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# $p$-ADIC MARKOV PROCESS AND THE PROBLEM OF THE FIRST RETURN OVER BALLS

# $p$-АДИЧНІ МАРКОВСЬКІ ПРОЦЕСИ ТА ЗАДАЧА ПЕРШОГО ПОВЕРНЕННЯ ДЛЯ КУЛЬ

We consider the pseudodifferential operator defined as $H^{\alpha}\varphi = \mathcal{F}^{-1}[(\langle\xi\rangle^{\alpha} - p^{r\alpha})\mathcal{F}\varphi]$, where $\langle\xi\rangle^{\alpha} = (\max\{|\xi|_p, p^r\})^{\alpha}$ and study the Markov process associated to this operator. We also study the first passage time problem associated to $H^{\alpha}$ for $r < 0$.

Розглядається псевдодиференціальний оператор $H^{\alpha}\varphi = \mathcal{F}^{-1}[(\langle\xi\rangle^{\alpha} - p^{r\alpha})\mathcal{F}\varphi]$, де $\langle\xi\rangle^{\alpha} = (\max\{|\xi|_p, p^r\})^{\alpha}$, та вивчається пов'язаний із цим оператором марковський процес. Також вивчається задача часу першого проходу для $H^{\alpha}$ при $r < 0$.

**1. Introduction.** Avetisov et al. have constructed a wide variety of models of ultrametric diffusion constrained by hierarchical energy landscapes (see [2, 3]). From a mathematical point of view, in these models the time-evolution of a complex system is described by a $p$-adic master equation (a parabolic-type pseudodifferential equation) which controls the time evolution of a transition function of a random walk on an ultrametric space, and the random walk describes the dynamics of the system in the space of configurational states which is approximated by an ultrametric space ($\mathbb{Q}_p$).

The problem of the first return in dimension 1 was studied by Avetisov, Bikulov and Zubarev in [4, 5], and in arbitrary dimension by Chacón-Cortés, Torresblanca-Badillo and Zúñiga-Galindo in [8, 15]. In these articles, pseudodifferential operators with radial symbols were considered. More recently, Chacón-Cortés [7] considers pseudodifferential operators over $\mathbb{Q}_p^4$ with nonradial symbol; he studies the problem of first return for a random walk $X(t, w)$ whose density distribution satisfies certain diffusion equation.

In this paper we define the operator

$$H^{\alpha}\varphi = \mathcal{F}^{-1}[(\langle\xi\rangle^{\alpha} - p^{r\alpha})\mathcal{F}\varphi]$$

for $\varphi \in \mathbf{S}(\mathbb{Q}_p)$, where $\langle\xi\rangle = \max\{|\xi|_p, p^r\}$. We also define the heat-kernel $Z_r$ as

$$Z_r(x,t) := \int\limits_{\mathbb{Q}_p} \chi(-x\xi)\, e^{-t\left(\langle\xi\rangle^{\alpha} - p^{r\alpha}\right)}\, d\xi. \tag{1.1}$$

Heat kernels of this type have been studied in [6], where it is shown that function

$$u(x,t) = Z_r(x,t) * \Omega(|x|_p) = \int\limits_{\mathbb{Q}_p} \chi(-x\xi) e^{-t(\langle\xi\rangle^{\alpha} - p^{r\alpha})}\Omega(|\xi|_p)\, d\xi$$

is a solution of the Cauchy problem

$$u \in C\big([0,\infty], \mathbf{S}(\mathbb{Q}_p)\big) \cap C^1\big([0,\infty], L^2(\mathbb{Q}_p)\big),$$

$$\frac{\partial u}{\partial t}(x,t) + \big(H^\alpha u\big)(x,t) = 0, \qquad x \in \mathbb{Q}_p, \quad t \in (0,T], \quad \alpha > 0,$$

$$u(x,0) = \Omega(|x|_p).$$

We show that $Z_r(x,t)$ is the transition density of a time and space homogeneous Markov process, which is bounded, right-continuous and has no discontinuities other than jumps, see Theorem 4.1.

In [12] Kochubei considers the Vladimirov operator restricted to a ball $B_N$ and studies a Cauchy problem. Despite he uses a different approach to the one given by Casas-Sánchez and Rodríguez-Vega in [6], the kernel $Z_r$ (1.1) is the same. On the other hand, Khrennikov and Kochubei (see [11]) show that the family of operators $Z_r * \cdot$ is a strongly continuous contraction semigroup on $L^1(B_r)$.

Among other properties, the kernel $Z_r(x,t)$ vanishes outside the ball of radius $p^{-r}$, which implies that the process remains supported in the same ball. For that reason, we are interested in the case $r < 0$, and thus $\mathbb{Z}_p \subseteq B_{-r}$. In these conditions it is possible to study the problem of the first return by a trajectory of the stochastic process entering the unit ball. In order to solve this problem we demand that $r < 0$ and the natural answer is that the trajectory is always recurrent. Observe that this problem is different to the one solved by Bikulov in [5], where the author define the stochastic quantity as the first passage time entering some domain $B_r(a)$, since his solution is not bounded, the answer depends on the range of $\alpha$, whereas we do not have conditions on $\alpha$. Our work can be seen as a continuation of the problem of first return for a stochastic process, considered by Avetisov in [4], since we use the same techniques, but a different symbol.

The article is organized as follows. In Section 2, we collect some facts about $p$-adic numbers. In Section 3, we define the pseudodifferential operator, we show it has an integral representation and solve the Cauchy problem based on the results of [6]. Section 4 is dedicated to the $p$-adic Markov process over balls. In Section 5, we study the problem of the first passage time entering the domain $\mathbb{Z}_p$, we conclude that the process is always recurrent with respect to the unit ball, see Theorem 5.1.

**2. Preliminaries.** In this section, we fix the notation and collect some $p$-adic facts that we will use through the article. For a detailed exposition on $p$-adic analysis the reader may consult [1, 14, 16].

***2.1. The field of p-adic numbers.*** Along this article $p$ will denote a prime number. The field of $p$-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0, & \text{if} \quad x = 0, \\ p^{-\gamma}, & \text{if} \quad x = p^r \dfrac{a}{b}, \end{cases}$$

where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := \operatorname{ord}(x)$, with $\operatorname{ord}(0) := +\infty$, is called the *$p$-adic order of $x$*.

Any $p$-adic number $x \neq 0$ has a unique expansion $x = p^{\operatorname{ord}(x)} \sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0, & \text{if} \quad x = 0 \quad \text{or} \quad \operatorname{ord}(x) \geq 0, \\ p^{\operatorname{ord}(x)} \sum_{j=0}^{-\operatorname{ord}(x)-1} x_j p^j, & \text{if} \quad \operatorname{ord}(x) < 0. \end{cases}$$

For $r \in \mathbb{Z}$, denote by $B_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq p^r\}$ *the ball of radius* $p^r$ *with center at* $a \in \mathbb{Q}_p$, and take $B_r(0) := B_r$.

***2.2. The Bruhat–Schwartz space.*** A complex-valued function $\varphi$ defined on $\mathbb{Q}_p$ is *called locally constant* if for any $x \in \mathbb{Q}_p$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for } x' \in B_{l(x)}. \tag{2.1}$$

The space of locally constant functions is denoted by $\mathcal{E}(\mathbb{Q}_p)$. A function $\varphi\colon \mathbb{Q}_p \to \mathbb{C}$ is called a *Bruhat–Schwartz function* (*or a test function*) if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat–Schwartz functions is denoted by $\mathbf{S}(\mathbb{Q}_p)$. For $\varphi \in \mathbf{S}(\mathbb{Q}_p)$, the largest of such numbers $l = l(\varphi)$ satisfying (2.1) is called *the exponent of local constancy of* $\varphi$.

Let $\mathbf{S}'(\mathbb{Q}_p)$ denote the set of all functionals (distributions) on $\mathbf{S}(\mathbb{Q}_p)$. All functionals on $\mathbf{S}(\mathbb{Q}_p)$ are continuous.

Set $\chi(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e., a continuous map from $\mathbb{Q}_p$ into $S$ (the unit circle) satisfying $\chi(y_0 + y_1) = \chi(y_0)\chi(y_1)$, $y_0, y_1 \in \mathbb{Q}_p$.

***2.3. Fourier transform.*** Given $\xi$ and $x \in \mathbb{Q}_p$, the Fourier transform of $\varphi \in \mathbf{S}(\mathbb{Q}_p)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int\limits_{\mathbb{Q}_p} \chi(\xi x)\varphi(x)dx \qquad \text{for} \qquad \xi \in \mathbb{Q}_p,$$

where $dx$ is the Haar measure on $\mathbb{Q}_p$ normalized by the condition $\operatorname{vol}(B_0) = 1$. The Fourier transform is a linear isomorphism from $\mathbf{S}(\mathbb{Q}_p)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$. We will also use the notation $\mathcal{F}_{x\to\xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of $\varphi$.

The Fourier transform $\mathcal{F}[f]$ of a distribution $f \in \mathbf{S}'(\mathbb{Q}_p)$ is defined by

$$(\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi]) \qquad \text{for all} \qquad \varphi \in \mathbf{S}(\mathbb{Q}_p).$$

The Fourier transform $f \to \mathcal{F}[f]$ is a linear isomorphism from $\mathbf{S}'(\mathbb{Q}_p)$ onto $\mathbf{S}'(\mathbb{Q}_p)$. Furthermore, $f = \mathcal{F}[\mathcal{F}[f](-\xi)]$.

**3. Pseudodifferential operators.**

**Definition 3.1.** *For all* $\alpha \in \mathbb{C}$, *we define the following pseudodifferential operator*:

$$H^\alpha\varphi = \mathcal{F}^{-1}[(\langle\xi\rangle^\alpha - p^{r\alpha})\mathcal{F}\varphi], \qquad \varphi \in \mathbf{S}(\mathbb{Q}_p),$$

*where* $\langle\xi\rangle^\alpha = \max\{|\xi|_p, p^r\}$.

It is clear that the map $H^\alpha\colon \mathbf{S}(\mathbb{Q}_p) \to \mathbf{S}(\mathbb{Q}_p)$ is continuous. Also it is possible to show that the pseudodifferential operator $H^\alpha$ has the following integral representation:

$$(H^\alpha\varphi)(x) =$$

$$= \frac{1-p^{\alpha}}{1-p^{\alpha+1}} \left[ p^{r(\alpha+1)} \int\limits_{|y|_p \le p^{-r}} (\varphi(x-y) - \varphi(x))dy - \right.$$

$$\left. - p^{(\alpha+1)} \int\limits_{|y|_p \le p^{-r}} \frac{\varphi(x-y)-\varphi(x)}{|y|_p^{\alpha+1}}\, dy \right].$$

**Definition 3.2.** *Set* $\alpha_k := \dfrac{2k\pi i}{ln\ p}, \ \ k \in \mathbb{Z},$

$$K_\alpha(x) := \begin{cases} \left[ \dfrac{1-p^{\alpha}}{1-p^{-\alpha-1}} |x|_p^{-\alpha-1} + p^{r(\alpha+1)} \dfrac{1-p^{\alpha}}{1-p^{\alpha+1}} \right] \Omega(p^r|x|_p) & \textit{for } \alpha \neq -1+\alpha_k, \\ (1-p^{-1})\Omega(p^r|x|_p)((1-r) - \log_p |x|_p) & \textit{for } \alpha = -1+\alpha_k, \end{cases}$$

*and, for* $\alpha = 0$, *we define* $K_0 = \delta$.

After some calculations it is possible to show the following result.

**Theorem 3.1.** *The Fourier transform (as a distribution) of* $K_\alpha$ *is given by* $\langle \xi \rangle^{\alpha}$ *for all* $\alpha \in \mathbb{C}$.

**Definition 3.3.** *For* $x \in \mathbb{Q}_p, \ t \in \mathbb{R}$, *the heat kernel is defined as*

$$Z_r(x,t) := \int\limits_{\mathbb{Q}_p} \chi(-x\xi)\, e^{-t\left(\langle \xi \rangle^{\alpha} - p^{r\alpha}\right)}\, d\xi.$$

The following properties are proved in [6].

**Lemma 3.1.** *For* $\alpha > 0, \ t > 0$, *the following assertions hold*:

(1) $Z_r(x,t) \in C(\mathbb{Q}_p, \mathbb{R}) \cap L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$ *for* $t > 0$,

(2) $Z_r(x,t) \ge 0$ *for all* $x \in \mathbb{Q}_p$,

(3) $\displaystyle\int_{\mathbb{Q}_p} Z_r(x,t)\, dx = \int_{|x|_p \le p^{-r}} Z_r(x,t)\, dx = 1,$

(4) $\lim_{t \to 0^+} Z_r(x,t) * \varphi(x) = \varphi(x)$ *for* $\varphi \in \mathbf{S}(\mathbb{Q}_p)$,

(5) $Z_r(x,t) * Z(x,t') = Z(x, t+t')$ *for* $t, \ t' > 0$,

(6) $Z_r(x,t) \le Ct|x|_p^{-1} \left( \langle px^{-1} \rangle^{\alpha} - p^{r\alpha} \right).$

Observe that thanks to (3) the heat kernel is concentrated in the ball $B_{-r}$.

If we set, for $\varphi \in \mathbf{S}(\mathbb{Q}_p)$,

$$u(x,t) := \begin{cases} Z_r(x,t) * \varphi(x), & \text{if } t > 0, \\ \varphi(x), & \text{if } t = 0, \end{cases} \tag{3.1}$$

then it is easy to see that $u(x,t) \in \mathbf{S}(\mathbb{Q}_p)$ for $t \ge 0$, and also it is possible to show that, for $t \ge 0$, $\alpha > 0$,

$$H^{\alpha}(u(x,t)) = \mathcal{F}^{-1}_{\xi \to x} \left[ (\langle \xi \rangle^{\alpha} - p^{r\alpha}) e^{-t(\langle \xi \rangle^{\alpha} - p^{r\alpha})} \widehat{\varphi}(\xi) \right].$$

**Theorem 3.2.** *Consider the following Cauchy problem*:

$$u \in C\big([0,\infty], \mathbf{S}(\mathbb{Q}_p)\big) \cap C^1\big([0,\infty], L^2(\mathbb{Q}_p)\big),$$

$$\frac{\partial u}{\partial t}(x,t) + \big(H^\alpha u\big)(x,t) = 0, \qquad x \in \mathbb{Q}_p, \quad t \in (0,T], \quad \alpha > 0,$$

$$u(x,0) = \varphi(x), \qquad \varphi \in \mathbf{S}(\mathbb{Q}_p).$$

*Then the function* $u(x,t)$ *defined in* (3.1) *is the solution.*

***Proof.*** See Theorem 3.14 in [6].

Another interpretation for the fundamental solution $Z_r$ was obtained in [12].

**4. *p*-Adic Markov process over balls.** The space $(\mathbb{Q}_p, |\cdot|_p)$ is a complete non-Archimedean metric space. Let $\mathcal{B}$ be the Borel $\sigma$-algebra of $p^r\mathbb{Z}_p$; thus $(p^r\mathbb{Z}_p, \mathcal{B}, dx)$ is a measure space. By using the terminology and results of [9] (Chapters 2, 3), we set

$$p(t,x,y) := Z_r(x-y,t), \quad t > 0, \;\; x, y \in p^r\mathbb{Z}_p,$$

and

$$P(t,x,B) = \begin{cases} \displaystyle\int_B p(t,x,y)\,dy & \text{for } t > 0, \quad x \in \mathbb{Q}_p, \quad B \in \mathcal{B}, \\ 1_B(x) & \text{for } t = 0. \end{cases}$$

In this case the Markov process remains in the ball $p^r\mathbb{Z}_p,\ r < 0$.

**Lemma 4.1.** *With the above notation the following assertions hold*:

(i) $p(t,x,y)$ *is a normal transition density*,

(ii) $P(t,x,B)$ *is a normal transition function*.

***Proof.*** The result follows from Lemma 3.1 (see [9] (Section 2.1) for further details).

**Lemma 4.2.** *The transition function* $P(t,x,B)$ *satisfies the following two conditions*:

(i) *for each* $u \geq 0$ *and compact* $B$,

$$\lim_{|x|_p \to \infty} \sup_{t \leq u} P(t,x,B) = 0;$$

(ii) *for each* $\epsilon > 0$ *and compact* $B$,

$$\lim_{t \to 0^+} \sup_{x \in B} P(t,x,\mathbb{Q}_p \setminus B_\epsilon(x)) = 0.$$

***Proof.*** (i) By Lemma 3.1 (6), we have

$$P(t,x,B) = \int\limits_B Z_r(x-y,t)dy \leq Ct \int\limits_B |x-y|_p^{-1} \left(\left\langle p(x-y)^{-1}\right\rangle^\alpha - p^{r\alpha}\right) dy =$$

$$= Ct|x|_p^{-1}\left(\left\langle px^{-1}\right\rangle^\alpha - p^{r\alpha}\right) \int\limits_B dy,$$

since, for $x \in \mathbb{Q}_p \setminus B$, we have $|x|_p = |x-y|_p$. Therefore, $\lim_{|x|_p \to \infty} \sup_{t \leq u} P(t,x,B) = 0$.

(ii) By using Lemma 3.1 (6), $\alpha > 0$, we have

$$P(t, x, \mathbb{Q}_p \setminus B_\epsilon(x)) \leq Ct \int\limits_{|x-y|>\epsilon} |x-y|_p^{-1} \left(\left\langle p(x-y)^{-1}\right\rangle^\alpha - p^{r\alpha}\right) dy =$$

$$= Ct \int\limits_{|z|>\epsilon} |z|_p^{-1} \left(\left\langle p\, z^{-1}\right\rangle^\alpha - p^{r\alpha}\right) dz.$$

If $p^{-r-1} \leq \epsilon < |z|_p$ or $\epsilon < p^{-r-1} \leq |z|_p$, then $\left\langle p\, z^{-1}\right\rangle^\alpha = p^{r\alpha}$ and

$$\int\limits_{|z|>\epsilon} |z|_p^{-1} \left(\left\langle p\, z^{-1}\right\rangle^\alpha - p^{r\alpha}\right) dz = 0.$$

Thus,

$$P(t, x, \mathbb{Q}_p \setminus B_\epsilon(x)) \leq Ct \int\limits_{|x-y|>\epsilon} |x-y|_p^{-1} \left(\left\langle p(x-y)^{-1}\right\rangle^\alpha - p^{r\alpha}\right) dy =$$

$$= Ct \int\limits_{p^{-r-1}>|z|>\epsilon} |z|_p^{-1} \left(|p\, z^{-1}|_p^\alpha - p^{r\alpha}\right) dz \leq$$

$$\leq Ctp^{-1} \int\limits_{p^{-r-1}>|z|>\epsilon} |z|_p^{-1-\alpha} dz =$$

$$= Ctp^{-1} C_1.$$

Therefore,

$$\lim_{t\to 0^+} \sup_{x\in B} P(t, x, \mathbb{Q}_p \setminus B_\epsilon(x)) = 0.$$

Lemma 4.2 is proved.

**Theorem 4.1.** *$Z_r(x,t)$ is the transition density of a time and space homogeneous Markov process in $p^r\mathbb{Z}_p$, called $\mathfrak{T}(t,\omega)$, which is bounded, right-continuous and has no discontinuities other than jumps.*

***Proof.*** The result follows from [9] (Theorem 3.6) by using that $(\mathbb{Q}_p, |x|_p)$ is a semicompact space, i.e., a locally compact Hausdorff space with a countable base, and $P(t,x,B)$ is a normal transition function satisfying conditions (i) and (ii).

**5. The first passage time.** From now on, we assume $r < 0$ and we study the problem of the first return to the domain $\mathbb{Z}_p$.

By Theorem 3.2, the function

$$u(x,t) = Z_r(x,t) * \Omega(|x|_p) = \int\limits_{\mathbb{Q}_p} \chi(-x\xi) e^{-t(\langle\xi\rangle^\alpha - p^{r\alpha})} \Omega(|\xi|_p)\, d\xi \tag{5.1}$$

is a solution of

$$\frac{\partial u}{\partial t}(x,t) + (H^{\alpha}u)(x,t) = 0, \qquad x \in \mathbb{Q}_p, \quad t > 0,$$

$$u(x,0) = \Omega(|x|_p).$$

Among other properties, the function $u(x,t) = Z_r(x,t) * \Omega(|x|_p),\ t \geq 0,$ is pointwise differentiable in $t$ and, by using the dominated convergence theorem, we can show that its derivative is given by the formula

$$\frac{\partial u}{\partial t}(x,t) = \int\limits_{\mathbb{Q}_p} \chi_p(-x\xi)\,(\langle\xi\rangle^{\alpha} - p^{r\alpha})\,e^{-t(\langle\xi\rangle^{\alpha} - p^{r\alpha})}\Omega(|\xi|_p)\,d\xi. \tag{5.2}$$

**Lemma 5.1.** *If* $\alpha > 0$ *and* $r < 0$, *then*

$$0 < -\int\limits_{1<|y|_p\leq p^{-r}} K_{\alpha}(y)dy < 1.$$

***Proof.*** We have

$$-\int\limits_{1<|y|_p\leq p^{-r}} K_{\alpha}(y)dy =$$

$$= \frac{1-p^{\alpha}}{1-p^{\alpha+1}}\left[p^{\alpha+1}\int\limits_{1<|y|_p\leq p^{-r}} \frac{1}{|y|_p^{\alpha+1}}dy - p^{r(\alpha+1)}\int\limits_{1<|y|_p\leq p^{-r}} dy\right] <$$

$$< \frac{1-p^{\alpha}}{1-p^{\alpha+1}}\left[p^{\alpha+1}\int\limits_{1<|y|_p} \frac{1}{|y|_p^{\alpha+1}}dy - p^{r(\alpha+1)}(p^{-r}-1)\right] =$$

$$= \frac{1-p^{-1}}{1-p^{-\alpha-1}} - \frac{1-p^{\alpha}}{1-p^{\alpha+1}}p^{r\alpha}(1-p^{r}) =$$

$$= 1 - \frac{1-p^{\alpha}}{1-p^{\alpha+1}}\left(1 + p^{r\alpha}(1-p^{r})\right) < 1.$$

Now

$$-\int\limits_{1<|y|_p\leq p^{-r}} K_{\alpha}(y)dy =$$

$$= \frac{1-p^{\alpha}}{1-p^{\alpha+1}}\left[p^{\alpha+1}\int\limits_{1<|y|_p\leq p^{-r}} \frac{1}{|y|_p^{\alpha+1}}dy - p^{r(\alpha+1)}\int\limits_{1<|y|_p\leq p^{-r}} dy\right] >$$

$$> \frac{p^{\alpha}(p-1)(1-p^{r\alpha})}{p^{\alpha+1}-1} + \frac{p^{r\alpha}(1-p^{\alpha})}{p^{\alpha+1}-1} > 0.$$

Lemma 5.1 is proved.

The rest of this section is dedicated to the study of the following random variable, by using the same techniques given in [4].

**Definition 5.1.** *The random variable* $\tau_{\mathbb{Z}_p}(\omega) : \mathfrak{Y} \to \mathbb{R}_+$ *defined by*

$$\inf\{t > 0 \mid \mathfrak{T}(t,\omega) \in \mathbb{Z}_p \textit{ and there exists } t' \textit{ such that } 0 < t' < t \textit{ and } \mathfrak{T}(t',\omega) \notin \mathbb{Z}_p\}$$

*is called the first passage time of a path of the random process* $\mathfrak{T}(t,\omega)$ *entering the domain* $\mathbb{Z}_p$.

**Lemma 5.2.** *The probability density function for a path of* $\mathfrak{T}(t,\omega)$ *to enter into* $\mathbb{Z}_p$ *at the instant of time* $t$, *with the condition that* $\mathfrak{T}(0,\omega) \in \mathbb{Z}_p$ *is given by*

$$g(t) = \int\limits_{1<|y|_p\leq p^{-r}} K_\alpha(y) u(y,t) dy. \tag{5.3}$$

***Proof.*** We first note that, for $x, y \in \Omega(|z|_p)$, we have

$$u(x-y,t) = \int\limits_{\mathbb{Z}_p} \chi_p(-(x-y)\cdot\xi) e^{-t(\langle\xi\rangle^\alpha - p^{r\alpha})}\, d\xi =$$

$$= \int\limits_{\mathbb{Z}_p} e^{-t(\langle\xi\rangle^\alpha - p^{r\alpha})}\, d\xi = \int\limits_{\mathbb{Z}_p} \chi_p(-x\cdot\xi) e^{-t(\langle\xi\rangle^\alpha - p^{r\alpha})}\, d\xi =$$

$$= u(x,t),$$

i.e., $u(x-y,t) - u(x,t) \equiv 0$ for $x,\ y \in \mathbb{Z}_p$.

The survival probability, by definition

$$S(t) := S_{\Omega(|x|_p)}(t) = \int\limits_{\mathbb{Z}_p} u(x,t) d^n x,$$

is the probability that a path of $\mathfrak{T}(t,\omega)$ remains in $\mathbb{Z}_p$ at the time $t$. Because there are no external or internal sources,

$$S'(t) = \begin{array}{l}\text{Probability that a path of } \mathfrak{T}(t,\omega)\\ \text{goes back to } \mathbb{Z}_p \text{ at the time } t\end{array} - \begin{array}{l}\text{Probability that a path of } \mathfrak{T}(t,\omega)\\ \text{exits } \mathbb{Z}_p \text{ at the time } t\end{array} =$$

$$= g(t) - C\cdot S(t) \text{ with } 0 < C \leq 1.$$

By using the derivative (5.2), we have

$$S'(t) = \int\limits_{\mathbb{Z}_p} \frac{\partial u(x,t)}{\partial t} dx =$$

$$= -\frac{1-p^\alpha}{1-p^{\alpha+1}} \left[ p^{r(\alpha+1)} \int\limits_{|x|_p\leq 1} \int\limits_{1<|y|_p\leq p^{-r}} (u(x-y,t) - u(x,t)) dy dx - \right.$$

$$-p^{(\alpha+1)} \int\limits_{|x|_p\le 1}\int\limits_{1<|y|_p\le p^{-r}} \frac{u(x-y,t)-u(x,t)}{|y|_p^{\alpha+1}}\,dydx\Bigg] =$$

$$= -\frac{1-p^{\alpha}}{1-p^{\alpha+1}}\Bigg[p^{r(\alpha+1)} \int\limits_{|x|_p\le 1}\int\limits_{1<|y|_p\le p^{-r}} u(x-y,t)\,dydx -$$

$$-p^{(\alpha+1)} \int\limits_{|x|_p\le 1}\int\limits_{1<|y|_p\le p^{-r}} \frac{u(x-y,t)}{|y|_p^{\alpha+1}}\,dydx\Bigg] +$$

$$+\frac{1-p^{\alpha}}{1-p^{\alpha+1}}\Bigg[p^{r(\alpha+1)} \int\limits_{|x|_p\le 1}\int\limits_{1<|y|_p\le p^{-r}} u(x,t)\,dydx -$$

$$-p^{(\alpha+1)} \int\limits_{|x|_p\le 1}\int\limits_{1<|y|_p\le p^{-r}} \frac{u(x,t)}{|y|_p^{\alpha+1}}\,dydx\Bigg].$$

Now if $y\in\Omega(p^r|y|_p)\setminus\Omega(|y|_p)$ and $x\in\Omega(|x|_p)$, then $u(x-y,t)=u(y,t)$, consequently,

$$S'(t) = \int\limits_{1<|y|_p\le p^{-r}} K_\alpha(y)u(y,t)dy +$$

$$+ \int\limits_{1<|y|_p\le p^{-r}} K_\alpha(y)dy \int\limits_{|x|_p\le 1} u(x,t)\,dx =$$

$$= g(t) - CS(t),$$

where

$$C = -\int\limits_{1<|y|_p\le p^{-r}} K_\alpha(y)dy.$$

Lemma 5.2 is proved.

**Proposition 5.1.** *The probability density function $f(t)$ of the random variable $\tau_{\mathbb{Z}_p}(\omega)$ satisfies the nonhomogeneous Volterra equation of the second kind*

$$g(t) = \int\limits_0^\infty g(t-\tau)f(\tau)d\tau + f(t).$$

***Proof.*** The result follows from Lemma 5.2 by using the argument given in the proof of Theorem 1 in [4].

**Proposition 5.2.** *The Laplace transform $G_r(s)$ of $g(t)$ is given by*

$$G_r(s) = \int\limits_{1<|y|_p\leq p^{-r}} K_\alpha(y) \int\limits_{|\xi|_p\leq 1} \frac{\chi_p(-\xi\cdot y)}{s+(\langle\xi\rangle^\alpha - p^{r\alpha})}\, d\xi dy.$$

***Proof.*** We first note that

$$e^{-st}K_\alpha(y)e^{-t(\langle\xi\rangle^\alpha - p^{r\alpha})}\Omega(|\xi|_p) \in \mathcal{L}^1\big((0,\infty)\times\Omega(p^r|\xi|_p)\setminus\Omega(|\xi|_p)\times\mathbb{Q}_p, dtdyd\xi\big)$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$. The announced formula follows now from (5.1) and (5.3) by using Fubini's theorem.

**Definition 5.2.** *We say that $\mathfrak{T}(t,\omega)$ is recurrent with respect to $\mathbb{Z}_p$ if*

$$P(\{\omega\in\mathfrak{Y} : \tau_{\mathbb{Z}_p}(\omega)<\infty\}) = 1. \tag{5.4}$$

*Otherwise, we say that $\mathfrak{T}(t,\omega)$ is transient with respect to $\mathbb{Z}_p$.*

The meaning of (5.4) is that every path of $\mathfrak{T}(t,\omega)$ is sure to return to $\mathbb{Z}_p$. If (5.4) does not hold, then there exist paths of $\mathfrak{T}(t,\omega)$ that abandon $\mathbb{Z}_p$ and never go back.

**Theorem 5.1.** *For all $\alpha>0$, the processes $\mathfrak{T}(t,\omega)$ is recurrent with respect to $\mathbb{Z}_p$.*

***Proof.*** By Proposition 5.1, the Laplace transform $F(s)$ of $f(t)$ equals $\dfrac{G_r(s)}{1+G_r(s)}$, where $G_r(s)$ is the Laplace transform of $g(t)$, and thus

$$F(0) = \int\limits_0^\infty f(t)dt = 1 - \frac{1}{1+G_r(0)}.$$

Hence in order to prove that $\mathfrak{T}(t,\omega)$ is recurrent is sufficient to show that

$$G_r(0) = \lim_{s\to 0} G_r(s) = \infty,$$

and to prove that it is transient that

$$G_r(0) = \lim_{s\to 0} G_r(s) < \infty,$$

$$G_r(s) = \int\limits_{1<|y|_p\leq p^{-r}}\ \int\limits_{|\xi|_p\leq p^r} \frac{K_\alpha(y)\chi(-\xi y)}{s} d\xi dy +$$

$$+ \int\limits_{1<|y|_p\leq p^{-r}}\ \int\limits_{p^r<|\xi|_p\leq 1} \frac{K_\alpha(y)\chi(-\xi y)}{s+|\xi|_p^\alpha - p^{r\alpha}} d\xi dy =$$

$$= \frac{p^r}{s}\int\limits_{1<|y|_p\leq p^{-r}} K_\alpha(y)dy + \int\limits_{1<|y|_p\leq p^{-r}}\ \int\limits_{p^r<|\xi|_p\leq 1} \frac{K_\alpha(y)\chi(-\xi y)}{s+|\xi|_p^\alpha - p^{r\alpha}} d\xi dy =$$

$$= \frac{p^r}{s}\int\limits_{1<|y|_p\leq p^{-r}} K_\alpha(y)dy + \sum_{k=1}^{-r}\sum_{m=0}^{k-1} \frac{p^{k-m}}{s+p^{-m\alpha}-p^{r\alpha}} \int\limits_{|u|_p=1} K_\alpha(p^{-k}u)du +$$

$$+\sum_{k=1}^{-r}\sum_{m=k}^{-r-1}\frac{p^{k-m}}{s+p^{-m\alpha}-p^{r\alpha}}\int\limits_{|u|_p=1}\int\limits_{|v|_p=1}K_\alpha(p^{-k}u)\chi(-p^{m-k}uv)dvdu.$$

Therefore, $\lim_{s\to 0}G_r(s)=\infty$ and the process $\mathfrak{T}(t,\omega)$ is recurrent.

Theorem 5.1 is proved.

The meaning of this result is that every path is sure to return to $\mathbb{Z}_p$, this always holds and the process is never transient, this agrees with the fact that the process is concentrated in $p^r\mathbb{Z}_p,\ r<0$.

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