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INEQUALITIES FOR COMPLEX RATIONAL FUNCTIONS
НЕРІВНОСТІ ДЛЯ КОМПЛЕКСНИХ РАЦІОНАЛЬНИХ ФУНКЦІЙ

For the rational function $r(z)=p(z)/H(z)$ having all its zeros in $|z|\leq 1$, it is known that

$$|r'(z)|\geq \frac{1}{2}|B'(z)||r(z)| \quad \text{for} \quad |z|=1,$$

where $H(z)=\prod_{j=1}^{n}(z-c_j)$, $|c_j|>1$, n is a positive integer, $B(z)=H^*(z)/H(z)$, and $H^*(z)=z^n\overline{H(1/\overline{z})}$. In this paper, we improve the above mentioned inequality for the rational function $r(z)$ with all zeros in $|z|\leq 1$ and a zero of order s at the origin. Our main results refine and generalize some known rational inequalities.

Для раціональної функції $r(z)=p(z)/H(z)$, що має всі нулі у $|z|\leq 1$, виконується нерівність

$$|r'(z)|\geq \frac{1}{2}|B'(z)||r(z)| \quad \text{для} \quad |z|=1,$$

де $H(z)=\prod_{j=1}^{n}(z-c_j)$, $|c_j|>1$, n — додатне ціле, $B(z)=H^*(z)/H(z)$ і $H^*(z)=z^n\overline{H(1/\overline{z})}$. У цій роботі вказану нерівність удосконалено для раціональної функції $r(z)$ із нулями у $|z|\leq 1$ та нулем порядку s у початку координат. Наші основні результати уточнюють та узагальнюють деякі відомі раціональні нерівності.

1. Introduction and statement of results. Everywhere in this paper we assume that $m, n\in\{1,2,\ldots\}$. Let $\mathcal{P}_m$ be a class of all polynomials of degree at most m. For c_j, $j=1,2,\ldots,n$, belong to the complex plane $\mathbb{C}$, we take $H(z)=\prod_{j=1}^{n}(z-c_j)$ and $B(z)=H^*(z)/H(z)$, where $H^*(z)=z^n\overline{H(1/\overline{z})}$. Here, $\overline{z}=\overline{x+iy}=x-iy$, x, y are real and i is the imaginary complex unit. Also, let

$$\mathcal{R}_{m,n}=\mathcal{R}_{m,n}(c_1,\ldots,c_n)=$$

$$=\left\{p(z)/H(z);\ p\in\mathcal{P}_m,\ H(z)=\prod_{j=1}^{n}(z-c_j),\ \text{where}\ \ |c_j|>1,\ j=1,\ldots,n\right\}$$

denote the class of rational functions with poles at $c_1,c_2,\ldots,c_n$. For $m=n$, we write $\mathcal{R}_n:=\mathcal{R}_{n,n}$.

Li et al. [4, 5] obtained Bernstein-type inequalities for the rational function $r(z)$. They proved that if the rational function $r(z)\in\mathcal{R}_n$ having all its zeros in $|z|\leq 1$, then, for $|z|=1$,

$$|r'(z)|\geq \frac{1}{2}|B'(z)r(z)|. \tag{1.1}$$

Let $\alpha\in\mathbb{C}$ is any fixed number. For a polynomial $p(z)$ of degree n, $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined by

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z).$$

Polynomial $D_\alpha p(z)$ is of degree less than or equal $n-1$ and

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right] = p'(z).$$

In this paper we first give a generalization and refinement of inequality (1.1) by proving the following theorem.

Theorem 1.1. *Let $r(z) \in \mathcal{R}_{m,n}$ has all its zeros in $|z| \le k$, where $k \le 1$, with a zero of order s at the origin and $m \le n \le \dfrac{2(m+sk)}{1+k}$. Then, for any γ with $|\gamma| \le 1$ and $|z| = 1$,*

$$\left|zr'(z) + \frac{n\gamma}{1+k} r(z)\right| \ge \frac{1}{2}\left(|B'(z)| + \frac{2(m+sk) - n(1+k) + 2n\operatorname{Re}\gamma}{1+k}\right)|r(z)|.$$

***Remark* 1.1.** In particular case, if we consider $p(z)$ as a polynomial of degree n, then, for rational function $r(z) = \dfrac{p(z)}{H(z)} = \dfrac{p(z)}{(z-\alpha)^n}$, we have

$$r'(z) = \left(\frac{p(z)}{(z-\alpha)^n}\right)' = -\frac{D_\alpha p(z)}{(z-\alpha)^{n+1}},$$

and, for $B(z) = \dfrac{H^*(z)}{H(z)}$,

$$B'(z) = n\frac{|\alpha|^2 - 1}{(z-\alpha)^2}\left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^{n-1}.$$

Hence, for $|z| = 1$, we have $|B'(z)| = \dfrac{n(|\alpha|^2-1)}{|z-\alpha|^2}$.

Now by taking $m = n$ and $c_j = \alpha$, $j = 1, 2, \ldots, n$, in Theorem 1.1, for $|z| = 1$, we get

$$\frac{\left|zD_\alpha p(z) + \dfrac{n\gamma(\alpha - z)}{1+k}p(z)\right|}{|z-\alpha|^{n+1}} \ge \frac{1}{2}\left(\frac{n(|\alpha|^2-1)}{|z-\alpha|^2} + \frac{2sk + n(1-k) + 2n\operatorname{Re}\gamma}{1+k}\right)\frac{|p(z)|}{|z-\alpha|^n}, \quad (1.2)$$

or

$$\left|zD_\alpha p(z) + \frac{n\gamma(\alpha - z)}{1+k}p(z)\right| \ge \frac{1}{2}\left(\frac{n(|\alpha|^2-1)}{|z-\alpha|} + \frac{2sk + n(1-k) + 2n\operatorname{Re}\gamma}{1+k}|z-\alpha|\right)|p(z)|,$$

or

$$\left|zD_\alpha p(z) + \frac{n\gamma(\alpha - z)}{1+k}p(z)\right| \ge \frac{1}{2}\left(\frac{n(|\alpha|^2-1)}{1+|\alpha|} + \frac{2sk + n(1-k) + 2n\operatorname{Re}\gamma}{1+k}(|\alpha|-1)\right)|p(z)|.$$

Therefore, we have the following result which is a refinement of the result due to Dewan and Mir [3].

***Corollary* 1.1.** *Let $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \le k$, where $k \le 1$, with a zero of order s at the origin. Then, for every α with $|\alpha| \ge 1$, any γ with $|\gamma| \le 1$ and $|z| = 1$, we have*

$$\left|zD_\alpha p(z) + \frac{n\gamma(\alpha - z)}{1+k}p(z)\right| \ge \frac{(n + sk + n\operatorname{Re}\gamma)(|\alpha|-1)}{1+k}|p(z)|. \quad (1.3)$$

By dividing both sides of (1.3) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result which is a generalization of the result due to Aziz and Shah [1].

***Corollary* 1.2.** *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k,\ k \le 1,$ with a zero of order s at the origin, then, for any γ with $|\gamma| \le 1$ and $|z| = 1,$*

$$\left|zp'(z) + \frac{n\gamma}{1+k}p(z)\right| \ge \frac{n + sk + n\mathrm{Re}\,\gamma}{1+k}|p(z)|.$$

If we take $m = n$ and $k = 1$ in Theorem 1.1, then we have the following refinement of inequality (1.1).

***Corollary* 1.3.** *If $r(z) \in \mathcal{R}_n$ having all its zeros in $|z| \le 1$ with a zero of order s at the origin, then, for any γ with $|\gamma| \le 1$ and $|z| = 1,$*

$$\left|zr'(z) + \frac{n\gamma}{2}r(z)\right| \ge \frac{1}{2}(|B'(z)| + s + n\mathrm{Re}\,\gamma)|r(z)|. \tag{1.4}$$

Next, we use $\min_{|z|=1}|r(z)|$ to obtain the more precise of inequality (1.4).

Theorem 1.2. *If $r(z) \in \mathcal{R}_n$ having all its zeros in $|z| \le 1,$ with a zero of order s at the origin, then, for any γ with $|\gamma| \le 1$ and $|z| = 1,$*

$$\left|zr'(z) + \frac{n\gamma}{2}r(z)\right| \ge$$

$$\ge \frac{1}{2}\left((|B'(z)| + s + n\mathrm{Re}\,\gamma)|r(z)| + (|B'(z)| + s + n\mathrm{Re}\,\gamma - |2s + n\gamma|)\min_{|z|=1}|r(z)|\right).$$

Let $\min_{|z|=1}|r(z)| = r(z_0)$. Similar to (1.2), if we take $c_j = \alpha,\ j = 1, 2, \ldots, n,$ in Theorem 1.2, we have

$$\frac{\left|zD_\alpha p(z) + \dfrac{n\gamma(\alpha - z)}{2}p(z)\right|}{|z-\alpha|^{n+1}} \ge \frac{1}{2}\left(\left(\frac{n(|\alpha|^2-1)}{|z-\alpha|^2} + s + n\mathrm{Re}\,\gamma\right)\frac{|p(z)|}{|z-\alpha|^n} + \right.$$

$$\left. + \left(\frac{n(|\alpha|^2-1)}{|z-\alpha|^2} + s + n\mathrm{Re}\,\gamma - |2s + n\gamma|\right)\frac{|p(z_0)|}{|z_0-\alpha|^n}\right),$$

or

$$\left|zD_\alpha p(z) + \frac{n\gamma(\alpha - z)}{2}p(z)\right| \ge \frac{1}{2}\left(\left(\frac{n(|\alpha|^2-1)}{|z-\alpha|} + (s + n\mathrm{Re}\,\gamma)|z-\alpha|\right)|p(z)| + \right.$$

$$\left. + \left(\frac{n(|\alpha|^2-1)}{|z-\alpha|} - (|2s+n\gamma| - (s + n\mathrm{Re}\,\gamma))|z-\alpha|\right)\frac{|z-\alpha|^n}{|z_0-\alpha|^n}|p(z_0)|\right),$$

or

$$\left|zD_\alpha p(z) + \frac{n\gamma(\alpha - z)}{2}p(z)\right| \ge \frac{1}{2}\left((n + s + n\mathrm{Re}\,\gamma)(|\alpha| - 1)|p(z)| + \right.$$

$$\left. + ((n - s + n|\gamma| - n\mathrm{Re}\,\gamma)|\alpha| - (n + s + n\mathrm{Re}\,\gamma - n|\gamma|))\left(\frac{|\alpha|-1}{|\alpha|+1}\right)^n|p(z_0)|\right).$$

Therefore, the next result is obtained for the polar derivative of a polynomial.

***Corollary* 1.4.** *Let $p(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$ with a zero of order s at the origin. Then, for every α with $|\alpha| \geq 1$ and any γ with $|\gamma| \leq 1$, $|z| = 1$,*

$$\left| zD_\alpha p(z) + \frac{n\gamma(\alpha - z)}{2} p(z) \right| \geq \frac{1}{2} \Bigg((n + s + n\mathrm{Re}\,\gamma)(|\alpha| - 1)|p(z)| +$$

$$+ \left[(n - s + n|\gamma| - n\mathrm{Re}\,\gamma)|\alpha| - (n + s + n\mathrm{Re}\,\gamma - n|\gamma|) \right] \left(\frac{|\alpha| - 1}{|\alpha| + 1} \right)^n \min_{|z|=1} |p(z)| \Bigg). \qquad (1.5)$$

By dividing both sides of inequality (1.5) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following refinement of a result which has been proved by Dewan and Hans [2] (Theorem 1).

***Corollary* 1.5.** *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, with a zero of order s at the origin, then, for any γ with $|\gamma| \leq 1$ and $|z| = 1$,*

$$\left| zp'(z) + \frac{n\gamma}{2} p(z) \right| \geq \frac{n + s + n\mathrm{Re}\,\gamma}{2} |p(z)| + \frac{n - s + n|\gamma| - n\mathrm{Re}\,\gamma}{2} \min_{|z|=1} |p(z)|.$$

By referring to the above theorems, the bounds which are obtained for rational function depends only on the zero of largest modulus and not on the other zeros even if some of them are close to the origin. Therefore, it would be interesting to obtain a bound which depends on the location of all the zeros of a rational function. In this connection, we use some known ideas in the literature and obtain the following interesting result.

Theorem 1.3. *If $r(z) = \dfrac{z^s p(z)}{H(z)} \in \mathcal{R}_n$ having all its zeros in $|z| \leq 1$, where $p(z) = b \prod_{i=1}^{n-s} (z - b_i)$, $|b_i| \leq 1, 0 \leq s \leq n$, and $H(z) = \prod_{j=1}^{n} (z - c_j)$ by $|c_j| > 1$, $j = 1, \ldots, n$, then, for $|z| = 1$,*

$$|r'(z)| \geq \frac{1}{2} \left(s + \frac{1 - \prod_{i=1}^{n-s} |b_i|}{1 + \prod_{i=1}^{n-s} |b_i|} + |B'(z)| \right) |r(z)|. \qquad (1.6)$$

Similar to (1.2), under the formula (1.6) in $c_j = \alpha$, $j = 1, 2, \ldots, n$, we have the following result.

***Corollary* 1.6.** *Let $p(z) = bz^s \prod_{i=1}^{n-s} (z - b_i)$, $0 \leq s \leq n$, be a polynomial of degree n having all its zeros in $|z| \leq 1$. Then, for every α with $|\alpha| \geq 1$ and $|z| = 1$,*

$$|D_\alpha p(z)| \geq \frac{|\alpha| - 1}{2} \left(n + s + \frac{1 - \prod_{i=1}^{n-s} |b_i|}{1 + \prod_{i=1}^{n-s} |b_i|} \right) |p(z)|. \qquad (1.7)$$

By dividing both sides of inequality (1.7) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following refinement of the result which proved by Zireh [7].

***Corollary* 1.7.** *If $p(z) = bz^s \prod_{i=1}^{n-s} (z - b_i)$, $0 \leq s \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq 1$, then, for $|z| = 1$,*

$$|p'(z)| \geq \frac{1}{2} \left(n + s + \frac{1 - \prod_{i=1}^{n-s} |b_i|}{1 + \prod_{i=1}^{n-s} |b_i|} \right) |p(z)|.$$

2. Proofs of the theorems. For the proofs of these theorems, we need the following lemma which is due to Ossermann [6, p. 3514] (Lemma 1).

Lemma 2.1. *Let* $f: D \to D$ *be an analytic function, where* $D = \{z \in \mathbb{C}: |z| < 1\}$. *Assume that* $f(0) = 0$ *and exists a continuous extendibility of* $f'(z)$ *with* $|z| < 1$ *to the point* b *with* $|b| = 1$. *Then*

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}.$$

***Proof of Theorem* 1.1.** By hypothesis we have

$$r(z) = \frac{z^s p(z)}{H(z)} = \frac{z^s \prod_{i=1}^{m-s}(z - b_i)}{\prod_{j=1}^{n}(z - c_j)},$$

where $b_i,\ |b_i| \leq k \leq 1,\ i = 1, \ldots, m - s$, are the zeros of $r(z)$. Therefore,

$$\frac{zr'(z)}{r(z)} + \frac{n\gamma}{1+k} = s + \frac{zp'(z)}{p(z)} - \frac{zH'(z)}{H(z)} + \frac{n\gamma}{1+k} =$$

$$= s + \sum_{i=1}^{m-s} \frac{z}{z - b_i} - \frac{zH'(z)}{H(z)} + \frac{n\gamma}{1+k}. \tag{2.1}$$

Now we have

$$\operatorname{Re}\left(\frac{2zH'(z)}{H(z)}\right) - n = \sum_{j=1}^{n} \frac{2z}{z - c_j} - 1 = \sum_{j=1}^{n} \frac{1 + \overline{z}c_j}{1 - \overline{z}c_j} =$$

$$= \sum_{j=1}^{n} \frac{1 - |c_j|^2}{|1 - \overline{z}c_j|^2} = -|B'(z)|,$$

which implies

$$\operatorname{Re}\left(\frac{zH'(z)}{H(z)}\right) = \frac{n - |B'(z)|}{2}. \tag{2.2}$$

Now, for $|z| = 1$ and $|b_i| \leq k$, where $k \leq 1$, we have

$$\operatorname{Re}\frac{z}{z - b_i} \geq \frac{1}{1+k}. \tag{2.3}$$

So, for $|z| = 1$, by (2.1), (2.2) and (2.3), we obtain

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)} + \frac{n\gamma}{1+k}\right) = s + \operatorname{Re}\left(\sum_{i=1}^{m-s} \frac{z}{z - b_i}\right) - \operatorname{Re}\left(\frac{zH'(z)}{H(z)}\right) + \frac{n\operatorname{Re}\gamma}{1+k} =$$

$$= s + \operatorname{Re}\left(\sum_{i=1}^{m-s} \frac{z}{z - b_i}\right) - \frac{n - |B'(z)|}{2} + \frac{n\operatorname{Re}\gamma}{1+k} \geq$$

$$\geq s + \frac{m-s}{1+k} - \frac{n - |B'(z)|}{2} + \frac{n\mathrm{Re}\gamma}{1+k} =$$

$$= \frac{|B'(z)|}{2} + \frac{2(m+sk) - n(1+k) + 2n\mathrm{Re}\gamma}{2(1+k)},$$

and, hence,

$$\left|\frac{zr'(z)}{r(z)} + \frac{n\gamma}{1+k}\right| \geq \mathrm{Re}\left(\frac{zr'(z)}{r(z)} + \frac{n\gamma}{1+k}\right) \geq$$

$$\geq \frac{|B'(z)|}{2} + \frac{2(m+sk) - n(1+k) + 2\mathrm{Re}\gamma}{2(1+k)},$$

from which we can obtain Theorem 1.1.

Proof of Theorem **1.2.** Let $m = \min_{|z|=1} |r(z)|$. If $m = 0$, then we have the result from Corollary 1.3. Suppose that $m > 0$, then $m \leq |r(z)|$ for $|z| = 1$. If $|\lambda| < 1$, then it follows by Rouche's theorem that the rational function $R(z) = r(z) + \lambda m z^s$ has all its zeros in $|z| \leq 1$ with a zero of order s at the origin. By applying inequality (1.4), for rational function $R(z)$ for $|z| = 1$, we obtain

$$\left|zR'(z) + \frac{n\gamma}{2}R(z)\right| \geq \frac{1}{2}(|B'(z)| + s + n\mathrm{Re}\gamma)|R(z)|$$

or

$$\left|zr'(z) + \lambda s m z^s + \frac{n\gamma}{2}(r(z) + \lambda m z^s)\right| \geq \frac{1}{2}(|B'(z)| + s + n\mathrm{Re}\gamma)|r(z) + \lambda m z^s|.$$

By using a suitable argument of λ, we have

$$|r(z) + \lambda m z^s| = |r(z)| + |\lambda m z^s|.$$

Using this equality for the right-hand side of above inequality, we get

$$\left|zr'(z) + \frac{n\gamma}{2}r(z)\right| + |\lambda| \left|s + \frac{n\gamma}{2}|m|z^s\right| \geq \frac{1}{2}(|B'(z)| + s + n\mathrm{Re}\gamma)(|r(z)| + |\lambda| m).$$

Since $|z| = 1$, then

$$\left|zr'(z) + \frac{n\gamma}{2}r(z)\right| \geq \frac{1}{2}((|B'(z)| + s + n\mathrm{Re}\gamma)|r(z)| + (|B'(z)| + s + n\mathrm{Re}\gamma - |2s + n\gamma|)|\lambda| m).$$

Now making $|\lambda| \to 1$, we get

$$\left|zr'(z) + \frac{n\gamma}{2}r(z)\right| \geq \frac{1}{2}((|B'(z)| + s + n\mathrm{Re}\gamma)|r(z)| + (|B'(z)| + s + n\mathrm{Re}\gamma - |2s + n\gamma|)m),$$

from which we can obtain Theorem 1.2.

Proof of Theorem **1.3.** By similar argument in Theorem 1.1, we can write

$$r(z) = \frac{z^s p(z)}{H(z)} = \frac{b z^s \prod_{i=1}^{n-s}(z - b_i)}{\prod_{j=1}^{n}(z - c_j)},$$

where $b_i,\ |b_i| \leq 1,\ i = 1, \ldots, n-s$, are the zeros of $r(z)$. Therefore,

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) = s + \operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right) - \operatorname{Re}\left(\frac{zH'(z)}{H(z)}\right). \tag{2.4}$$

Now we calculate $\operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right)$. Since $p(z)$ is a polynomial of degree $(n-s)$, which has all its zeros in $|z| \leq 1$ and implies $q(z) = z^{n-s}\overline{p(1/\overline{z})} \neq 0$ in $|z| < 1$, then $S(z) = \frac{zp(z)}{q(z)}$ is analytic function in $|z| \leq 1$, where $S(0) = 0$ with $|S(z)| = 1$ for $|z| = 1$. Applying Lemma 2.1 to $S(z)$, we conclude that

$$|S'(z)| \geq \frac{2}{1 + |S'(0)|}. \tag{2.5}$$

Now, for $S(z) = \frac{zp(z)}{q(z)}$,

$$\frac{zS'(z)}{S(z)} = 1 + \frac{zp'(z)}{p(z)} - \frac{zq'(z)}{q(z)}. \tag{2.6}$$

Since $q(z) = z^{n-s}\overline{p(1/\overline{z})}$, then

$$q'(z) = (n-s)z^{(n-s-1)}\overline{p(1/\overline{z})} - z^{(n-s-2)}\overline{p'(1/\overline{z})}.$$

Also, for $|z| = 1$,

$$\frac{zq'(z)}{q(z)} = (n-s) - \overline{\left(\frac{zp'(z)}{p(z)}\right)}. \tag{2.7}$$

From (2.6) and (2.7), we get

$$\frac{zS'(z)}{S(z)} = -(n-s-1) + 2\operatorname{Re}\frac{zp'(z)}{p(z)} \quad \text{for } |z| = 1. \tag{2.8}$$

Also

$$S(z) = \frac{zp(z)}{q(z)} = z\frac{b}{\overline{b}}\prod_{i=1}^{n-s}\left(\frac{z - b_i}{1 - \overline{b_i}z}\right),$$

hence, for $|z| = 1$,

$$\frac{zS'(z)}{S(z)} = 1 + \sum_{i=1}^{n-s}\frac{z}{z - b_i} + \sum_{i=1}^{n-s}\frac{\overline{b_i}z}{1 - \overline{b_i}z} = 1 + \sum_{i=1}^{n-s}\frac{1 - |b_i|^2}{|z - b_i|^2}.$$

It means that $\frac{zS'(z)}{S(z)}$ is positive and $\frac{zS'(z)}{S(z)} = \left|\frac{zS'(z)}{S(z)}\right|$ for $|z| = 1$. Since $|S(z)| = 1$ for $|z| = 1$, hence,

$$\frac{zS'(z)}{S(z)} = |S'(z)| \quad \text{for } |z| = 1. \tag{2.9}$$

By using (2.8) and (2.9), we have

$$|S'(z)| = -(n-s-1) + 2\mathrm{Re}\frac{zp'(z)}{p(z)} \quad \text{for } |z| = 1. \tag{2.10}$$

As

$$S(z) = z\frac{b}{\bar{b}}\prod_{i=1}^{n-s}\left(\frac{z-b_i}{1-\overline{b}_i z}\right),$$

hence

$$|S'(0)| = \prod_{i=1}^{n-s}|b_i|. \tag{2.11}$$

From (2.5), (2.10) and (2.11), we get

$$\mathrm{Re}\frac{zp'(z)}{p(z)} \geq \frac{n-s-1}{2} + \frac{1}{1+\prod_{i=1}^{n-s}|b_i|} \qquad \text{for} \quad |z| = 1. \tag{2.12}$$

By combining the relations (2.2), (2.4) and (2.12), the required result is obtained.

Theorem 1.3 is proved.

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