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ON THE THEORY OF GENERALIZED TOEPLITZ KERNELS *

ПРО ТЕОРІЮ УЗАГАЛЬНЕНИХ ЯДЕР ТЕПЛІЦА

A new proof of integral representation of the generalized Toeplitz kernels is given. This proof is based on the spectral theory of corresponding differential operator which acts in the Hilbert space constructed from the kernel of this sort. A theorem on conditions which are imposed on the kernel and guarantee the selfadjointness of considered operator (i.e., the uniqueness of measure in the representation) is obtained.

Наводиться нове доведення інтегрального зображення узагальнених ядер Тепліца. Це доведення базується на спектральній теорії відповідного диференціального оператора, що діє в гільбертовому просторі, побудованому за таким ядром. Одержано теорему про умови на ядро, які забезпечують самоспряженість цього оператора (тобто єдиність міри в зображенні).

0. Introduction. Let $I = (-l, l)$, $0 < l \leq \infty$, and $I \times I \ni \langle x, y \rangle \mapsto K(x, y) \in \mathbb{C}^1$ be a bounded measurable (with respect to Lebesgue measure $dx\,dy$) kernel. Recall that this kernel K is called positive definite if for every $f \in C^\infty_{\text{fin}}(I)$,

$$\iint\limits_{I\times I} K(x, y)\, f(y)\, \overline{f(x)}\, dx\, dy \geq 0.$$

It is obvious that in this inequality, it is possible to take f to be continuous with compact support or integrable on I with respect to dx, etc. If K is continuous, then its positive definitness is equivalent to the requirement that all matrices $(K(x_j, x_k))_{j,k=1}^N$, for arbitrary different points $x_1, \ldots, x_n \in I$, $N = 1, 2, \ldots$, be nonnegative.

This kernel is called a Toeplitz kernel if the function $(-2l, 2l) \ni t \mapsto k(t) \in \mathbb{C}^1$ exists such that

$$K(x, y) = k(x - y), \quad x, y \in I \tag{0.1}$$

(such a function k is said to be a positive definite function). For Toeplitz kernel (or for positive definite function), the following classical integral representation of Bochner – Krein: K is a Toeplitz kernel iff

$$K(x, y) = k(x - y) = \int\limits_{\mathbb{R}^1} e^{i\lambda(x-y)}\, d\sigma(\lambda), \quad x, y \in I, \tag{0.2}$$

where $d\sigma(\lambda)$ is a nonnegative bounded Borel measure on $\mathbb{R}^1$ (see e.g. [1], Ch. 8, § 3, Subsect. 3). In the case $I = \mathbb{R}^1$, this measure is determined by K uniquely.

In 1979 M. Cotlar and C. Sadosky [2] have introduced an essential generalization of Toeplitz kernel. Namely, let $I = \mathbb{R}^1$, a positive definite kernel K is, by definition, a generalized Toeplitz kernel if, instead of one function k in (0.1), we have four functions $k_{\alpha\beta}(t)$, $\alpha, \beta = 1, 2$, such that

$$K(x, y) = k_{\alpha\beta}(x - y), \quad x \in I_\alpha, \quad y \in I_\beta, \tag{0.3}$$

where $I_1 = [0, \infty)$, $I_2 = (-\infty, 0)$.

They give, for such kernels, a generalization of representation (0.2) and they, together with R. Arocena [3 – 6], have obtained applications of this notion to dilations of operators, etc.

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The above mentioned generalization of Toeplitz kernels dealt witn the case $I = \mathbb{R}^1$, i.e. $l = \infty$. R. Bruzual in [7] generalized these constructions to the case $l < \infty$, when in (0.3), $I_1 = [0, l)$, $I_2 = (-l, 0)$ ("Krein situation"). In particular, he has developed a theory of local semigroups of contractions and using this theory has proved a generalization of integral representation (0.2) for such a type of generalized Toeplitz kernel. Note here also the articles [8, 9], containing interesting results about such kernels.

On the other hand, Yu. M. Berezansky in 1956 has developed a general approach to the integral representation of positive definite kernels K, which was based on the theory of generalized eigenfunction expansion of differential (and other) operators in the space constructed from K (see [10, 11] and book [1], Chapter 8).

The aim of this article is to show that the integral representation of type (0.2) for generalized Toeplitz kernels in the cases $0 < l < \infty$ and $l = +\infty$, can be obtained by using, in a quite natural way, the above mentioned generalized eigenfunction approach. This approach gives a possibility to find conditions on K that imply uniqueness of the measure $d\sigma(\lambda)$ in the representation of type (0.2), to develop a theory of extension of a generalized Toeplitz kernel K on $(-l, l)\times(-l, l)$ to a kernel on $\mathbb{R}^1 \times \mathbb{R}^1$, of the same type similar to stated in [1] (Chapter 8, § 3, Subsect. 8, 9), etc. Note that M. Cotlar and C. Sadosky in [2] have pointed out possibility to apply the generalized eigenfunction approach to the theory of generalized Toeplitz kernels in the case $l = \infty$.

In this article, we also use books [12, 13] and text book [14] in which the theory of rigged spaces, the needed theorems about eigenfunction expansions and positive definite kernels are explained in a more modern way, but actually we only use results from [1]. Communication [15] contains a short information about results given in this article.

1. Formulations of results. Let I be an interval of the form $I = (-l, l)$ $0 < l \le \infty$, and let $I_1 = I \cap [0, \infty)$, $I_2 = I \cap (-\infty, 0)$. Denote $\forall\, \alpha, \beta = 1, 2$,

$$I_{\alpha\beta} = \{t = x - y \mid x \in I_\alpha,\ y \in I_\beta\}, \tag{1.1}$$

i.e. $I_{11} = I_{22} = (-l, l)$, $I_{12} = (0, 2l)$, $I_{21} = (-2l, 0)$.

Consider a *bounded* positive definite kernel

$$I \times I \ni \langle x, y\rangle \mapsto K(x, y) \in \mathbb{C}^1.$$

This kernel is, by definition, a generalized Toeplitz (g.T.) kernel if there exist four continuous functions $I_{\alpha\beta} \ni t \mapsto k_{\alpha\beta}(t) \in \mathbb{C}^1$ such that

$$K(x, y) = k_{\alpha\beta}(x - y), \quad \langle x, y\rangle \in I_\alpha \times I_\beta, \quad \alpha, \beta = 1, 2. \tag{1.2}$$

Any positive definite kernel is Hermitian $(K(x, y) = \overline{K(y, x)},\ \langle x, y\rangle \in I \times I)$, therefore representation (1.2) gives:

$$\begin{aligned} k_{\alpha\alpha}(t) &= \overline{k_{\alpha\alpha}(-t)}, \quad t \in I_{\alpha\alpha}, \quad \alpha = 1, 2; \\ k_{12}(t) &= \overline{k_{21}(-t)}, \quad t \in I_{12}. \end{aligned} \tag{1.3}$$

For every $\alpha, \beta = 1, 2$, the restriction $K \restriction (I_\alpha \times I_\beta)$ is a continuous function $k_{\alpha\beta}(x - y)$, hence *the function K is continuous on $I \times I$* (and bounded by definition). The boundedness of K gives the boundedness of every $k_{\alpha\beta}$ on $I_{\alpha\beta}$.

The main result of the article is the following.

Theorem 1. *For every generalited Toeplitz kernel, the following integral representation takes place*:

$$K(x, y) = \int_{\mathbb{R}^1} e^{i\lambda(x-y)} \sum_{\alpha,\beta=1}^{2} \kappa_\alpha(x)\kappa_\beta(y)\, d\sigma_{\alpha\beta}(\lambda), \quad \langle x, y\rangle \in I \times I. \tag{1.4}$$

Here κ_α *is the characteristic function of the interval* I_α, $\alpha = 1, 2$, *and* $d\sigma(\lambda) = (d\sigma_{\alpha\beta}(\lambda))_{\alpha,\beta=1}^2$ *is a finite nonnegative matrix-valued Borel "spectral" measure on* $\mathbb{R}^1$ ($d\sigma_{11}(\lambda)$ *and* $d\sigma_{22}(\lambda)$ *are nonnegative finite scalar measures,* $d\sigma_{12}(\lambda) == \overline{d\sigma_{21}(\lambda)}$ *has bounded variation on* $\mathbb{R}^1$).

Conversely, every kernel of form (1.4) *with a finite nonnegative matrix-valued measure* $d\sigma(\lambda)$ *is a generalized Toeplitz kernel.*

Remark **1.1.** The proof of this theorem, which will be given in Subsect. 2, 3 and 5, shows that it holds true for a more general situation, namely, if the functions $I_{\alpha\beta} \ni t \mapsto k_{\alpha\beta}(t) \in \mathbb{C}^1$ are only measurable. In this case, the corresponding integral (1.4) is, as before, continuous. Therefore it is possible to conclude that *such a still* g.T. *kernel* K, *for almost all* $\langle x, y\rangle \in I \times I$, *coincides with a continuous* g.T. *kernel given by the integral in* (1.4). Moreover, in [8] it is proved that the difference between K and this integral is a positive definite kernel (defined for almost all x, y).

Remark **1.2.** Let $l < \infty$ and K be a g.T. kernel on $I \times I$. For K we have representation (1.4) in which the integral is a g.T. kernel on $\mathbb{R}^1 \times \mathbb{R}^1$. Therefore it is possible to say that *every g.T. kernel on* $I \times I$ *can be extend to a g.T. kernel on* $\mathbb{R}^1 \times \mathbb{R}^1$.

Remark **1.3.** The measure $d\sigma(\lambda)$ in (1.4) is defined by K, as rule, not uniquely. Therefore, we have many extensions of K from $I \times I$ onto $\mathbb{R}^1 \times \mathbb{R}^1$.

The uniqueness of the measure $d\sigma(\lambda)$ is connected with the selfadjointness of some operator A generated by K. More exactly, this operator A is constructed from the differential expression $-i\dfrac{d}{dx}$ acting in the Hilbert space H_K generated by K (see (2.1)). It is the closure in H_K of operator (2.9).

We will give some sufficient condition on K which involves the selfadjointness of A and, therefore, uniqueness of the extension of K from $I \times I$ to $\mathbb{R}^1 \times \mathbb{R}^1$. For a connection between selfadjointness of the operator A and uniqueness of the measure $d\sigma(\lambda)$ see [1], Chapter 8, § 1, Theorem 1.1.

Let $l < \infty$. By definition, a g.T. kernel is nondegenerate, since it follows from the equality

$$\iint_{I\times I} K(x, y)\, f(y)\, \overline{f(x)}\, dx\, dy = 0, \tag{1.5}$$

where $f \in L^2(I, dx)$, that $f = 0$.

The folowing result takes place.

Theorem 2. *Let* $l < \infty$. *Assume that a generalized Toeplitz kernel* K *is nondegenerated and the corresponding functions* $k_{11}(t)$, $k_{22}(t)$, $t \in (-l, l)$, *are infinitely differentiable in some neighborhood of zero. Then this kernel can be extended uniquely to a generalized Toeplitz kernel on* $\mathbb{R}^1 \times \mathbb{R}^1$, *i.e. the operator* A *is selfadjoint in the space* H_K, *if*

$$\sum_{m=1}^{\infty} \left((-1)^m k_{\alpha\alpha}^{(2m)}(0)\right)^{-1/2m} = \infty, \qquad \alpha = 1, 2. \tag{1.6}$$

This theorem will be proved in Subsect. 4. It is a generalization to g.T. kernels of the well known result concerning positive definite functions (see e.g. [1], Chapter 8, § 3, Theorem 3.13).

Note that, in Subsect. 2–4, we will investigate the case $l < \infty$. The case $l = \infty$ will be examined in Subsect. 5.

***Remark* 1.4.** For ordinary positive definite functions $k(t)$ on $\mathbb{R}^1$, the measure $d\sigma(\lambda)$ in representation (0.2) $(x - y = t \in \mathbb{R}^1)$ is defined uniquely. *For* g.T. *kernels on* $\mathbb{R}^1 \times \mathbb{R}^1$ *such a uniqueness can fail to hold.*

It is possible to construct the corresponding example in the following way. Let $\varphi_1, \varphi_2 \in L^1(\mathbb{R}^1, d\lambda)$ be two different functions for which

$$\int_{\mathbb{R}^1} e^{i\lambda t} \varphi_1(\lambda)\, d\lambda = \int_{\mathbb{R}^1} e^{i\lambda t} \varphi_2(\lambda)\, d\lambda, \quad t \in (0, \infty). \tag{1.7}$$

Put $\forall\, \lambda \in \mathbb{R}^1$ and $\forall\, \alpha, \beta,\ j = 1, 2,\ d\sigma^{(j)}_{\alpha\beta}(\lambda) = \tau^{(j)}_{\alpha\beta}(\lambda)\, d\lambda$, where $\tau^{(j)}_{\alpha\alpha}(\lambda) = |\varphi_1(\lambda)| + |\varphi_2(\lambda)|$, $\tau^{(j)}_{12}(\lambda) = \varphi_j(\lambda)$, $\tau^{(j)}_{21}(\lambda) = \overline{\tau^{(j)}_{12}(\lambda)} = \overline{\varphi_j(\lambda)}$. Every matrix $\tau^{(j)}(\lambda) = (\tau^{(j)}_{\alpha\beta}(\lambda))^2_{\alpha,\beta=1}$, $j = 1, 2$, is positive definite, because $\tau^{(j)}_{11}(\lambda) \geq 0$ and $|\tau^{(j)}_{12}(\lambda)|^2 = |\varphi_j(\lambda)|^2 \leq \tau^{(j)}_{11}(\lambda)\tau^{(j)}_{22}(\lambda)$. Therefore, the matrix-valued measures $d\sigma^{(j)}(\lambda) = (d\sigma^{(j)}_{\alpha\beta}(\lambda))^2_{\alpha,\beta=1}$ are nonnegative, finite and distinct, but according to (1.4) and (1.7) the corresponding g.T. kernels $K(x, y)$ are the same. □

For real-valued g.T. kernels on $\mathbb{R}^1 \times \mathbb{R}^1$, it is easy to prove the uniqueness of $d\sigma(\lambda)$ in representation (1.4).

Note that Theorem 2 still holds in the case $l = \infty$. The proof is the same as in Subsect. 4, with changes from Subsect. 5.

2. Hilbert spaces and differential operators connected with generalized Toeplitz kernel. Using a given g.T. kernel K for the case $l < \infty$, we introduce a quasiscalar product

$$(f, g)_{H_K} = \iint_{I\times I} K(x, y)\, f(y)\, \overline{g(x)}\, dx\, dy, \quad f, g \in L^2, \tag{2.1}$$

where $L^2 = L^2(I, dx)$, dx is the Lebesgue measure (K is bounded, therefore integrals (2.1) exists). Identifying all $f \in L^2$ for which $(f, f)_{H_K} = 0$ with zero and then completing the set of the corresponding classes

$$\hat{f} = \{h \in L^2 \mid (f - h, f - h)_{H_K} = 0\}, \quad f \in L^2, \tag{2.2}$$

we obtain a space H_K in which our operators will act. Vectors from H_K are denoted by $F, G, \ldots$.

Consider the rigging (chain)

$$W^{-1}_{2,0}(I) \supset L^2 \supset W^1_{2,0}(I), \tag{2.3}$$

where $W^1_{2,0}(I)$ is the subspace of the Sobolev space $W^1_2(I)$ consisting of functions $u \in W^1_2(I)$ for which $u(0) = 0$. This rigging is quasinuclear, i.e. the imbedding $W^1_{2,0}(I) \hookrightarrow L^2$ is quasinuclear. Using (2.3) it is possible to construct a quasinuclear rigging

$$H_{K,-} \supset H_K \supset H_{K,+}, \tag{2.4}$$

where the space $H_{K,+}$ consists of classes $\hat{u}$, $u \in W^1_{2,0}(I)$, with the corresponding scalar product. This scalar product $(\hat{u}, \hat{v})_{H_{K,+}}$ is equal to $(u_N, v_N)_{W^1_{2,0}(I)}$, where u_N

is a special unique vector from $W^1_{2,0}(I)$, belonging to $\hat{u}$. For details of the above construction see [13], Chapter 5, § 5, Subsect. 5.1 (note that in our case, $G_+ = G_0 = G_- = L^2$) or in [1], Chapter 8, § 1.

***Remark* 2.1.** Note that in the case of a nondegenerate g.T. kernel K (i.e., if $(f,f)_{H_K} \neq 0$ for every $0 \neq f \in L^2$) $\hat{f} = f$ and the space $H_{K,+} = W^1_{2,0}(I)$. Denote by I_{L^2} and I_{H_K} the standard operator I (connected with the chain) for chains (2.3) and (2.4), respectively. Then in our case, it follows from (2.3), (2.4) that:

$$H_{K,-} = I^{-1}_{H_K} H_{K,+} = I^{-1}_{H_K} W^1_{2,0}(I) = I^{-1}_{H_K} I_{L_2} W^{-1}_{2,0}(I). \tag{2.5}$$

Denote by $\kappa_{\alpha\beta}(x,y)$ the characteristic function of the set $I_\alpha \times I_\beta$ and introduce the kernels

$$K_{\alpha\beta}(x,y) = \kappa_{\alpha\beta}(x,y)K(x,y), \quad \langle x,y\rangle \in I\times I, \quad \alpha,\beta = 1,2. \tag{2.6}$$

Using (1.2) we can write:

$$K(x,y) = \sum_{\alpha,\beta=1}^{2} K_{\alpha\beta}(x,y) = \sum_{\alpha,\beta=1}^{2} \kappa_{\alpha\beta}(x,y)\, k_{\alpha\beta}(x-y), \quad \langle x,y\rangle \in I\times I. \tag{2.7}$$

Representation (2.7) permits to rewrite expression (2.1) in the form

$$(f,g)_{H_K} = \sum_{\alpha,\beta=1}^{2} \iint_{I_\alpha\times I_\beta} k_{\alpha\beta}(x-y)\, f(y)\, \overline{g(x)}\, dx\, dy, \quad f,g\in L^2. \tag{2.8}$$

Introduce now operators connected with our problem. Denote by $C_0^\infty(I)$ the set of all functions u from $C^\infty(I)$ which are equal to zero in some neighborhoods (depending on u) of the points $-l,\ 0,\ l$. On such finite functions, we define the operator

$$\operatorname{Dom}(A') = C_0^\infty(I) \ni u \mapsto A'u = -i\frac{d}{dx} =: (\mathcal{L}u)(x). \tag{2.9}$$

Lemma 2.1. *The operator A' is Hermitian wth respect to quasiscalar product* (2.1), *i.e.*

$$(A'u,v)_{H_K} = (u,A'v)_{H_K}, \quad u,v\in C_0^\infty(I). \tag{2.10}$$

Proof. Using representation (2.8) for (2.1) we get

$$(A'u,v)_{H_K} = \sum_{\alpha,\beta=1}^{2} \iint_{I_\alpha\times I_\beta} k_{\alpha\beta}(x-y)(-iu'(y))\,\overline{v(x)}\, dx\, dy, \quad u,v\in C_0^\infty(I). \tag{2.11}$$

Fix some α, β and functions $u,v\in C_0^\infty(I)$, and consider the integral

$$\iint_{I_\alpha\times I_\beta} k_{\alpha\beta}(x-y)\, u'(y)\,\overline{v(x)}\, dx\, dy. \tag{2.12}$$

Extend in an arbitrary way the function $K_{\alpha\beta}(x,y)$ from $I_{\alpha\beta}$ onto $\mathbb{R}^1$ as a bounded function and extend the functions $u(y)$ and $v(x)$ to be zero for $y\in\mathbb{R}^1\setminus I_\beta$, $x\in\mathbb{R}^1\setminus I_\alpha$. Because the functions from $C_0^\infty(I)$ are equal to zero in some neighborhoods of the points $-l,\ 0,\ l$, these extended u, v belongs to $C^\infty_{\text{fin}}(I)$ and we can rewrite integral (2.12) in following way:

$$\iint\limits_{I_\alpha \times I_\beta} k_{\alpha\beta}(x-y)\, u'(y)\, \overline{v(x)}\, dx\, dy = \iint\limits_{\mathbb{R}^1 \times \mathbb{R}^1} k_{\alpha\beta}(x-y)\, u'(y)\, \overline{v(x)}\, dx\, dy =$$

$$= \int\limits_{\mathbb{R}^1} k_{\alpha\beta}(t) \left(\int\limits_{\mathbb{R}^1} u'(x-t)\, \overline{v(x)}\, dx \right) dt = -\int\limits_{\mathbb{R}^1} k_{\alpha\beta}(t) \left(\int\limits_{\mathbb{R}^1} u(x-t)\, \overline{v'(x)}\, dx \right) dt =$$

$$= -\iint\limits_{I_\alpha \times I_\beta} k_{\alpha\beta}(x-y)\, u(y)\, \overline{v'(x)}\, dx\, dy \tag{2.13}$$

(we used above integration by parts formula).

Applying equality (2.13) to each term in (2.11) gives (2.10). □

Note that, for smooth functions $k_{\alpha\beta}$, equality (2.10) follows from (2.11) using directly the integration by parts formula.

The Hermitness of A' in $(\cdot,\cdot)_{H_K}$ gives that this operator in a natural way can be extended to smooth classes (2.2); $A'\hat{u} = (A'u)\hat{}$ (the proof of this simple assertion can be found in [13], Chapter 5, § 5, Subsect. 5.2 or in [1], Chapter 8, § 1).

So, as a result we have, in the Hilbert space H_K, a densely defined Hermitian operator A', let A be its closure, $A = (A')\tilde{}$. The operator A may or may not be selfajoint (see below, Subsect. 4). But it has equal deficient numbers. This assertion follows from the following.

Lemma 2.2. *The map*

$$\hat{f} \mapsto \hat{f}^* := \widehat{f^*}, \quad \textit{where} \quad f \in L^2, \quad f^*(x) = \overline{f(-x)}, \tag{2.14}$$

is an involution in the space H_K and the operator A is real with respect to this involution:

$$AF^* = (AF)^*, \qquad F \in \mathrm{Dom}(A). \tag{2.15}$$

Proof. For every $\alpha, \beta = 1, 2$, $\kappa_{\alpha\beta}(x,y) = \kappa_{\beta\alpha}(-x,-y)$, $\langle x, y\rangle \in I \times I$, and $k_{\alpha\beta}(x-y) = \overline{k_{\beta\alpha}(-x+y)}$, $x \in I_\alpha$, $y \in I_\beta$ (see (1.3)). Therefore, formula (2.7) gives

$$K(x,y) = \overline{K(-x,-y)}, \qquad \langle x, y\rangle \in I \times I. \tag{2.16}$$

Using definition (2.14) and (2.1), (2.16) we conclude that

$$(f^*, g^*)_{H_K} = \overline{(f,g)_{H_K}}, \qquad (f^*)^* = f, \quad f, g \in L^2. \tag{2.17}$$

This equality and (2.2) show that the definition $\hat{f} \mapsto \widehat{f^*}$ is correct and, therefore, its continuous extension to all H_K gives the involution $H_K \ni F \mapsto F^* \in H_K$.

It is necessary to check equality (2.15) only for $F = \hat{u}$, $u \in C_0^\infty(I)$. Using (2.9) we get: $(A'u^*)(x) = -i\dfrac{d}{dx}\overline{u(-x)} = ((A'u)(x))^*$ and, therefore, $A'\hat{u}^* = (A'\hat{u})^*$, which gives (2.15) for $F = \hat{u}$. □

Thus, the operator A is real with respect to the involution $*$ and, therefore, has equal deficient number. We fix some selfadjoint extension B of this operator in the space H_K and construct the generalized eigenfunction expansion for B. This expansion gives a proof of Theorem 1.

3. Spectral projection theorem and a proof of Theorem 1. For the application of the spectral projection theorem (i. e. of the theory of generalized eigenvectors) to the operator B, it is necessary to construct an extension of rigging (2.4).

Turn $C_0^\infty(I)$ into a linear topological space by introducing the convergence $C_0^\infty(I) \ni u_n \to u \in C_0^\infty(I)$ which is uniform for the functions and all their derivatives that have uniformly bounded supports (i.e. there exist some neighborhoods of the points $-l,\ 0,\ l$ inside of which $u_n(x) = 0$ for all n).

Consider the space of classes

$$D = (C_0^\infty(I))^{\hat{}} \tag{3.1}$$

and endow it with the quotient topology via the map $u \mapsto \hat{u}$. As result, we construct an extension of chain (2.4):

$$H_{K,-} \supset H_K \supset H_{K,+} \supset D; \tag{3.2}$$

the imbedding $D \hookrightarrow H_{K,+}$ is dense and continuous.

Chain (3.2) is standardly connected with the operator $B : D \subset \mathrm{Dom}(B)$ and the restriction $B \upharpoonright D$ acts continuously from D into $H_{K,+}$ (moreover, $B \upharpoonright D$ is given by (2.9) and, therefore, acts continuously in D).

So, it is possible to apply the spectral projection theorem to B and rigging (3.2). We will treat it in the following way. In our general situation when K may be degenerate, we apply some corollary of this theorem (Theorem 5.1 from [13], Chapter 5, § 5; we use only its special case for a single selfadjoint operator and the spaces $G_+ = G_0 = G_- = L^2,\ H_+ = W_{2,0}^1(I)$; the involution $*$ is now the usual complex conjugation; see also [1], Chapter 8, § 1, Theorem 1.1). After its formulation, we will explain in what manner we get the corresponding result from the standard spectral projection theorem in the case when g.T. kernel K is nondegenerated.

The above mentioned Theorem 5.1 in the necessary special case asserts the following.

Proposition 3.1. *For the kernel K, the following representation holds:*

$$K = \int_{\mathbb{R}^1} \Omega(\lambda)\, d\rho(\lambda). \tag{3.3}$$

Here $\Omega(\lambda) \in H_- \otimes H_-$, $H_- = W_{2,0}^{-1}(I)$, is an elementary positive definite kernel and the norm $\|\Omega(\lambda)\|_{H_-\otimes H_-}$ is bounded with respect to λ; the measure ρ is a Borel nonnegative finite measure on the axis $\mathbb{R}^1$. The integral in (3.3) is convergent in the norm of the space $H_- \otimes H_-$.

The positive definiteness of the kernel $\Omega(\lambda)$, $\lambda \in \mathbb{R}^1$, means that $\forall\, u \in H_+ = W_{2,0}^1(I)$

$$(\Omega(\lambda), u \otimes \overline{u})_{L^2 \otimes L^2} \geq 0. \tag{3.4}$$

The elementary character of $\Omega(\lambda)$ consists in validity of the following equality:

$$(\Omega(\lambda), v \otimes \overline{(A'u)})_{L^2\otimes L^2} = (\Omega(\lambda), (A'v) \otimes \overline{u})_{L^2\otimes L^2} =$$
$$= \lambda(\Omega(\lambda), v \otimes \overline{u}), \qquad u, v \in C_0^\infty(I). \tag{3.5}$$

Observe that, in terms of the tensor product, expression (2.1) has the form

$$(f, g)_{H_K} = (K, g \otimes \bar{f})_{L^2\otimes L^2}, \qquad f, g \in L^2. \tag{3.6}$$

Remark **3.1.** Explain in what way one can deduce for the case of a nondegenerated g.T. kernel K, representation (3.3) from the usual spectral projection theorem (see e.g. [14], Chapter 15, § 2, Theorem 2.1). We apply this theorem to the operator B standardly connected with the chain (3.2); now $\hat{f} = f$ and, therefore, $D = C_0^\infty(I)$.

Using this theorem, it is possible to assert that the following statement takes place.

Proposition 3.2. *On the axis $\mathbb{R}^1$ there exists a Borel nonnegative finite measure ρ (the spectral measure of B) for which the following Parseval equality holds:*

$$(u, v)_{H_K} = \int_{\mathbb{R}^1} (P(\lambda)u, v)_{H_K} \, d\rho(\lambda), \qquad u, v \in H_{K,+} = W_2^1(I). \tag{3.7}$$

Here $P(\lambda)$ is defined, for ρ-almost every $\lambda \in \mathbb{R}^1$, and it is an operator-valued function values of which are operators from $H_{K,+}$ into $H_{K,-}$. The corresponding Hilbert–Schmidt norm $\| P(\lambda) \|_{H.S.} \le 1$.

The operator $P(\lambda)$ "projects" onto generalized eigenvectors of the operator B corresponding to the "eigenvalue" λ in the following sense: $\forall\, u \in H_{K,+}$

$$(P(\lambda)u, A'v)_{H_K} = \lambda (P(\lambda)u, v)_{H_K}, \tag{3.8}$$

$$v \in D = C_0^\infty(I) \qquad (B \upharpoonright C_0^\infty(I) = A').$$

This operator is nonnegative with respect to chain (3.2), *i.e.*

$$(P(\lambda)u, u)_{H_K} \ge 0, \qquad u \in H_{K,+}. \tag{3.9}$$

Proposition 3. 1 follows from the preceding one in the case of a nondegenerate K.

Indeed, fix $\lambda \in \mathbb{R}^1$. Using the kernel theorem (see e.g. [14], Chapter 14, § 6, Theorem 6.3, and Remark 6.2 ; * now is the complex conjugation) we can assert that the continuons bilinear form

$$W_{2,0}^1(I) \oplus W_{2,0}^1(I) \ni \langle u, v \rangle \mapsto a_\lambda(u, v) := (P(\lambda)u, v)_{H_K} \in \mathbb{C}^1$$

has the representation

$$(P(\lambda)u, v)_{H_k} = a_\lambda(u, v) = (\Omega(\lambda), v \otimes \overline{u})_{L^2 \otimes L^2}, \tag{3.10}$$

where $\Omega(\lambda) \in (W_{2,0}^{-1}(I) \otimes W_{2,0}^{-1}(I))$ (it is essential, of course, that the imbedding $W_{2,0}^1(I) \hookrightarrow L^2$ is quasinuclear).

Equalities (3.6) and (3.10) make it possible to rewrite representation (3.7) in form (3.3). This generalized kernel $\Omega(\lambda)$ is positive definite: inequality (3.4) follows from (3.9). Equalities (3.10) and (3.8) give: $\forall\, u, v \in C_0^\infty(I)$

$$(\Omega(\lambda), (A'v) \otimes \overline{u})_{L^2 \otimes L^2} = (P(\lambda)u, A'v)_{H_K} =$$

$$= \lambda (P(\lambda)u, v)_{H_K} = \lambda (\Omega(\lambda), v \otimes \overline{u})_{L^2 \otimes L^2}. \tag{3.11}$$

Now we prove the last equality in (3.5). For the proof, it is necessary to note that the form a_λ from (3.10) is positive and, consequently, Hermitian: $\forall\, u, v \in W_{2,0}^1(I)$ $a_\lambda(u, v) = \overline{a_\lambda(v, u)}$. Therefore using (3.11) $\forall\, u, v \in C_0^\infty(I)$

$$(\Omega(\lambda), v \otimes \overline{(A'u)})_{L^2 \otimes L^2} = a_\lambda(A'u, v) = \overline{a_\lambda(v, A'u)} =$$

$$= \overline{(\Omega(\lambda), (A'u) \otimes \overline{v})_{L^2 \otimes L^2}} = \lambda \overline{(\Omega(\lambda), u \otimes \overline{v})_{L^2 \otimes L^2}} =$$

$$= \lambda \overline{a_\lambda(v, u)} = \lambda a_\lambda(u, v) = \lambda (\Omega(\lambda), v \otimes \overline{u})_{L^2 \otimes L^2}.$$

Thus, (3.5) is proved. The boundedness of the norm $\| \Omega(\lambda) \|_{H_- \otimes H_-}$ follows from the estimate $\| P(\lambda) \|_{H.S.} \le 1$.

Indeed, let $\boldsymbol{I}$ be a unitary operator connected with the chain

$$H_- \otimes H_- \supset L^2 \otimes L^2 \supset H_+ \otimes H_+, \qquad H_+ = W_{2,0}^1(I) = H_K,$$

and $\boldsymbol{I}_{H_K}$ connected with (2.4) (note that $\boldsymbol{I} = \boldsymbol{I}_{L^2} \otimes \boldsymbol{L}_{L^2}$, see Remark 2.1). Then we can rewrite (3.10) as the equality:

$$(I\Omega(\lambda), v\otimes\overline{u})_{H_+\otimes H_+} = (\Omega(\lambda), v\otimes\overline{u})_{L^2\otimes L^2} =$$
$$= (P(\lambda)u, v)_{H_K} = (I_{H_K}P(\lambda)u, v)_{H_+}, \quad u, v\in H_+. \tag{3.12}$$

Let $(e_j)_{j=1}^{\infty}$ be some orthonormal real basis in the space H_+, then $(e_j\otimes e_k)_{j,k=1}^{\infty}$ is an orthonormal basis in $H_+\otimes H_+$. Taking in (3.12) $v = e_j$, $u = e_k$, we get

$$\|I\Omega(\lambda)\|^2_{H_+\otimes H_+} = \sum_{j,k=1}^{\infty}|(I\Omega(\lambda), e_j\otimes e_k)_{H_+\otimes H_+}|^2 =$$
$$= \sum_{j,k=1}^{\infty}|(I_{H_K}P(\lambda)e_k, e_j)_{H_K}|^2 = \sum_{k=1}^{\infty}\|I_{H_K}P(\lambda)e_k\|^2_{H_+} = \|I_{H_K}P(\lambda)\|^2_{H.S.}. \tag{3.13}$$

The operator $P(\lambda)\colon H_+\to H_{K,-}$, $\|P(\lambda)\|_{H.S.}\le 1$, therefore, $\|I_{H_K}P(\lambda)\|_{H.S.}\le$ ≤ 1 and equality (3.13) gives that $\|I\Omega(\lambda)\|_{H_+\otimes H_+}\le 1$, i.e. $\|\Omega(\lambda)\|_{H_-\otimes H_-}\le 1$ for an arbitrary λ.

Proposition 3.2 is proved. □

Note that, for a degenerate g.T. kernel K, the proof of Proposition 3.1 is the same as above but technicaly it is more complicated.

The proof of Theorem 1 is based on Proposition 3.1 and the following assertion.

Let $G\subset\mathbb{R}^1$ be a finite open interval and $\xi\in W_2^{-1}(G)$ be a generalized solution, inside G, of the equation $\mathcal{L}\xi = \lambda\xi$ ($\mathcal{L} = \mathcal{L}^+$ is given by (2.9), $\lambda\in$ $\in\mathbb{C}^1$), i.e. the following equality holds:

$$(\xi, \mathcal{L}v)_{L^2(G)} = \lambda(\xi, v)_{L^2(G)}, \quad v\in C^{\infty}_{\text{fin}}(G). \tag{3.14}$$

Then, automatically, $\xi\in C^{\infty}(\tilde{G})$ and has the form

$$\xi(x) = c\,e^{i\lambda x}, \quad x\in\tilde{G}, \tag{3.15}$$

where $c\in\mathbb{C}^1$ is some constant.

(This result is a special case of the theorem about smoothness, up to the boundary, of a generalized solution of ordinary differential equation (see e.g. [14], Chapter 16, § 6, Theorem 6.1)).

***Proof of Theorem* 1.** Denote by $H_{\alpha,+}$ the subspace of $H_+ = W^1_{2,0}(I)$ consisting of functions from H_+ which are equal to zero on $I\setminus I_\alpha$, and let

$$H_{\alpha\beta,+} = H_{\alpha,+}\otimes H_{\beta,+}\subset H_+\otimes H_+, \quad \alpha,\beta = 1, 2.$$

Note that for $u\in H_+$, the function $u(x)\kappa_\alpha(x)\in H_{\alpha,+}$ ($\kappa_\alpha(x)$ is the characteristic function of I_α). Let

$$H_{\alpha,-}\supset L^2(I_\alpha)\supset H_{\alpha,+}, \quad \alpha = 1, 2, \tag{3.16}$$

be the rigging connected with the spaces $L^2(I_\alpha)$ and $H_{\alpha,+}$.

Fix $\lambda\in\mathbb{R}^1$ and denote by $\Omega_{\alpha\beta}(\lambda)$ the restriction of the generalized function $\Omega(\lambda)\in H_-\otimes H_-$ to $H_{\alpha\beta,+}$, i.e.

$$(\Omega_{\alpha\beta}(\lambda), v_\alpha\otimes\overline{u_\beta})_{L^2(I_\alpha)\otimes L^2(I_\beta)} = (\Omega(\lambda), v_\alpha\otimes\overline{u_\beta})_{L^2\otimes L^2}, \tag{3.17}$$
$$v_\alpha\in H_{\alpha,+}, \quad u_\beta\in H_{\beta,+}, \quad \alpha,\beta = 1, 2.$$

Evidently, we have the equality

$$(\Omega(\lambda), v \otimes \overline{u})_{L^2 \otimes L^2} =$$

$$= \sum_{\alpha,\beta=1}^{2} (\Omega_{\alpha\beta}(\lambda), \kappa_\alpha(x) v(x) \kappa_\beta(y) \overline{u(y)})_{L^2(I_\alpha) \otimes L^2(I_\beta)}, \quad u, v \in H_+. \tag{3.18}$$

We will find the expression for $\Omega_{\alpha\beta}(\lambda)$; below $\alpha, \beta = 1, 2$ are fixed. Note at first that the bilinear form

$$H_{\beta,+} \oplus H_{\alpha,+} \ni \langle u_\beta, v_\alpha \rangle \mapsto a(u_\beta, v_\alpha) := (\Omega_{\alpha\beta}(\lambda), v_\alpha \otimes \overline{u_\beta})_{L^2(I_\alpha) \otimes L^2(I_\beta)} \tag{3.19}$$

is continuous. Indeed, because $\| \Omega(\lambda) \|_{H_- \otimes H_-}$, $\lambda \in \mathbb{R}^1$, is bounded, we have using definition (3.17):

$$| a(u_\beta, v_\alpha) | = |(\Omega(\lambda), v_\alpha \otimes \overline{u_\beta})_{L^2 \otimes L^2}| \le$$

$$\le \|\Omega(\lambda)\|_{H_- \otimes H_-} \|v_\alpha \otimes \overline{u_\beta}\|_{H_+ \otimes H_+} \le c \|u_\beta\|_{H_{\beta,+}} \|v_\alpha\|_{H_{\alpha,+}}.$$

Using chains (3.16) we can assert that there exist such continuous operators $R : H_{\beta,+} \to H_{\alpha,-}$ and $S : H_{\alpha,+} \to H_{\beta,-}$ that we have the representations

$$a(u_\beta, v_\alpha) = (R u_\beta, v_\alpha)_{L^2(I_\alpha)} = (u_\beta, S v_\alpha)_{L^2(I_\beta)}, \tag{3.20}$$

$$u_\beta \in H_{\beta,+}, \quad v_\alpha \in H_{\alpha,+}.$$

From (3.20), (3.19), (3.17), and (3.5) we can conclude that $\xi = R u_\beta \in H_{\alpha,-}$ is a generalized solution, inside I_α, of the equation $\mathscr{L} \xi = \lambda \xi$. Namely, we have the corresponding equality (3.14): $\forall\, v_\alpha \in C^\infty_{\text{fin}}(G) \subset C^\infty_0(I)$

$$(\xi, \mathscr{L} v_\alpha)_{L^2(I_\alpha)} = (R u_\beta, \mathscr{L} v_\alpha)_{L^2(I_\alpha)} = a(u_\beta, \mathscr{L} v_\alpha) =$$

$$= (\Omega(\lambda), (\mathscr{L} v_\alpha) \otimes \overline{u_\beta})_{L^2 \otimes L^2} = \lambda (\Omega(\lambda), v_\alpha \otimes \overline{u_\beta})_{L^2 \otimes L^2} =$$

$$= \lambda (\Omega_{\alpha\beta}(\lambda), v_\alpha \otimes \overline{u_\beta})_{L^2(I_\alpha) \otimes L^2(I_\beta)} = \lambda\, a(u_\beta, v_\alpha) = \lambda (\xi, v_\alpha)_{L^2(I_\alpha)}.$$

Therefore, the above-mentioned assertion gives that $R u_\beta = \xi \in C^\infty_{\text{fin}}(\tilde{I})$ and, according to (3.15),

$$(R u_\beta)(x) = c_1(u_\beta) e^{i\lambda x}, \quad x \in \tilde{I}_\alpha, \quad u_\beta \in H_{\beta,+}, \tag{3.21}$$

where the constant $c = c_1(u_\beta)$ depends on ξ, i.e. on u_β. Linearity and continuity of R gives that the functional $H_{\beta,+} \ni u_\beta \mapsto c_1(u_\beta) \in \mathbb{C}^1$ is linear and continuous.

Quite analogously we get the representation

$$(S v_\alpha)(y) = c_2(v_\alpha) e^{i\lambda y}, \quad y \in \tilde{I}_\beta, \quad v_\alpha \in H_{\alpha,+}.$$

Equality (3.2) gives that

$$c_1(u_\beta) \int_{I_\alpha} e^{i\lambda x} \overline{v_\alpha(x)}\, dx = \overline{c_2(v_\alpha)} \int_{I_\beta} u_\beta(y) e^{-i\lambda y}\, dy,$$

$$u_\beta \in H_{\beta,+}, \quad v_\alpha \in H_{\alpha,+}.$$

From this equality, it is easy to conclude that, with some constant $\tau \in \mathbb{C}^1$,

$$c_1(u_\beta) = \tau \int_{I_\beta} u_\beta(y) e^{-i\lambda y}\, dy, \quad u_\beta \in H_{\beta,+} \tag{3.22}$$

(it follows from the following general assertion: let l_1, l_2 (m_1, m_2) be some linear functionals on linear spaces U (V), respectively, $l_2, m_2 \neq 0$. Suppose that $l_1(u)\overline{m_1(v)} = l_2(u)\overline{m_2(v)}$, $u \in U$, $v \in V$, then $\exists\, \tau \in \mathbb{C}^1$ such that $l_1 = \tau l_2$, $m_1 = \tau m_2$).

Substituting (3.22) into (3.21) and using (3.19), (3.20) we get:

$$(\Omega_{\alpha\beta}(\lambda), v_\alpha \otimes \overline{u_\beta})_{L^2(I_\alpha)\otimes L^2(I_\beta)} = \tau \iint\limits_{I_\alpha \times I_\beta} e^{i\lambda(x-y)}\overline{v_\alpha(x)}\, u_\beta(y)\, dx\, dy,$$

$$u_\beta \in H_{\beta,+}, \quad v_\alpha \in H_{\alpha,+}.$$

This equality means that $\Omega_{\alpha\beta}(\lambda)$ is a smooth function $\Omega_{\alpha\beta}(\lambda; x, y)$ and

$$\Omega_{\alpha\beta}(\lambda; x, y) = \tau_{\alpha\beta}(\lambda)\, e^{i\lambda(x-y)}, \quad x \in \tilde{I}_\alpha, \quad y \in \tilde{I}_\beta, \quad \alpha, \beta = 1, 2 \tag{3.23}$$

(the constant τ depends on λ, α, β).

Let $u, v \in H_+ = W^1_{2,0}(I)$. Then representations (3.18) and (3.23) give:

$$(\Omega(\lambda), v \otimes \overline{u})_{L^2 \otimes L^2} = \sum_{\alpha,\beta=1}^{2} \tau_{\alpha\beta}(\lambda) \iint\limits_{I_\alpha \times I_\beta} e^{i\lambda(x-y)}\overline{v(x)}\, u(y)\, dx\, dy =$$

$$= \iint\limits_{I \times I} \left(\sum_{\alpha,\beta=1}^{2} e^{i\lambda(x-y)} \kappa_\alpha(x)\kappa_\beta(y)\tau_{\alpha\beta}(\lambda) \right) \overline{v(x)}\, u(y)\, dx\, dy. \tag{3.24}$$

The arbitrariness of the functions $u, v \in W^1_{2,0}(I)$ in (3.24) shows that $\Omega(\lambda)$ is an ordinary kernel, $\Omega(\lambda; x, y)$, and

$$\Omega(\lambda; x, y) = \sum_{\alpha,\beta=1}^{2} e^{i\lambda(x-y)} \kappa_\alpha(x)\kappa_\beta(y)\tau_{\alpha\beta}(\lambda), \quad x, y \in I. \tag{3.25}$$

Note that the matrix $\tau(\lambda) = (\tau_{\alpha\beta}(\lambda))_{\alpha,\beta=1}^{2}$ is nonnegative definite for every $\lambda \in \mathbb{R}^1$. Indeed, from (3.24) and (3.4) we can conclude:

$$\sum_{\alpha,\beta=1}^{2} \tau_{\alpha\beta}(\lambda) c_\alpha \overline{c_\beta} = (\Omega(\lambda), \overline{u} \otimes u)_{L^2 \otimes L^2} \geq 0,$$

$$c_\alpha = \int\limits_{I_\alpha} e^{i\lambda x} u(x)\, dx, \quad u \in H_+, \quad \alpha = 1, 2.$$

This inequality shows that $\tau(\lambda)$ is nonnegative definite, because the numbers c_α are arbitrary.

The nonnegativeness of $\tau(\lambda)$ gives

$$\tau_{11}(\lambda) \geq 0, \quad \tau_{22}(\lambda) \geq 0, \quad |\tau_{12}(\lambda)|^2 \leq \tau_{11}(\lambda)\tau_{22}(\lambda), \quad \lambda \in \mathbb{R}^1. \tag{3.26}$$

Using the measure ρ from Proposition 3.1 we introduce the matrix-valued nonnegative Borel measure $d\sigma(\lambda)$ on $\mathbb{R}^1$:

$$d\sigma(\lambda) = \tau(\lambda)\, d\rho(\lambda) := (\tau_{\alpha\beta}(\lambda)\, d\rho(\lambda))_{\alpha,\beta=1}^{2} = (d\sigma_{\alpha\beta}(\lambda))_{\alpha,\beta=1}^{2}. \tag{3.27}$$

After substituting representation (3.25) into (3.3) we get the required formula (1.4) (we use the measure (3.27)):

$$K(x, y) = \int\limits_{\mathbb{R}^1} e^{i\lambda(x-y)} \sum_{\alpha,\beta=1}^{2} \kappa_\alpha(x)\kappa_\beta(y)\, d\sigma_{\alpha\beta}(\lambda), \quad x, y \in I. \tag{3.28}$$

Let us investigate the convergence of integrals (3.28). According to (3.26), the measures $d\sigma_{11}(\lambda)$, $d\sigma_{22}(\lambda)$ are nonnegative. Taking in (3.28) $x = y \in I_1$ and using (1.2) we get $\sigma_{11}(\mathbb{R}^1) = k_{11}(0) < \infty$. Analogously, $\sigma_{22}(\mathbb{R}^1) = k_{22}(0) < \infty$. The measure $d\sigma_{12}(\lambda) = \overline{d\sigma_{21}(\lambda)}$ has bounded variation. The proof follows from (3.27), (3.26), and the estimate

$$\int_{\mathbb{R}^1} |\tau_{12}(\lambda)|\, d\rho(\lambda) \le \int_{\mathbb{R}^1} (\tau_{11}(\lambda))^{1/2} (\tau_{22}(\lambda))^{1/2}\, d\rho(\lambda) \le$$

$$\le \left(\int_{\mathbb{R}^1} \tau_{11}(\lambda)\, d\rho(\lambda) \int_{\mathbb{R}^1} \tau_{22}(\lambda)\, d\rho(\lambda) \right)^{1/2} = (\sigma_{11}(\mathbb{R}^1)\, \sigma_{22}(\mathbb{R}^1))^{1/2} < \infty.$$

These properties of the measures $d\sigma_{\alpha\beta}(\lambda)$ involve the absolute convergence of 4 integrals in (3.28) and their continuity with respect to $\langle x, y\rangle \in I \times I$.

So, we proved representation (1.4). The inverse statement is evident: every integral (1.4) has form (1.2) with continuous functions $k_{\alpha\beta}(t)$ and is a bounded positive definite kernel, because $\forall f \in C^{\infty}_{\text{fin}}(I)$

$$\iint_{I\times I} K(x, y)\, f(y)\, \overline{f(x)}\, dx\, dy =$$

$$= \iint_{I\times I} \left(\int_{\mathbb{R}^1} e^{i\lambda(x-y)} \sum_{\alpha,\beta=1}^{2} \kappa_{\alpha}(x)\, \kappa_{\beta}(y)\, d\sigma_{\alpha\beta}(\lambda) \right) f(y)\, \overline{f(x)}\, dx\, dy =$$

$$= \int_{\mathbb{R}^1} \left(\sum_{\alpha,\beta=1}^{2} \overline{\int_{I_\alpha} e^{-\lambda x} f(x)\, dx} \int_{I_\beta} e^{-i\lambda y} f(y)\, dy \right) d\rho(\lambda) =$$

$$= \int_{\mathbb{R}^1} \left| \int_{I} e^{-\lambda x} f(x)\, dx \right|^2 d\rho(\lambda) \ge 0. \quad \square$$

4. Proof of Theorem 2. At first we will introduce some additional general constructions. Let $l < \infty$ and $I \times I \ni \langle x, y\rangle \mapsto K(x, y) \in \mathbb{C}^1$ be an arbitrary bounded continuous positive definite nondegenerate kernel. Using K we introduce the Hilbert space H_K (see the beginning of Subsect. 2) and investigate the question about whether the derivatives $\frac{d^m}{dx^m}\delta_z = \delta_z^{(m)}$ of the δ-function δ_z concentrated at the point $z \in I$, $m = 0, 1, \ldots$, $\delta_z^{(0)} = \delta_z$, belong to H_K.

Let us recall the notion of δ_z and its derivatives. Let $\Phi = C^{\infty}_{\text{fin}}(I)$ be the usual space of infinitely differentiable finite with respect to I functions on I. Denote by

$$\Phi' \supset L^2 \supset \Phi \tag{4.1}$$

the corresponding chain of spaces. The generalized function $\delta_z^{(m)}$ is defined by the expression

$$(\delta_z^{(m)}, \varphi)_{L^2} = (-1)^m\, \overline{\varphi^{(m)}(z)}, \qquad \varphi \in \Phi, \quad z \in I, \quad m = 0, 1, \ldots .$$

We will prove the following simple lemma (see also [1], Chapter 8, § 3, Subsect. 7).

Lemma 4.1. *Let $z \in I$, $l = 0, 1, \ldots$, and the function $I \times I \ni \langle x, y\rangle \mapsto$ $\mapsto K(x, y) \in \mathbb{C}^1$ be $2l$ times continuously differentiable in some neighborhood of* *the point* $\langle z, z\rangle$. *Then for every* $m = 0, \dots, l$ *the derivative* $\delta_z^{(m)} \in H_K$ *in the following sense.*

There exists a sequence $(\omega_{z,\varepsilon_n}(x))_{n=1}^{\infty}$, $0 < \varepsilon_n \to 0$, *of functions* $\omega_{z,\varepsilon_n} \in \Phi$ *with support belonging to* $\{x \in I \mid |x-z| \le \varepsilon_n\}$ *for which their derivatives* $\omega_{z,\varepsilon_n}^{(m)} \in \Phi \subset H_K$ *tend in* Φ' *to* $\delta_z^{(m)}$ *and tend in* H_K *to some vector from* H_K *which is also denoted by* $\delta_z^{(m)}$.

Suppose that the condition about smoothness of K *is fulfilled for* $z, \zeta \in I$. *Then the following formula holds:*

$$(\delta_z^{(j)}, \delta_\zeta^{(k)})_{H_K} = (-1)^{j+k}\left(\frac{\partial^{j+k}}{\partial x^k\,\partial y^j} K\right)(\zeta, z), \quad j, k = 0, \dots, l. \tag{4.2}$$

Proof. Denote by $\omega_{z,\varepsilon}(x)$ the nonnegative function from $C_{\text{fin}}^{\infty}(I) = \Phi$ which is equal to zero for $|x-z| \ge \varepsilon > 0$ and for which $\int_I \omega_{z,\varepsilon}(x)\,dx = 1$. Let $0 < \varepsilon_n \to 0$, $n \to \infty$. Then using the integration by parts formula we conclude that $\forall\, \varphi \in \Phi$

$$(\omega_{z,\varepsilon_n}^{(m)}, \varphi)_{L^2} \to (-1)^m\, \overline{\varphi^{(m)}(z)},$$

i.e. $\omega_{z,\varepsilon_n}^{(m)} \to \delta_z^{(m)}$ in the sense of chain (4.1).

On the other hand, with the help of integration by parts formula taking into account the smoothness of K near point $\langle z, z\rangle$ we get

$$\lim_{n,p\to\infty} (\omega_{z,\varepsilon_n}^{(m)}, \omega_{z,\varepsilon_p}^{(m)})_{H_K} =$$

$$= \lim_{n,p\to\infty} \iint\limits_{I\times I} \left(\frac{\partial^{2m}}{\partial x^m\,\partial y^m} K\right)(x, y)\, \omega_{z,\varepsilon_n}(y)\, \omega_{z,\varepsilon_p}(x)\, dx\, dy = \left(\frac{\partial^{2m}}{\partial x^m\,\partial y^m} K\right)(z, z). \tag{4.3}$$

It follows from this fromula that $\|\omega_{z,\varepsilon_n}^{(m)} - \omega_{z,\varepsilon_p}^{(m)}\|_{H_K}^2 \to 0$ if $n, p \to \infty$, i.e. the sequence $(\omega_{z,\varepsilon_n}^{(m)})_{n=1}^{\infty}$ is fundamental and, therefore, tends to some vector from H_K which we will denote by $\delta_z^{(m)}$.

For the proof of formula (4.2), it is necessary to calculate

$$(\delta_z^{(j)}, \delta_\zeta^{(k)})_{H_K} = \lim_{n,p\to\infty} (\omega_{z,\varepsilon_n}^{(m)}, \omega_{z,\varepsilon_p}^{(m)})_{H_K}.$$

This calculation is the same as (4.3) and gives (4.2). □

Note that *the vectors* δ_z *where* z *runs over a dense set from* I, *make a total set in* H_K (a proof is found in [1], Chapter 8, § 3, Subsect. 7).

We will apply now the above stated results to a g.T. kernel K for which the functions $k_{11}(t)$, $k_{22}(t)$, $t \in (-l, l)$, are infinitely differentiable in some neighborhood of 0. Formula (1.2) shows that in this case for every $z \in I$, $z \ne 0$, the function $I \times I \ni \langle x, y\rangle \mapsto K(x, y) \in \mathbb{C}^1$ is infinitely differentiable in some neighborhood of the point $\langle z, z\rangle$ and, therefore, Lemma 4.1 is now applicable for every such z and $m = 0, 1, \dots$.

Consider the operator A, which is equal to the closure of operator (2.9) in H_K. It is easy to understand that *every* $\delta_z^{(m)}$, *where* $z \in I$, $z \ne 0$, *belongs to* $\operatorname{Dom}(A)$ *and*

$$A\,\delta_z^{(m)} = -i\,\delta_z^{(m+1)}, \quad m = 0, 1, \dots. \tag{4.4}$$

Indeed, for n sufficiently large, the functions from Lemma 4.1 $\omega_{z,\varepsilon_n}^{(m)} \in C_0^{\infty}(I) = \operatorname{Dom}(A')$ and $A'\omega_{z,\varepsilon_n}^{(m)} = -i\,\omega_{z,\varepsilon_n}^{(m+1)}$. According to this lemma, in the space H_K, $\lim_{n\to\infty} \omega^{(m)}_{z,\varepsilon_n} = \delta^{(m)}_z$ and $\lim_{n\to\infty} A'\omega^{(m)}_{z,\varepsilon_n} = -i\delta^{(m+1)}_z$. This means that $\delta^{(m)}_z$ belongs to $\operatorname{Dom}(\tilde{A}')$ and $\tilde{A}'\delta^{(m)}_z = -i\delta^{(m+1)}_z$, i.e. (4.4) is proved.

From this assertion we conclude that $\forall z \in I,\ z \neq 0$,

$$\delta_z \in \bigcap_{m=1}^{\infty} \operatorname{Dom}(A^m) \quad \text{and} \quad A^m \delta_z = (-i)^m \delta^{(m)}_z. \tag{4.5}$$

Calculate the norms $\|A^m \delta_z\|_{H_K}$. Let $z \in I,\ z \neq 0$. Then, according to (4.5), (4.2) and (1.2), we get:

$$\forall z > 0 \quad \|A^m \delta_z\|^2_{H_K} = (\delta^{(m)}_z, \delta^{(m)}_z)_{H_K} = \left(\frac{\partial^{2m}}{\partial x^m \partial y^m} K\right)(z,z) = (-1)^m k^{(2m)}_{11}(0); \tag{4.6}$$

$$\forall z < 0 \quad \|A^m \delta_z\|^2_{H_K} = (-1)^m k^{(2m)}_{22}(0), \quad m = 0, 1, \ldots .$$

For the proof of Theorem 2, we now apply the quasianalytical criterion of selfadjointness (see e.g. [14], Chapter 13, § 9): *let A be a closed Hermitian operator in a Hilbert space H, a vector $f \in \bigcap_{m=1}^{\infty} \operatorname{Dom}(A^m)$ is called quasianalytical if*

$$\sum_{m=1}^{\infty} \|A^m f\|_H^{-1/m} = \infty. \tag{4.7}$$

The operator A is selfadjoint iff H contains a total set of its quasianalytical vectors.

Taking into account that the set $\{\delta_z \mid z \in I,\ z \neq 0\}$ is total in H_K (see above) and formulas (4.6), (4.7) we can assert the selfadjointness of the operator A if condition (1.6) is fulfilled:

$$\sum_{m=1}^{\infty} \left((-1)^m k^{(2m)}_{\alpha\alpha}(0)\right)^{-1/2m} = \infty, \quad \alpha = 1, 2. \quad \square$$

5. Remarks to the case $l = \infty$. In this case, instead of chain (2.3), it is necessary to construct a longer chain (see [13], Chapter 5, § 5, Subsect. 5.1 and also [1], Chapter 8, § 1).

Namely, let $\mathbb{R}^1 \ni p(x) \geq 1$ be an infinitely differentiable weight for which $p^{-1}(x)$ is integrable on $\mathbb{R}^1$ with respect to the Lebesgue measure dx. In the notations of the above cited book, we construct the chain

$$G_- \supset G_0 \supset G_+, \quad G_0 = L^2(\mathbb{R}^1, dx), \quad G_+ = L^2(\mathbb{R}^1, p(x)\,dx). \tag{5.1}$$

The g.T. kernel is, as earlier, bounded, therefore $K \in G_- \otimes G_-$ and we can construct, instead of (2.1), the quasiscalar product

$$(f, g)_{H_K} = \iint_{\mathbb{R}^1 \times \mathbb{R}^1} K(x,y) f(y) \overline{g(x)}\, dx\, dy, \quad f, g \in G_+, \tag{5.2}$$

generating the Hilbert space H_K.

The role of chain (2.3) is now played by the following chain with the zero space $H_0 = G_0$:

$$H_- \supset H_0 \supset H_+. \tag{5.3}$$

Here $H_+ = W_{2,0}^1(\mathbb{R}^1, q(x)\,dx)$ is the subspace consisting of the functions from $W_2^1(\mathbb{R}^1, q(x)\,dx)$ which are equal to zero at the point $x = 0$. Here $q(x) \geq p(x)$, $x \in \mathbb{R}^1$, is an infinitely differentiable weight with properties: $\exists\, C > 0$ such that

$$|q'(x)| \leq C q(x), \quad x \in \mathbb{R}^1, \quad \text{and} \quad \int_{\mathbb{R}^1} \frac{p(x)}{q(x)}\,dx < \infty.$$

It follows from [12] (Chapter 1, § 3, Theorem 3.6) that the imbedding $W_2^1(\mathbb{R}^1, q(x)\,dx) \hookrightarrow L^2(\mathbb{R}^1, p(x)\,dx)$ is quasinuclear. Therefore the imbedding $H_+ \hookrightarrow G_+$ will also be quasinuclear.

Introduce the space $H_{K,+}$ as a space of classes $\hat{u}$, with respect to (5.2), of functions $u \in H_+$ and construct the chain

$$H_{K,-} \supset H_K \supset H_{K,+}. \tag{5.4}$$

The imbedding $H_{K,+}$ is now also quasinuclear and we can take this rigging (5.4) instead of (2.4).

After such a change the proof of Theorem 1 is preserved.

The formulation of Theorem 2 in the case $l = \infty$ is the same as in the case $l < \infty$. Its proof is also preserved, but it is necessary to use chains (5.3), (5.4) instead of (2.3), (2.4).

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