+1}(k) = \frac{\mu+n}{\mu+n+1} q_{l,n}(k) + \frac{1}{\mu+n+1} p_{l,n}(k-1)$$

and

$$q_{l,n+1}(k) = \frac{\mu+n}{\mu+n+1} q_{l,n}(k) + \frac{1}{\mu+n+1} p_{l,n}(k).$$

From the first of these two equations

$$q_{l,n}(k) = \frac{\mu+n+1}{\mu+n} p_{l,n+1}(k) - \frac{1}{\mu+n} p_{l,n}(k-1), \tag{6}$$

by which the second becomes

$$\frac{\mu+n+2}{\mu+n+1}p_{l,n+2}(k) - \frac{p_{l,n+1}(k-1)}{\mu+n+1} = \frac{p_{l,n}(k)}{\mu+n+1} + p_{l,n+1}(k) - \frac{p_{l,n}(k-1)}{\mu+n+1},$$

or, equivalently,

$$p_{l,n+2}(k) = p_{l,n+1}(k) + \frac{\left[p_{l,n+1}(k-1) - p_{l,n}(k-1)\right] - \left[p_{l,n+1}(k) - p_{l,n}(k)\right]}{\mu+n+2}. \tag{7}$$

The crux of the argument is to come up with a reasonable conjecture from the recursion in (7) for the form of $p_{l,n}(k)$, which is given by $p^*_{l,n}(k)$ above. Having this, we now proceed to prove the desired identity $p_{l,n}(\cdot) \equiv p^*_{l,n}(\cdot)$ by induction, which for each $m \geq l$ produces $p_{l,m+2}(k)$ from $p_{l,m+1}(\cdot)$ and $p^*_{l,m}(\cdot)$ for all nonnegative integers k.

First, for all $k = 0, 1, 2, \ldots,$ we must consider

$$p_{l,l}(k) = \mathrm{P}\{X_l X_{l+1} + X_{l+1} = k\}$$

and

$$p_{l,l+1}(k) = \mathrm{P}\{X_l X_{l+1} + X_{l+1}X_{l+2} + X_{l+2} = k\}$$

in a direct fashion. Clearly, $p_{l,l}(k) = 0 = p^*_{l,l}(k)$ for all $k > 2$ and $p_{l,l+1}(k) = 0 = p^*_{l,l+1}(k)$ for all $k > 3$. Also,

$$p_{l,l}(0) = \mathrm{P}\{X_{l+1} = 0\} =$$

$$= \frac{\mu+l-1}{\mu+l} = 1 - \frac{1}{\mu+l-1} + \frac{1}{(\mu+l-1)(\mu+l)} = p^*_{l,l}(0),$$

$$p_{l,l}(1) = \mathrm{P}\{X_l = 0,\ X_{l+1} = 1\} =$$

$$= \frac{\mu+l-2}{\mu+l-1}\frac{1}{\mu+l} = \frac{1}{\mu+l-1} - \frac{2}{(\mu+l-1)(\mu+l)} = p^*_{l,l}(1)$$

and

$$p_{l,l}(2) = \mathrm{P}\{X_l = 1,\ X_{l+1} = 1\} = \frac{1}{\mu+l-1}\frac{1}{\mu+l} = p^*_{l,l}(2)$$

by the formula for the right-hand sides, and one can check similarly that the expressions for

$$p_{l,l+1}(0) = \mathrm{P}\{X_{l+1} = 0,\ X_{l+2} = 0\} + \mathrm{P}\{X_l = 0,\ X_{l+2} = 0\},$$

$$p_{l,l+1}(1) = \mathrm{P}\{X_{l+1} = 0,\ X_{l+2} = 1\} + \mathrm{P}\{X_l = 1,\ X_{l+1} = 1,\ X_{l+2} = 0\},$$

$$p_{l,l+1}(2) = \mathrm{P}\{X_l = 0,\ X_{l+1} = 1,\ X_{l+2} = 1\},$$

and

$$p_{l,l+1}(3) = \mathrm{P}\{X_l = 1,\ X_{l+1} = 1,\ X_{l+2} = 1\}$$

also agree with $p^*_{l,l+1}(0)$, $p^*_{l,l+1}(1)$, $p^*_{l,l+1}(2)$, and $p^*_{l,l+1}(3)$, respectively. Thus we have $p_{l,n}(\cdot) = p^*_{l,n}(\cdot)$ for $n = l, l+1$.

Now we suppose that $p_{l,m}(k) = p^*_{l,m}(k)$ and $p_{l,m+1}(k) = p^*_{l,m+1}(k)$ for all $k = 0, 1, 2, \ldots$ for some integer $m \geq l$. Then by (7) and this induction hypothesis,

$$p_{l,m+2}(k) = p^*_{l,m+1}(k) + \frac{\left[p^*_{l,m+1}(k-1) - p^*_{l,m}(k-1)\right] - \left[p^*_{l,m+1}(k) - p^*_{l,m}(k)\right]}{\mu+m+2} =$$

$$= p^*_{l,m+1}(k) + \frac{1}{\mu+m+2}\frac{(-1)^{m+k+l}}{\prod_{r=l}^{m+2}(\mu+r-1)}\left[\binom{m+3-l}{k-1} + \binom{m+3-l}{k}\right] =$$

$$= p^*_{l,\,m+1}(k) + \frac{(-1)^{m+2+k+l}}{\prod_{r=l}^{m+3}(\mu+r-1)}\binom{m+4-l}{k} =$$

$$= \sum_{j=l+k-2}^{m+2} \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1}(\mu+r-1)}\binom{j+2-l}{k} = p^*_{l,\,m+2}(k)$$

for all $k = 0, 1, 2, \ldots$. This proves the first statement in (3).

Since X_{n+1} converges in probability to zero as $n \to \infty$, the second statement in (4) follows directly from the first. Finally, from (4),

$$E\left(S^{N_l}\right) = \sum_{k=0}^{\infty} s^k \mathrm{P}\{N_l = k\} = \sum_{k=0}^{\infty} s^k \sum_{j=l+k-2}^{\infty} \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1}(\mu+r-1)}\binom{j+2-l}{k} =$$

$$= \sum_{j=l-2}^{\infty} \sum_{k=0}^{j+2-l} s^k \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1}(\mu+r-1)}\binom{j+2-l}{k} =$$

$$= \sum_{j=l-2}^{\infty} \frac{(-1)^{j+l}}{\prod_{r=l}^{j+1}(\mu+r-1)} \sum_{k=0}^{j+2-l} \binom{j+2-l}{k}(-s)^k =$$

$$= \sum_{j=l-2}^{\infty} \frac{(-1)^{j+2-l}}{\prod_{r=l}^{j+1}(\mu+r-1)}(1-s)^{j+2-l} = \sum_{j=l-2}^{\infty} \frac{1}{\prod_{r=l}^{j+1}(\mu+r-1)}(s-1)^{j+2-l}$$

for all $s \in [0, 1)$, which proves the third statement in (5).

Remark. For any probabilities

$$p_n = \mathrm{P}\{X_n = 1\} = 1 - \mathrm{P}\{X_n = 0\} \in [0, 1], \quad n \in \mathbf{N},$$

the first part of the proof gives the general recursion

$$p_{l,\,n+2}(k) = p_{l,\,n+1}(k) + \left[p_{n+3}p_{l,\,n+1}(k-1) - p_{n+2}(1-p_{n+3})p_{l,\,n+1}(k-1)\right] -$$
$$- \left[p_{n+3}p_{l,\,n+1}(k) - p_{n+2}(1-p_{n+3})p_{l,\,n+1}(k)\right]$$

for all $n \ge l$ and $k = 0, 1, 2, \ldots$, an as extension of (7). So, we see that Theorem 1 is about an „easy" case when the common value $p_{n+3} = p_{n+2}(1-p_{n+3})$ can be factored out from the two differences, which happens if and only if $p_{n+3} = p_{n+2}/(1+p_{n+2})$ for every $n \ge l$ and the starting values of p_l and p_{l+1} make it possible to piece the induction together. It would be of interest to know whether in a „difficult" case, when $p_{n+3} \ne p_{n+2}/(1+p_{n+2})$ for some or all $n \ge l$, it is still possible to derive a closed solution of the recursive formula. The most prominent concrete example of this would be when $p_n = \lambda/(\mu+n-1)^\alpha$, $n \in \mathbf{N}$, for some parameters $\alpha, \lambda > 0$ and $\mu \ge \lambda^{1/\alpha}$, when

$$p_{l,\,n+2}(k) = p_{l,\,n+1}(k) + \frac{\lambda}{(\mu+n+2)^\alpha}\left[p_{l,\,n+1}(k-1) - \frac{(\mu+n+2)^\alpha - \lambda}{(\mu+n+1)^\alpha}p_{l,\,n}(k-1)\right] -$$
$$- \frac{\lambda}{(\mu+n+2)^\alpha}\left[p_{l,\,n+1}(k) - \frac{(\mu+n+2)^\alpha - \lambda}{(\mu+n+1)^\alpha}p_{l,\,n}(k)\right]$$

for $n \ge l$ and $k = 0, 1, 2, \ldots$, as a special generalization of (7). This recursion is what one ought to solve in order to obtain an extension of (3). Even for $\alpha = 1$ the ensuing results would generalize those in Theorem 1, i.e. the case $\alpha = 1 = \lambda$, of for a class of distributions containing the family in (2) for $\mu = \lambda$.

Proof of Theorem 2. For integers $m \ge l \ge 1$, set $N_{l,\,m} := \sum_{n=l}^{m} X_n X_{n+1} \ge 0$. Since $N_{l,\,m} \uparrow N_l$ almost surely as $m \to \infty$, by the monotone convergence theorem we have $E(N_l) = \lim_{m\to\infty} E(N_{l,m})$ and $E(N_l^2) = \lim_{m\to\infty} E(N_{l,m}^2)$. Since, with prime denoting left-hand-side derivative,

$$E(N_l) = f'_{l,\lambda}(1) = \frac{\lambda}{\mu+l-1}$$

and

$$E(N_l^2) = f''_{l,\lambda}(1) + f'_{l,\lambda}(1) = \frac{2\lambda^2}{(\mu+l-1)(\mu+l)} + \frac{\lambda}{\mu+l-1}$$

for all $l \in \mathbf{N}$, the equations

$$\frac{\lambda}{\mu+l-1} = E(N_l) = \lim_{m\to\infty} E(N_{l,m}) = \sum_{n=l}^{\infty} p_n p_{n+1}$$

and

$$E(N_l^2) = \lim_{m\to\infty} E(N_{l,m}^2) = E\left(\left[\sum_{n=l}^{\infty} X_n X_{n+1}\right]^2\right) =$$

$$= \sum_{n=l}^{\infty} E(X_n^2 X_{n+1}^2) + 2\sum_{n=l}^{\infty} E(X_n X_{n+1}^2 X_{n+2}) + 2\sum_{n=l}^{\infty}\sum_{j=n+2}^{\infty} E(X_n X_{n+1} X_j X_{j+1}) =$$

$$= \sum_{n=l}^{\infty} p_n p_{n+1} + 2\sum_{n=l}^{\infty} p_n p_{n+1} p_{n+2} + 2\sum_{n=l}^{\infty}\sum_{j=n+2}^{\infty} p_n p_{n+1} p_j p_{j+1}$$

imply

$$p_l p_{l+1} = \sum_{n=l}^{\infty} p_n p_{n+1} - \sum_{n=l+1}^{\infty} p_n p_{n+1} = \frac{\lambda}{\mu+l-1} - \frac{\lambda}{\mu+l} = \frac{\lambda}{(\mu+l-1)(\mu+l)}$$

and

$$2\sum_{n=l}^{\infty} p_n p_{n+1} p_{n+2} + 2\sum_{n=l}^{\infty} p_n p_{n+1} \frac{\lambda}{\mu+n+1} = \frac{2\lambda^2}{(\mu+l-1)(\mu+l)}$$

for every $l \in \mathbf{N}$. The latter equations in turn imply

$$p_l p_{l+1} p_{l+2} + p_l p_{l+1} \frac{\lambda}{\mu+l-1} = \frac{\lambda^2}{(\mu+l-1)(\mu+l)} - \frac{\lambda^2}{(\mu+l+1)(\mu+l)} =$$

$$= \frac{2\lambda^2}{(\mu+l-1)(\mu+l)(\mu+l+1)},$$

which, combined with the former equations, yield

$$p_l = \frac{1}{p_{l+1} p_{l+2}}\left[\frac{2\lambda^2}{(\mu+l-1)(\mu+l)(\mu+l+1)} - p_l p_{l+1} \frac{\lambda}{\mu+l+1}\right] =$$

$$= \frac{(\mu+l)(\mu+l+1)}{\lambda} \frac{\lambda^2}{(\mu+l-1)(\mu+l)(\mu+l+1)} = \frac{\lambda}{\mu+l-1}$$

for all $l \in \mathbf{N}$. Finally, confronting this with the first set of equations, we get $\lambda^2 = (\mu + l - 1)(\mu + l)\, p_l p_{l+1} = \lambda$. Hence $\lambda = 1$ necessarily, and so $p_l = 1/(\mu + l - 1)$ for all $l \in \mathbf{N}$.

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