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STOCHASTIC DYNAMICS AS A LIMIT OF HAMILTONIAN DYNAMICS OF HARD SPHERES

СТОХАСТИЧНА ДИНАМІКА ЯК ГРАНИЦЯ ГАМІЛЬТОНОВОЇ ДИНАМІКИ ПРУЖНИХ КУЛЬ

We consider the stochastic dynamics that is the Boltzmann–Grad limit of the Hamiltonian dynamics of a system of hard spheres. A new concept of averages over states of stochastic systems is introduced, in which the contribution of the hypersurfaces on which stochastic point particles interact is taken into account. We give a rigorous derivation of the infinitesimal operators of the semigroups of evolution operators.

Визначена стохастична динаміка, яка є границею Больцмана – Греда від гамільтонової динаміки системи пружних куль. Введено нову концепцію середніх від спостережуваних за станами стохастичних систем. В ньому враховуються вклади від гіперповерхонь, на яких взаємодіють точкові стохастичні частки. Дано строге визначення інфінітезімальних операторов для півгрупи еволюційних операторів.

Introduction. The stochastic dynamics that corresponds to the Boltzmann hierarchy was recently proposed in papers [1, 2].

In the present paper, we prove that the stochastic dynamics is a certain limit of averages over the sphere of the Hamiltonian dynamics of system of hard spheres as their diameter tends to zero (the Boltzmann-Grad limit). We define the domain (set) of interaction in which the stochastic dynamics differs from the Hamiltonian dynamics of the free particles.

By using the concept of the domain of interaction, we define the operator of evolution for the stochastic dynamics and its infinitesimal operator. We prove that the operator of evolution of stochastic particles and its infinitesimal operator are the limits of the averages of the operator of evolution of a system of hard spheres and its infinitesimal operator, respectively, over the sphere as its diameter tends to zero.

Thus, in this paper, we present the rigorous derivation of the new concept of the stochastic dynamics of a system of point-particles as the limit of the average over the sphere of the Hamiltonian dynamics of system of hard spheres as its diameter tends to zero.

The crucial circumstance in this new concept of the stochastic dynamics and associated with it an average of observables over states of system of point-particles, that are governed by the stochastic dynamics, consists in taking into account the hypersurfaces of lower dimension than the phase space. In the traditional statistical mechanics hypersurfaces of lower dimension are neglected. In the next publications we will show that in solutions of the Boltzmann equation and the Boltzmann hierarchy the same hypersurfaces of lower dimension are taken into account. Namely, the terms of the series of iterations for the Boltzmann equation, as well as for the Boltzmann hierarchy, can be represented through the integrals over the hypersurfaces of lower dimension on which the stochastic particles interact.

Thus, the new concept of the stochastic dynamics corresponds to the Boltzmann equation and the Boltzmann hierarchy.

1. Stochastic trajectories as the limit of the Hamiltonian trajectories of hard spheres as diameter tends to zero. First, consider two hard spheres with diameter a

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and mass 1. Denote by $(q_1, p_1) = x_1$, $(q_2, p_2) = x_2$ the positions of their centers and their momenta; x_1 and x_2 are their phase points.

We fix the initial momenta p_1, p_2 and consider the position q_1^0, q_2^0, such that the vector $q_1^0 - q_2^0$ is parallel to the vector $p_1 - p_2$ and $(p_1 - p_2) \cdot (q_1^0 - q_2^0) < 0$. Then for given q_2^0 consider the semisphere $q_2^0 - a\eta$, where $|\eta| = 1$, $\eta \in S_-^2$, $\eta \cdot (p_1 - p_2) < 0$. As the initial position of the first sphere, we take the point q_1^0 and, as the initial positions of the second sphere, we take the points $q_2^0 - a\eta$, $\eta \in S_-^2$, (Fig. 1).

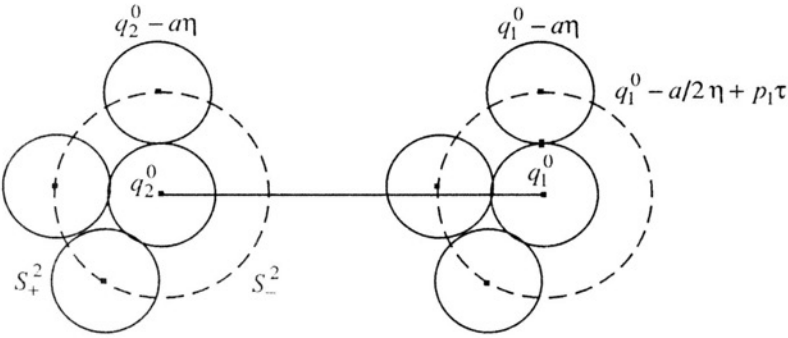

Fig. 1

We consider a positive (increasing time) $t \geq 0$ and $t = 0$ is initial time. It is obvious that, at the time

$$\tau = \frac{|q_2^0 - q_2^0|}{|p_1 - p_2|}, \tag{1.1}$$

the particles collide and touch each other at the point $q_1^0 - \frac{a}{2}\eta + p_1\tau$. After the elastic collision, their momenta are

$$p_1^* = p_1 - \eta\eta \cdot (p_1 - p_2),$$
$$p_2^* = p_2 - \eta\eta \cdot (p_1 - p_2), \qquad \eta \in S_-^2. \tag{1.2}$$

The corresponding Hamiltonian trajectory is defined as follows:

$$Q_1(t) = q_1 + p_1 t, \quad P_1(t) = p_1, \quad Q_2(t) = q_2 + p_2 t, \quad P_2(t) = p_2, \quad t \leq \tau,$$
$$Q_1(t) = q_1 + p_1\tau + p_1^*(t - \tau), \quad P_1(t) = p_1^*, \tag{1.3}$$
$$Q_2(t) = q_2 + p_2\tau + p_2^*(t - \tau), \quad P_2(t) = p_2^*, \quad t > \tau,$$

for all $q_2 = q_2^0 - a\eta$, $\eta \in S_-^2$, and fixed $q_1 = q_1^0$. Denote by

$$X^a(t) = (Q_1(t), P_1(t), Q_2(t), P_2(t))$$

the trajectories (1.3) of two hard spheres.

We defined by (1.3) the bunch of trajectories which is characterized by the vector $\eta \in S_-^2$. Now let the diameter a tends to zero. Then for $t \leq \tau$, the entire bunch of trajectories (1.3) is shrunk to the single trajectory

$$Q_1(t) = q_1^0 + p_1 t, \quad P_1(t) = p_1,$$
$$Q_2(t) = q_2^0 + p_2 t, \quad P_2(t) = p_2, \quad t \leq \tau. \tag{1.4}$$

For $t > \tau$, the arbitrary trajectory (1.3) with fixed $\eta \in S_-^2$ converges to the trajectory

$$Q_1(t) = q_1^0 + p_1\tau + p_1^*(t-\tau), \quad P_1(t) = p_1^*,$$
$$Q_2(t) = q_2^0 + p_2\tau + p_2^*(t-\tau), \quad P_2(t) = p_2^*, \quad t > \tau, \tag{1.5}$$

with the same $\eta \in S_-^2$.

Now we define the stochastic trajectories for two point-particles. We suppose that particles move as free ones until their positions coincide:

$$Q_1(t) = q_1 + p_1 t, \quad P_1(t) = p_1,$$
$$Q_2(t) = q_2 + p_2 t, \quad P_2(t) = p_2. \tag{1.6}$$

If their positions coincide at time τ,

$$Q_1(\tau) = Q_2(\tau),$$

then they collide instantaneously and their momenta change jumpwise

$$P_1(\tau+0) = P_1(\tau) + \eta\eta\cdot(P_1(\tau) - P_2(\tau)) = p_1^*,$$
$$P_2(\tau+0) = P_2(\tau) + \eta\eta\cdot(P_1(\tau) - P_2(\tau)) = p_2^*, \tag{1.7}$$

if $\eta\cdot(p_1 - p_2) \le 0$, where $|\eta| = 1$, $\eta \in S_-^2$. If the vector $\eta \in S_+^2$, $\eta\cdot(p_1 - p_2) \ge 0$, then, after collisions, the momenta p_1, p_2 do not change. We suppose that the vector $\eta \in S^2 = S_-^2 \cup S_+^2$ is a random one with the constant density of probability $\chi(\eta) = 1/4\pi$, and the particles after collision may have vectors p_1^*, p_2^* with $\eta \in S_-^2$ with the same constant probability. Thus, the stochasticity consists in the fact that, with the same probability, the particles after collision have the momenta p_1^*, p_2^* defined by (1.2) with random vector $\eta \in S_-^2$.

For $t > \tau$, the particles again move as free ones

$$Q_1(t) = q_1 + p_1\tau + p_1^*(t-\tau), \quad P_1(t) = p_1^*,$$
$$Q_2(t) = q_2 + p_2\tau + p_2^*(t-\tau), \quad P_2(t) = p_2^*, \quad t > \tau, \quad \eta \in S_-^2,$$
$$Q_1(t) = q_1 + p_1 t, \quad P_1(t) = p_1,$$
$$Q_2(t) = q_2 + p_2 t, \quad P_2(t) = p_2, \quad t > \tau, \quad \eta \in S_+^2. \tag{1.8}$$

It is obvious that the limit trajectories of hard spheres (as $a \to 0$) (1.4), (1.5) coincide with the stochastic trajectories (1.6) – (1.8) for the same $\eta \in S_-^2$, $q_1 = q_1^0$, $q_2 = q_2^0$.

The stochastic dynamics is defined by the stochastic trajectories, while the Hamiltonian dynamics is defined by the Hamiltonian trajectories.

If $q_1 = q_2$ at initial time $t = 0$, then the stochastic trajectories are defined as follows:

$$Q_1(t) = q_1 + p_1^* t, \quad P_1(t) = p_1^*,$$
$$Q_2(t) = q_1 + p_2^* t, \quad P_2(t) = p_2^*, \quad t > 0, \tag{1.9}$$

where p_1^*, p_2^* are defined by (1.2) with $\eta \in S_-^2$. If $\eta \in S_+^2$, then

$$Q_1(t) = q_1 + p_1 t, \quad P_1(t) = p_1,$$
$$Q_2(t) = q_2 + p_2 t, \quad P_2(t) = p_2, \quad t > 0. \tag{1.10}$$

Let us stress that, for the stochastic dynamics, the state of particles at the instant of collision is defined not only by their positions and momenta, but also by a random vector $\eta \in S^2 = S^2_- \cup S^2_+$ with constant density of probability on S^2_-. Note that the union of the points (1.9) of the stochastic trajectories

$$X(t) = (Q_1(t), P_1(t), Q_2(t), P_2(t))$$

with respect to the random $\eta \in S^2_-$, $t > 0$, and q_1, p_1, p_2 forms a set of the same dimension as the phase space.

If hard spheres touch each other at initial time $t = 0$, i.e., $q_2 = q_1 - a\eta$, then for $\eta \in S^2_-$,

$$Q_1(t) = q_1 + p_1^* t, \quad P_1(t) = p_1^*,$$
$$Q_2(t) = q_1 - a\eta + p_2^* t, \quad P_2(t) = p_2^*, \quad t > 0. \tag{1.11}$$

If $\eta \in S^2_+$, then

$$Q_1(t) = q_1 + p_1 t, \quad P_1(t) = p_1,$$
$$Q_2(t) = q_1 - a\eta + p_2 t, \quad P_2(t) = p_2, \quad t > 0. \tag{1.12}$$

Note that there exists another possibility to obtain the stochastic trajectories. Namely, fix the positions q_1^0 and q_2^0 of two hard spheres, fix the difference of their momenta $p_1^0 - p_2^0$ that is parallel to the vector $q_1^0 - q_2^0$. Let

$$\tau = \frac{|q_2^0 - q_2^0|}{|p_1^0 - p_2^0|} > 0$$

be the instant of collision.

Consider the all initial momenta (p_1, p_2) such that spheres collide as time τ and the unit vectors η in direction of their centers at the instant of collision belong to the semisphere S^2_- (Fig. 2). The initial momenta (p_1, p_2) depend on η, $(p_1, p_2) = (p_1(\eta), p_2(\eta))$ and there is one to one correspondence between η and $(p_1(\eta), p_2(\eta))$. After collisions the hard spheres have momenta (1.2) with $\eta \in S^2_-$. The Hamiltonian trajectories with the above described initial data are again represented by formula (1.3) with $q_1 = q_1^0$, $q_2 = q_2^0$. We will say that the collection of this trajectories is the cone of trajectories

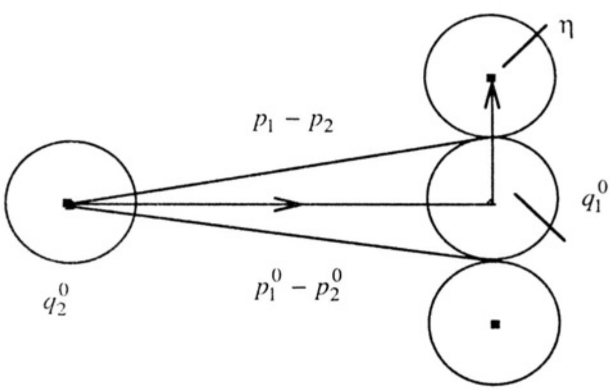

Fig. 2

It is obvious that momenta $(p_1(\eta), p_2(\eta))$ are continuous functions on a and tend to (p_1^0, p_2^0) as $a \to 0$. The limit as $a \to 0$ of the cone of trajectories of hard spheres (1.3) coincide with the stochastic trajectories (1.6) – (1.8) for the same $\eta \in \in s S_-^2$, $q_1 = q_1^0$, $q_2 = q_2^0$, $p_1 = p_1^0$, $p_2 = p_2^0$.

Now we proceed to the problem of convergence of the Hamiltonian trajectories of two hard spheres (1.3) to the stochastic trajectories (1.4), (1.5) when the diameter a tends to zero. For this purpose, we associate the Hamiltonian trajectory with the stochastic trajectory with the same vector η.

Lemma 1. *The Hamiltonian trajectory of two hard spheres* (1.3) *converges to the stochastic trajectory* (1.4), (1.5) (*or* (1.6) – (1.8)) *with the same* η *uniformly with respect to time, and the following estimate holds:*

$$\| X^a(t) - X(t) \| \le a, \quad t \ge 0, \tag{1.13}$$

where $\|\cdot\|$ *is the Euclidean norm in the phase space.*

Proof. Estimate (13.1) follows directly from definitions (1.3) – (1.8) of the Hamiltonian and the stochastic trajectories. Lemma 1 holds for the bunch and the cone of trajectories.

***Remark* 1.** It follows from the Hamiltonian dynamics of hard spheres that particles can collide only if $(q_1 - q_2)\cdot(p_1 - p_2) \le 0$ or $\eta\cdot(p_1 - p_2) \le 0$ (for initial data $q_2 = q_2^0 - a\eta$, $q_1 = q_1^0$, $\eta \in S_-^2$). This means that, in the stochastic dynamics, particles also collide only if $\eta\cdot(p_1 - p_2) \le 0$. But after collision, we have $\eta\cdot(p_1^* -- p_2^*) \le -\eta\cdot(p_1 - p_2) \ge 0$, thus $\eta \in S_+^2$ with respect to the momenta p_1^*, p_2^*. In order to avoid the repeated collision at the same instant of time, we require that for $\eta \in S_+^2$ momenta do not change.

Let us show that the convergence of the Hamiltonian trajectories to the stochastic ones implies the following weak convergence: For continuous functions $f_2(x_1, x_2) = = f_2(x)$, consider the functional

$$\frac{1}{4\pi a^2} \int_0^t dt' \int_{S_-^2} d\eta\, a^2 f_2(X^a(t', x^a)),$$
$$x^a = (q_1^0, p_1, q_2^0 - a\eta, p_2), \quad \eta \in S_-^2, \tag{1.14}$$

for the Hamiltonian dynamics. Here, $d\eta$ is an element of the unit sphere $|\eta| = 1$, and $X^a(t, x^a)$ denote trajectories (1.3) with the initial data x^a. Thus, functional (1.14) is an average over the sphere $a\eta$, $\eta \in S^2$ of the bunch of trajectories of hard spheres with the initial data $x^a = (q_1^0, p_1, q_2^0 - a\eta, p_2)$, $\eta \in S_-^2$ for fixed (q_1^0, p_1, q_2^0, p_2).

Consider also the functional

$$\frac{1}{4\pi} \int_0^t dt' \int_{S_-^2} d\eta\, f_2(X(t', x)), \quad x = (q_1^0, p_1, q_2^0, p_2), \quad \eta \in S_-^2, \tag{1.15}$$

for the stochastic dynamics. Here, $X(t, x)$ are the stochastic trajectories (1.4), (1.5) with the initial data $x = (q_1^0, p_1, q_2^0, p_2)$, $\eta \in S_-^2$. Now we show that there exists the limit of functional (1.14) as $a \to 0$, and it coincides with functional (1.15).

Lemma 2. *The following formula holds*

$$\lim_{a\to 0} \frac{1}{4\pi a^2} \int_0^t dt' \int_{S_-^2} d\eta\, a^2 f_2(X^a(t', x^a)) = \frac{1}{4\pi} \int_0^t dt' \int_{S_-^2} d\eta\, f_2(X(t', x))\chi(\eta)$$

uniformly with respect to t.

Proof. We have

$$\lim_{a\to 0} \frac{1}{4\pi a^2} \left| \int_0^t dt' \int_{S_-^2} d\eta\, a^2 [f_2(X^a(t', x^a)) - f_2(X(t', x))] \right| \le$$

$$\le \lim_{a\to 0} \frac{1}{4\pi} \int_0^t dt' \int_{S_-^2} d\eta \left| f_2(X^a(t', x^a)) - f_2(X(t', x)) \right| = 0 \qquad (1.16)$$

because, as it follows from Lemma 1,

$$\lim_{a\to 0} [f_2(X^a(t', x^a)) - f_2(X(t', x))] = 0$$

uniformly with respect to $0 \le t' \le t$ and $\eta \in S_-^2$. Thus, according to Lemma 1 and 2, the stochastic trajectories are the uniform and weak limit of the Hamiltonian trajectories in the above described sense.

Note that in the case of the cone of trajectories, we integrate in (1.14) over the semisphere $\eta \in S_-^2$ and the initial momenta $(p_1(\eta), p_2(\eta))$ depend on η as it was described above. *The limit of the Hamiltonian dynamics of hard spheres is the stochastic dynamics, but not the free one.*

It is easy to extend the all above obtained results for negative (decreasing) time, $t \le \le 0$. Namely, it is sufficient to replace the semispheres S_-^2 and S_+^2 by the semispheres S_+^2 and S_-^2, respectively, in all constructions for positive (increasing) time, $t \ge 0$. For example, if $q_1 = q_2$ at the initial time $t = 0$, then the stochastic trajectories are defined as follows:

$$Q_1(t) = q_1 + p_1^* t, \quad P_1(t) = p_1^*,$$
$$Q_2(t) = q_2 + p_2^* t, \quad P_2(t) = p_2^*, \quad t < 0, \qquad (1.17)$$

where p_1^*, p_2^* are defined as in (1.2) for $\eta \in S_+^2$. If $\eta \in S_-^2$, then

$$Q_1(t) = q_1 + p_1 t, \quad P_1(t) = p_1,$$
$$Q_2(t) = q_1 + p_2 t, \quad P_2(t) = p_2, \quad t < 0. \qquad (1.18)$$

The union of the points $X(t)$ (1.17) with respect to the random vectors $\eta \in S_+^2$, $t < 0$, and q_1, p_1, p_2 forms a set of the same dimension as the phase space.

If hard spheres touch each other at the initial time $t = 0$, i.e., $q_2 = q_1 - a\eta$, then for $\eta \in S_+^2$,

$$Q_1(t) = q_1 + p_1^* t, \quad P_1(t) = p_1^*,$$
$$Q_2(t) = q_1 - a\eta + p_2^* t, \quad P_2(t) = p_2^*, \quad t < 0. \qquad (1.19)$$

If $\eta \in S_-^2$, then

$$Q_1(t) = q_1 + p_1 t, \quad P_1(t) = p_1,$$
$$Q_2(t) = q_1 - a\eta + p_2 t, \quad P_2(t) = p_2, \quad t < 0. \tag{1.20}$$

It is obvious that the Hamiltonian trajectories of hard spheres (1.19), (1.20) converge as $a \to 0$ to the stochastic trajectories (1.17), (1.18) with the same η.

2. New representation of the Hamiltonian and stochastic trajectories. In this section, we introduce a new representation of the trajectories of interacting hard spheres and interacting stochastic point-particles. In both cases, there also exist free trajectories for noninteracting hard spheres and stochastic point-particles.

In order to interact, the positions of centers of hard spheres q_1, q_2 must belong at some time $\tau \geq 0$ to the hypersurface

$$q_1 = q_2 + a\eta, \quad \eta \in S^2 \tag{2.1}$$

with arbitrary p_1, p_2. For given fixed p_1, p_2, q_1, hypersurface (2.1) is parametrized by the vector $\eta \in S^2$, $\eta = \dfrac{q_1 - q_2}{|q_1 - q_2|}$.

In order to interact, the positions of the stochastic particles must belong at some time $\tau \geq 0$ to the hypersurface

$$q_1 = q_2, \quad \eta \in S^2 \tag{2.2}$$

with arbitrary p_1, p_2. In the case of the stochastic dynamics, in every point of hypersurface (2.2), there exist random vectors $\eta \in S^2$ with the constant probability density $\chi(\eta) = 1/4\pi$. In this case, hypersurface (2.2) is characterized by points $q_1 = q_2, p_1, p_2$ and the random vectors η.

Consider the initial positions such that hard spheres interact at some time τ on the interval $[0, t]$. In order to obtain them it is necessary to shift the points $q_1 = q_2 + a\eta$ "backward" in time on the time interval $[-\tau, 0]$. According to the dynamics of hard spheres, we obtain

$$(q_1 - p_1\tau, p_1, q_1 - a\eta - p_2\tau, p_2), \quad \eta \in S^2_-, \ (p_1 - p_2)\cdot\eta \leq 0,$$
$$(q_1 - p_1^*\tau, p_1^*, q_1 - a\eta - p_2^*\tau, p_2^*), \quad \eta \in S^2_+, \ (p_1 - p_2)\cdot\eta \geq 0. \tag{2.3}$$

Denote the collection of points (2.3) with arbitrary q_1, p_1, p_2, $\eta \in S^2_-$, $\eta \in S^2_+$, $0 \leq$ $\leq \tau \leq t$ by D^a_{-t}. Obviously, the domain D^a_{-t} is of full Lebesgue measure. We define the infinitesimal and entire volume of D^a_{-t} in the next section. Hard spheres with initial data (2.3) from the domain D^a_{-t} interact (touch each other) at time $t = \tau$ during the "forward" evolution on the time interval $[0, t]$. For $t = \tau - 0$, points (2.3) are shifted along the Hamiltonian trajectory to the points

$$(q_1, p_1, q_1 - a\eta, p_2), \quad \eta \in S^2_-,$$
$$(q_1, p_1^*, q_1 - a\eta, p_2^*), \quad \eta \in S^2_+. \tag{2.4}$$

For $t = \tau + 0$, they turn into the points

$$(q_1, p_1^*, q_1 - a\eta, p_2^*), \quad \eta \in S^2_-,$$
$$(q_1, p_1, q_1 - a\eta, p_2), \quad \eta \in S^2_+. \tag{2.5}$$

Finally, for $t > \tau$, points (2.3) turn into the points

$$\begin{aligned} &(q_1 + p_1^*(t-\tau),\ p_1^*, q_1 - a\eta + p_2^*(t-\tau),\ p_2^*), \qquad \eta \in S_-^2, \\ &(q_1 + p_1(t-\tau), p_1, q_1 - a\eta + p_2(t-\tau), p_2), \qquad \eta \in S_+^2. \end{aligned} \tag{2.6}$$

Note that points (2.4) are the states of hard spheres before collision because $\eta \cdot (p_1 - - p_2) \le 0$ for $\eta \in S_-^2$, and $\eta \cdot (p_1^* - p_2^*) \le -\eta \cdot (p_1 - p_2) \le 0$ for $\eta \in S_+^2$, i.e., the vectors $\eta \in S_+^2$ belong to the semisphere S_-^2 with respect to the momenta p_1^*, p_2^*. Analogously, points (2.5) are the states after collision because $\eta \cdot (p_1 - p_2) \ge 0$ for $\eta \in S_+^2$, and $\eta \cdot (p_1^* - p_2^*) \le -\eta \cdot (p_1 - p_2) \ge 0$ for $\eta \in S_-^2$, i.e., the vectors $\eta \in S_-^2$ belong to the semisphere S_+^2 with respect to the momenta p_1^*, p_2^*.

Note that the collection of points (2.6) with respect to $\eta \in S_-^2$, $\eta \in S_+^2$, $0 \le \tau \le t$, q_1, p_1, p_2 is of the same dimension as the phase space. We call the domain D_{-t}^a the domain of interaction of two hard spheres.

Now consider the stochastic particles and again shift the points $q_1 = q_2$, $\eta \in S^2$, p_1, p_2 of hypersurface (2.2) "backward" in time on the interval $[-\tau, 0]$. According to the dynamics of stochastic particles, we obtain

$$\begin{aligned} &(q_1 - p_1\tau, p_1, q_1 - p_2\tau, p_2), \qquad \eta \in S_-^2, \\ &(q_1 - p_1^*\tau,\ p_1^*, q_1 - p_2^*\tau,\ p_2^*), \qquad \eta \in S_+^2. \end{aligned} \tag{2.7}$$

Denote the collection of points (2.7) with arbitrary q_1, p_1, p_2, $\eta \in S_-^2$, $\eta \in S_+^2$, $0 \le \le \tau \le t$ by D_{-t}. Obviously, the set D_{-t} has the same dimension as the phase space and is of full Lebesgue measure with respect to the variables q_1, p_1, p_2, η, τ. We define the infinitesimal and entire volume of D_{-t} in the next section. We call the set D_{-t} the set of interaction of two stochastic particles.

Stochastic particles with initial data from the set D_{-t} interact at time τ during the forward evolution on the time interval $[0, t]$ because for $t = \tau$, particles touch each other. For $t = \tau - 0$, points (2.7) are shifted along the stochastic trajectories to the points

$$\begin{aligned} &(q_1, p_1, q_1, p_2), \qquad \eta \in S_-^2, \\ &(q_1,\ p_1^*, q_1,\ p_2^*), \qquad \eta \in S_+^2. \end{aligned} \tag{2.8}$$

For $t = \tau + 0$, they turn into the points

$$\begin{aligned} &(q_1,\ p_1^*, q_1,\ p_2^*), \qquad \eta \in S_-^2, \\ &(q_1, p_1, q_1, p_2), \qquad \eta \in S_+^2. \end{aligned} \tag{2.9}$$

For $t > \tau$, points (2.7) of the set D_{-t} turn into the points

$$\begin{aligned} &(q_1 + p_1^*(t-\tau),\ p_1^*, q_1 + p_2^*(t-\tau),\ p_2^*), \qquad \eta \in S_-^2, \\ &(q_1 + p_1(t-\tau), p_1, q_1 + p_2(t-\tau), p_2), \qquad \eta \in S_+^2. \end{aligned} \tag{2.10}$$

It is obvious that the collection of points (2.10) with respect to $\eta \in S_-^2$, $\eta \in S_+^2$, $0 \le \tau \le t$, q_1, p_1, p_2 is of the same dimension as the phase space.

Thus, we have obtained a new representation of all Hamiltonian trajectories of hard spheres (2.6) and of all stochastic trajectories of stochastic particles (2.10) that interact at some time τ on the interval $[0, t]$. We have also obtained a complete description of the initial phase points (2.3) and (2.7) for the trajectories of interacting hard spheres and interacting stochastic particles, respectively. We have shown that the points of the domain of interaction D_{-t}^a and the set of interaction D_{-t} are the states before collisions for the "forward" motion of hard spheres and stochastic particles, respectively.

All result above can easily be reformulated for negative (decreasing) time. The domain D_t^a is the collection of points

$$(q_1 + p_1^*\tau,\, p_1^*,\, q_1 - a\eta + p_2^*\tau,\, p_2^*), \quad \eta\in S_-^2,\ \eta\cdot(p_1 - p_2) \le 0,$$
$$(q_1 + p_1\tau, p_1, q_1 - a\eta + p_2\tau, p_2), \quad \eta\in S_+^2,\ \eta\cdot(p_1 - p_2) \ge 0. \tag{2.11}$$

For $-t < 0$, points (2.11) turn into the points

$$(q_1 + p_1(-t+\tau), p_1, q_1 - a\eta + p_2(-t+\tau), p_2), \quad \eta\in S_-^2,$$
$$(q_1 + p_1^*(-t+\tau),\, p_1^*,\, q_1 - a\eta + p_2^*(-t+\tau),\, p_2^*), \quad \eta\in S_+^2, \quad 0\le\tau\le t, \tag{2.12}$$

during the "backward" evolution on the interval $[-t, 0]$.

The set D_t is the collection of points

$$(q_1 + p_1^*\tau,\, p_1^*,\, q_1 + p_2^*\tau,\, p_2^*), \quad \eta\in S_-^2,$$
$$(q_1 + p_1\tau, p_1, q_1 + p_2\tau, p_2), \quad \eta\in S_+^2,\ 0\le \tau\le t. \tag{2.13}$$

For $-t < 0$, points (2.13) turn into the points

$$(q_1 + p_1(-t+\tau), p_1, q_1 + p_2(-t+\tau), p_2), \quad \eta\in S_-^2,$$
$$(q_1 + p_1^*(-t+\tau),\, p_1^*,\, q_1 + p_2^*(-t+\tau),\, p_2^*), \quad \eta\in S_+^2, \quad 0\le\tau\le t \tag{2.14}$$

during the "backward" evolution on the interval $[-t, 0]$. Note that the sets D_{-t} (2.7) and D_t (2.13) have the following characteristic property: The vectors of differences of positions are parallel to the vectors of differences of momenta. Indeed, for D_{-t}, we have

$$q_1 - p_1\tau - q_1 + p_2\tau = \tau(p_2 - p_1), \quad \eta\in S_-^2,$$
$$q_1 - p_1^*\tau - q_1 + p_2^*\tau = \tau(p_1^* - p_2^*), \quad \eta\in S_+^2.$$

For D_t, we have

$$q_1 + p_1^*\tau - q_1 - p_2^*\tau = \tau(p_1^* - p_2^*), \quad \eta\in S_-^2,$$
$$q_1 + p_1\tau - q_1 - p_2\tau = \tau(p_2 - p_1), \quad \eta\in S_+^2.$$

In the previous section, it was pointed out that, according to the stochastic dynamics, particles may interact only if the vector of difference of their initial positions is parallel to the vector of difference of their initial momenta. We have shown above that all points of the set D_{-t} taken as initial data for the "forward" evolution satisfy this condition. Analogously, all points of the domain D_t taken as initial data for the "backward" evolution also satisfy this condition.

***Remark* 2.** It should be stressed that the collection of points (x_1, x_2) (2.7) or (2.13) of the set D_{-t} or D_t, $0 \le \tau \le t$, has lower dimension in the phase space because it consists of points such that the differences of their positions are parallel to the differences of their momenta. It is obvious that points (2.7) with $\eta \in S_-^2$ do not change in the phase space as η varies on S_-^2.

It is quite easy to check that the Jacobian of the transformation $(x_1, x_2) \to (q_1 - p_1\tau, p_1, q_1 - p_2\tau, p_2)$, $\eta \in S_-^2$, $\tau \in [0, t)$, $q_1 \in R^3$, $p_1 \in R^3$, $p_2 \in R^3$, is equal to zero. One can also check that the Jacobian of the transformation $(x_1, x_2) \to (q_1 - p_1^*\tau, p_1^*, q_1 - p_2^*\tau, p_2^*)$, $\eta \in S_+^2$, $\tau \in [0, t)$, $q_1 \in R^3$, $p_1 \in R^3$, $p_2 \in R^3$, is also equal to zero. In the case of hard spheres, it was established that the corresponding Jacobian of the transformation $(x_1, x_2) \to (q_1 - p_1\tau, p_1, q_1 - a\eta - p_2\tau, p_2)$, $\eta \in S_-^2$, $0 \le \tau \le t$, or $(x_1, x_2) \to (q_1 - p_1^*\tau, p_1^*, q_1 - a\eta - p_2^*\tau, p_2^*)$, $\eta \in S_+^2$, $0 \le \tau \le t$ is different from zero and equal to $a^2 |\eta \cdot (p_1 - p_2)|$ [4]. This means that the domain D_{-t}^a has full Lebesgue measure in the phase space (with respect to x_1, x_2).

3. Functionals. Consider the following functional:

$$(S_2^a(t) f_2 - S_2^0(t) f_2, \varphi_2) =$$

$$= \frac{1}{2} \int [S_2^a(t) f_2(x_1, x_2) - S_2^0(t) f_2(x_1, x_2)] \varphi_2(x_1, x_2) dx_1 dx_2, \quad t \ge 0, \tag{3.1}$$

where f_2 is a continuously differentiable function, φ_2 is a test function, $S_2^a(t)$ is the operator of evolution of two hard spheres $S_2^a(t) f_2(x_1, x_2) = f_2(X^a(t, x_1, x_2))$, and $S_2^0(t)$ is the operator of evolution of free point-particles $S_2^0(t) f_2(x_1, x_2) = f_2(X^0(t, x_1, x_2))$. By $X^a(t, x_1, x_2)$ and $X^0(t, x_1, x_2)$ we denote the trajectories of two hard spheres and two free particles, respectively. The properties of the operators $S_2^a(t)$ and $S_2^0(t)$ in the space L_1 are described in detail in [3, 4]. For what follows, it is convenient to define the operator $S_2^a(t)$ on forbidden configurations $|q_1 - q_2| < a$ as the operator of free evolution.

The function

$$S_2^a(t) f_2(x_1, x_2) - S_2^0(t) f_2(x_1, x_2) \tag{3.2}$$

is different from zero in the domain D_{-t}^a because, for initial data from D_{-t}^a, hard spheres interact at time τ on the interval $[0, t]$ and, according to (2.6), we have

$$\begin{aligned} S_2^a(t) f_2(x_1, x_2) - S_2^0(t) f_2(x_1, x_2) &= \\ = f_2(q_1 + p_1^*(t - \tau), p_1^*, q_1 - a\eta + p_2^*(t - \tau), p_2^*) &- \\ - f_2(q_1 + p_1(t - \tau), p_1, q_1 - a\eta + p_2(t - \tau), p_2), &\quad \eta \in S_-^2, \\ S_2^a(t) f_2(x_1, x_2) - S_2^0(t) f_2(x_1, x_2) &= \\ = f_2(q_1 + p_1(t - \tau), p_1, q_1 - a\eta + p_2(t - \tau), p_2) &- \\ - f_2(q_1 + p_1^*(t - \tau), p_1^*, q_1 - a\eta + p_2^*(t - \tau), p_2^*), &\quad \eta \in S_+^2. \end{aligned} \tag{3.3}$$

For initial data outside the domain D^a_{-t}, function (3.2) is equal to zero because hard spheres do not interact and their trajectories coincide with free ones.

Thus, functional (3.1) can be represented in the following form:

$$(S_2^a(t)f_2 - S_2^0(t)f_2, \varphi_2) =$$

$$= \frac{1}{2} \int\limits_{D^a_{-t}} [S_2^a(t)f_2(x_1, x_2) - S_2^0(t)f_2(x_1, x_2)]\varphi_2(x_1, x_2)\, dx_1\, dx_2 =$$

$$= \frac{1}{2} \int\limits_0^t d\tau \int\limits_{S_-^2} d\eta \int dq_1\, dp_1\, dp_2\, a^2 |\eta \cdot (p_1 - p_2)| \times$$

$$\times [f_2(q_1 + p_1^*(t-\tau), p_1^*, q_1 - a\eta + p_2^*(t-\tau), p_2^*) -$$

$$- f_2(q_1 + p_1(t-\tau), p_1, q_1 - a\eta + p_2(t-\tau), p_2)] \times$$

$$\times \varphi_2(q_1 - p_1\tau, p_1, q_1 - a\eta - p_2\tau, p_2) +$$

$$+ \frac{1}{2} \int\limits_0^t d\tau \int\limits_{S_+^2} d\eta \int dq_1\, dp_1\, dp_2\, a^2 |\eta \cdot (p_1 - p_2)| \times$$

$$\times [f_2(q_1 + p_1(t-\tau), p_1, q_1 - a\eta + p_2(t-\tau), p_2) -$$

$$- f_2(q_1 + p_1^*(t-\tau), p_1^*, q_1 - a\eta + p_2^*(t-\tau), p_2^*)] \times$$

$$\times \varphi_2(q_1 - p_1^*\tau, p_1^*, q_1 - a\eta - p_2^*\tau, p_2^*) =$$

$$= \int\limits_0^t d\tau \int\limits_{S_-^2} d\eta \int dq_1\, dp_1\, dp_2\, a^2 |\eta \cdot (p_1 - p_2)| \times$$

$$\times [f_2(q_1 + p_1^*(t-\tau), p_1^*, q_1 - a\eta + p_2^*(t-\tau), p_2^*) -$$

$$- f_2(q_1 + p_1(t-\tau), p_1, q_1 - a\eta + p_2(t-\tau), p_2)] \times$$

$$\times \varphi_2(q_1 - p_1\tau, p_1, q_1 - a\eta - p_2\tau, p_2) =$$

$$= \int\limits_0^t d\tau \int\limits_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, a^2 |\eta \cdot (p_1 - p_2)| \delta(q_1 - q_2 - a\eta) \times$$

$$\times [f_2(q_1 + p_1^*(t-\tau), p_1^*, q_2 + p_2^*(t-\tau), p_2^*) -$$

$$- f_2(q_1 + p_1(t-\tau), p_1, q_2 + p_2(t-\tau), p_2)]\, \varphi_2(q_1 - p_1\tau, p_1, q_2 - p_2\tau, p_2) =$$

$$= \int\limits_0^t d\tau \int\limits_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, a^2 |\eta \cdot (p_1 - p_2)| \delta(q_1 + p_1\tau - q_2 - p_2\tau - a\eta) \times$$

$$\times [f_2(q_1 + p_1\tau + p_1^*(t-\tau), p_1^*, q_2 + p_2\tau + p_2^*(t-\tau), p_2^*) -$$

$$- f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2)]\, \varphi_2(q_1, p_1, q_2, p_2). \tag{3.4}$$

In (3.4), we have used the variables τ, η, q_1, p_1, p_2 in the domain D^a_{-t} and corresponding Jacobian [5]; in the second term with S_+^2, we have used the variables p_1^*, p_2^* instead of p_1, p_2, taking into account that the corresponding Jacobian is equal to one. The infinitesimal volume of the domain D_{-t}^a in the variables τ, η, q_1, p_1, p_2 is equal to $dx_1\,dx_2 = d\tau\,d\eta\,dq_1\,dp_1\,dp_2\,a^2|\eta\cdot(p_1-p_2)|$.

One can see that the term with S_+^2 coincides with the term with S_-^2 because the points from D_{-t}^a with S_+^2 represent the states before a collision in the "forward" evolution as well as the points from D_{-t}^a with S_-^2. Indeed, in the second term in (3.4),

$$\eta\cdot(p_1^* - p_2^*) = -\eta\cdot(p_1-p_2) \le 0, \qquad \eta\in S_+^2,$$

and, thus, $\eta\in S_-^2$ with respect to the momenta p_1^*, p_2^*. In the first term $\eta\cdot(p_1 - p_2)\le 0$, $\eta\in S_-^2$, and we also have points from D_{-t}^a in the states before a collision. These circumstances explain the factor $1/2$ in functionals (3.1), (3.4).

Let us define the derivative of functional (3.1), (3.4) with respect to time at $t=0$. From (3.4), we have

$$\frac{d}{dt}(S_2^a(t)f_2 - S_2^0(t)f_2, \varphi_2)\big|_{t=0} = \int_{S_-^2} d\eta \int dq_1\,dq_2\,dp_1\,dp_2\,a^2|\eta\cdot(p_1-p_2)| \times$$

$$\times\,[f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1,p_1,q_2,p_2)]\,\delta(q_1-q_2-a\eta)\,\varphi_2(q_1,p_1,q_2,p_2) \qquad (3.5)$$

for arbitrary test functions φ_2. This means that

$$\frac{d}{dt}(S_2^a(t)f_2(x_1,x_2) - S_2^0(t)f_2(x_1,x_2))\big|_{t=0} =$$

$$= a^2\int_{S^2} d\eta\,\delta(q_1-q_2-a\eta)|\eta\cdot(p_1-p_2)|\Theta(-\eta\cdot(p_1-p_2)) \times$$

$$\times\,[f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1,p_1,q_2,p_2)] =$$

$$= \delta(|q_1-q_2|-a)\left|\frac{(q_1-q_2)}{|q_1-q_2|}\cdot(p_1-p_2)\right|\Theta\left(-\frac{(q_1-q_2)}{|q_1-q_2|}\cdot(p_1-p_2)\right)$$

$$\times\,[f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1,p_1,q_2,p_2)] \qquad (3.6)$$

in the sense of generalized functions. From (3.6), we get

$$\frac{d}{dt}S_2^a(t)f_2(x_1,x_2)\big|_{t=0} = \sum_{i=1}^{2} p_i\frac{\partial}{\partial q_i}f_2(x_1,x_2) +$$

$$+\; a^2\int_{S^2} d\eta\,\delta(q_1-q_2-a\eta)|\eta\cdot(p_1-p_2)|\Theta(-\eta\cdot(p_1-p_2)) \times$$

$$\times\,[f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1,p_1,q_2,p_2)]. \qquad (3.7)$$

We have obtained a well-known expression [3, 4] with the boundary condition according to which, in the first term on the right-hand side of (3.7), one must replace p_1, p_2 by p_1^*, p_2^* for $q_1-q_2-a\eta=0$, $\eta\in S_-^2$.

Consider the derivative of functional (3.1), (3.4) with respect to time $t\neq 0$. We have

$$\frac{d}{dt}(S_2^a(t)f_2 - S_2^0(t)f_2, \varphi_2) =$$

$$= \int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, a^2 |\eta\cdot(p_1-p_2)|\,\delta(q_1+p_1\tau-q_2-p_1\tau-a\eta)\times$$

$$\times\left[\left(p_1^*\frac{\partial}{\partial q_1}+p_2^*\frac{\partial}{\partial q_2}\right)f_2(q_1+p_1\tau+p_1^*(t-\tau),\ p_1^*, q_2+p_2\tau+p_2^*(t-\tau),\ p_2^*) -\right.$$

$$\left.-\left(p_1\frac{\partial}{\partial q_1}+p_2\frac{\partial}{\partial q_2}\right)f_2(q_1+p_1t, p_1, q_2+p_2t, p_2)\right]\varphi_2(q_1,p_1,q_2,p_2) +$$

$$+\int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, a^2|\eta\cdot(p_1-p_2)|\,\delta(q_1+p_1t-q_2-p_2t-a\eta)\times$$

$$\times[f_2(q_1+p_1t,\ p_1^*, q_2+p_2t,\ p_2^*) - f_2(q_1+p_1t, p_1, q_2+p_2t, p_2)]\times$$

$$\times\varphi_2(q_1,p_1,q_2,p_2). \tag{3.8}$$

This formula means that

$$\frac{d}{dt}S_2^a(t)f_2(x_1,x_2) = \left(p_1\frac{\partial}{\partial q_1}+p_2\frac{\partial}{\partial q_2}\right)S_2^0(t)f_2(x_1,x_2)$$

if $q_1-q_2\neq(p_2-p_1)\tau+a\eta$ for all $0\le\tau\le t$, i.e. for $(x_1,x_2)\notin D_{-t}^a$.

If $q_1-q_2=(p_2-p_1)\tau+a\eta$ for some $0\le\tau\le t$, i.e. $(x_1,x_2)\in D_{-t}^a$, then (3.8) implies

$$\frac{d}{dt}S_2^a(t)f_2(x_1,x_2) = \left(p_1^*\frac{\partial}{\partial q_1}+p_2^*\frac{\partial}{\partial q_2}\right)\times$$

$$\times f_2(q_1+p_1\tau+p_1^*(t-\tau),\ p_1^*, q_2+p_2\tau+p_2^*(t-\tau),\ p_2^*) +$$

$$+\int_{S_-^2} d\eta\, a^2|\eta\cdot(p_1-p_2)|\,\delta(q_1+p_1t-q_2-p_2t-a\eta)\times$$

$$\times[f_2(q_1+p_1t,\ p_1^*, q_2+p_2t,\ p_2^*) - f_2(q_1+p_1t, p_1, q_2+p_2t, p_2)].$$

Consider the functional

$$\frac{1}{4\pi a^2}(S_2^a(t)f_2 - S_2^0(t)f_2, \varphi_2), \tag{3.8}$$

which is the average of functional (3.1) over the sphere $q_1-q_2-a\eta=0$. It is easy to see that, for continuous functions f_2 and φ_2, the following limit exists:

$$\lim_{a\to 0}\frac{1}{4\pi a^2}(S_2^a(t)f_2 - S_2^0(t)f_2, \varphi_2) =$$

$$= \frac{1}{4\pi}\frac{1}{2}\int_0^t d\tau\int_{S_-^2} d\eta\int dq_1\, dp_1\, dp_2\,|\eta\cdot(p_1-p_2)|\times$$

$$\times[f_2(q_1+p_1^*(t-\tau),\ p_1^*, q_1+p_2^*(t-\tau),\ p_2^*) -$$

$$-f_2(q_1+p_1(t-\tau), p_1, q_1+p_2(t-\tau), p_2)]\varphi_2(q_1-p_1\tau, p_1, q_1-p_2\tau, p_2) +$$

$$+\frac{1}{4\pi}\frac{1}{2}\int_0^t d\tau\int_{S_+^2} d\eta\int dq_1\, dp_1\, dp_2\,|\eta\cdot(p_1-p_2)|\times$$

$$\times\ [f_2(q_1+p_1(t-\tau),p_1,q_1+p_2(t-\tau),p_2)\ -$$
$$-\ f_2(q_1+\ p_1^*(t-\tau),\ p_1^*,q_1+\ p_2^*(t-\tau),\ p_2^*)]\,\varphi_2(q_1-\ p_1^*\tau,\ p_1^*,q_1-\ p_2^*\tau,\ p_2^*)\ =$$
$$=\ \frac{1}{4\pi}\int_0^t d\tau\int_{S_-^2} d\eta\int dq_1\,dp_1\,dp_2\,|\eta\cdot(p_1-p_2)|\ \times$$
$$\times\ [f_2(q_1+\ p_1^*(t-\tau),\ p_1^*,q_1+\ p_2^*(t-\tau),\ p_2^*)\ -$$
$$-\ f_2(q_1+p_1(t-\tau),p_1,q_1+p_2(t-\tau),p_2)]\,\varphi_2(q_1-p_1\tau,p_1,q_1-p_2\tau,p_2)\ =$$
$$=\ \frac{1}{4\pi}\int_0^t d\tau\int_{S_-^2} d\eta\int dq_1\,dp_1\,dq_2\,dp_2\,|\eta\cdot(p_1-p_2)|\,\delta(q_1-q_2)\ \times$$
$$\times\ [f_2(q_1+\ p_1^*(t-\tau),\ p_1^*,q_2+\ p_2^*(t-\tau),\ p_2^*)\ -$$
$$-\ f_2(q_1+p_1(t-\tau),p_1,q_2+p_2(t-\tau),p_2)]\,\varphi_2(q_1-p_1\tau,p_1,q_2-p_2\tau,p_2)\ =$$
$$=\ \frac{1}{4\pi}\int_0^t d\tau\int_{S_-^2} d\eta\int dq_1\,dp_1\,dq_2\,dp_2\,|\eta\cdot(p_1-p_2)|\,\delta(q_1+p_1\tau-q_2-p_2\tau)\ \times$$
$$\times\ [f_2(q_1+p_1\tau+\ p_1^*(t-\tau),\ p_1^*,q_2+p_2\tau+\ p_2^*(t-\tau),\ p_2^*)\ -$$
$$-\ f_2(q_1+p_1t,p_1,q_2+p_2t,p_2)]\,\varphi_2(q_1,p_1,q_2,p_2)\ =$$
$$=\ \frac{1}{4\pi}\int_0^t d\tau\int_{S_-^2} d\eta\int dq_1\,dp_1\,dq_2\,dp_2\,|\eta\cdot(p_1-p_2)|\,\delta(q_1+p_1\tau-q_2-p_2\tau)\ \times$$
$$\times\quad [S_2(t)f_2(x_1,x_2)-S_2^0(t)f_2(x_1,x_2)]\,\varphi_2(x_1,x_2)\ =\ (S_2(t)f_2-S_2^0(t)f_2,\varphi_2),\qquad(3.9)$$

where $S_2(t)$ is the evolution operator of the stochastic dynamics, $S_2(t)f_2(x_1,x_2) = f_2(X(t,x_1,x_2))$, and $X(t,x_1,x_2)$ is the stochastic trajectory.

It follows from the definition of the stochastic dynamics (2.10) on the set D_{-t} (2.7) that

$$\lim_{a\to 0}\frac{1}{4\pi a^2}(S_2^a(t)f_2-S_2^0(t)f_2,\varphi_2)\ =\ (S_2(t)f_2-S_2^0(t)f_2,\varphi_2)\ =$$
$$=\ M\int_{D_{-t}}[S_2(t)f_2(x_1,x_2)-S_2^0(t)f_2(x_1,x_2)]\,\varphi_2(x_1,x_2)\,dx_1\,dx_2,\qquad(3.10)$$

M is the operation of averaging with respect to the random vector $\eta\in S_-^2$. The infinitesimal volume $dx_1\,dx_2 = d\tau\,d\eta\,dq_1\,dp_1\,dp_2\,|\eta\cdot(p_1-p_2)|$ coincides with the infinitesimal volume of the domain D_{-t}^a with $a=1$.

It is obvious that the function

$$S_2(t)f_2(x_1,x_2)-S_2^0(t)f_2(x_1,x_2)\ =$$
$$=\begin{cases} f_2(q_1+\ p_1^*(t-\tau),\ p_1^*,q_1+\ p_2^*(t-\tau),\ p_2^*)-\\ -\ f_2(q_1+\ p_1(t-\tau),\ p_1,q_1+\ p_2(t-\tau),\ p_2), & \eta\in S_-^2,\\ f_2(q_1+\ p_1(t-\tau),\ p_1,q_1+\ p_2(t-\tau),\ p_2)-\\ -\ f_2(q_1+\ p_1^*(t-\tau),\ p_1^*,q_1+\ p_2^*(t-\tau),\ p_2^*), & \eta\in S_+^2,\end{cases}$$
$$(x_1,x_2)\in D_{-t},$$

is different from zero for $(x_1, x_2) \in D_{-t}$ because, for initial data $(x_1, x_2) \in D_{-t}$, stochastic particles interact at $t = \tau$, and stochastic trajectories differ from free ones.

In the set D_{-t}, we used the variables τ, η, q_1, p_1, p_2 and the infinitesimal volume $|\eta \cdot (p_1 - p_2)|\, d\tau\, d\eta\, dq_1\, dp_1\, dp_2$. This volume was defined directly in formula (3.9) and coincides with the infinitesimal volume $a^2|\eta \cdot (p_1 - p_2)|\, d\tau\, d\eta\, dq_1\, dp_1\, dp_2$ used in (3.4) for $a^2 = 1$. It can be defined independently because the distance between particles in the set D_{-t} in the direction of the vector η is equal to $|\eta \cdot (p_1 - p_2)|$. The infinitesimal volume depends on the random vector η, and the factor $1/4\pi$ in (3.9) is connected with the averaging operation M with respect to η.

Using formulas (3.9), (3.10) we define the functional $(S_2(t) f_2, \varphi_2)$ as follows

$$(S_2(t)f_2, \varphi_2) = (S_2^0(t) f_2, \varphi_2) + (S_2(t)f_2 - S_2^0(t) f_2, \varphi_2) =$$

$$= \int dq_1\, dp_1\, dq_2\, dp_2\, f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2)\varphi_2(q_1, p_1, q_2, p_2) +$$

$$+ \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta \cdot (p_1 - p_2)|\, \delta(q_1 + p_1\tau - q_2 - p_2\tau) \times$$

$$\times [f_2(q_1 + p_1\tau + p_1^*(t-\tau),\ p_1^*, q_2 + p_2\tau + p_2^*(t-\tau),\ p_2^*) -$$

$$- f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2)]\, \varphi_2(q_1, p_1, q_2, p_2). \tag{3.11}$$

According to (3.11) the functional $(S_2(t)f_2, \varphi_2)$ consists from two terms. The first one coincides with the corresponding functional of the free dynamics. The second one takes into account the hypersurfaces $q_1 + p_1\tau = q_2 + p_2\tau$, $0 \le \tau \le t$, on which the stochastic particles interact and where the stochastic dynamics differs from the free dynamics. On these hypersurfaces we use the measure $\delta(q_1 + p_1\tau - q_2 - p_2\tau) \times$ $\times |\eta \cdot (p_1 - p_2)|\, dq_1\, dp_1\, dq_2\, dp_2$ and the averaging procedure with respect the random vector η. The second functional was obtained as the limit as $a \to 0$ of the average over sphere of the corresponding functional (3.4) for hard spheres. Stress that it is a crucial point in the definition (3.11) of the functional $(S_2(t)f_2, \varphi_2)$ because in traditional statistical mechanics sets of lower phase than phase space are neglected.

Remark **3.** Note that the integrand in functional (3.9) regarded as a function of the phase points (x_1, x_2) is defined on the set D_{-t} that consists of points with the following characteristic property: the vectors of the differences of positions are parallel to the vectors of the differences of momenta. This means that the set D_{-t} is of lower dimension than the dimension of the domain D_{-t}^a with $a = 1$ or of the entire phase space. But the integrand regarded as a function of τ, η, q_1, p_1, p_2 is different from zero on the domain $0 \le \tau \le t$, $\eta \in S^2$, $q_1 \in R^3$, $p_1 \in R^3$, $p_2 \in R^3$, and functional (3.9) exists for test functions $\varphi_2(x_1, x_2)$ with compact support and for continuous functions f_2.

The main difference between functionals (3.4) and (3.9) is that the integrand of functional (3.4)

$$[f_2(q_1 + p_1^*(t-\tau),\ p_1^*, q_1 - a\eta + p_2^*(t-\tau),\ p_2^*) -$$

$$- f_2(q_1 + p_1(t-\tau), p_1, q_1 - a\eta + p_2(t-\tau), p_2)] \times$$

$$\times \varphi_2(q_1 - p_1\tau, p_1, q_1 - a\eta - p_2\tau, p_2)$$

depends on the points

$$(x_1, x_2) = (q_1 + p_1^*(t-\tau),\ p_1^*, q_1 - a\eta + p_2^*(t-\tau),\ p_2^*)$$

or

$$(x_1, x_2) = (q_1 + p_1(t-\tau), p_1, q_1 - a\eta + p_2(t-\tau), p_2)$$

and the corresponding Jacobian of the transformation $(x_1, x_2) \to (\tau, \eta, q_1, p_1, p_2)$ is different from zero and equal to $a^2|\eta \cdot (p_1 - p_2)|$ in both cases.

The integrand of functional (3.9)

$$[f_2(q_1 + p_1^*(t-\tau),\ p_1^*, q_1 + p_2^*(t-\tau),\ p_2^*) -$$

$$- f_2(q_1 + p_1(t-\tau), p_1, q_1 + p_2(t-\tau), p_2)]\,\varphi_2(q_1 - p_1\tau, p_1, q_1 - p_2\tau, p_2)$$

depends on the points

$$(x_1, x_2) = (q_1 + p_1^*(t-\tau),\ p_1^*, q_1 + p_2^*(t-\tau),\ p_2^*)$$

or

$$(x_1, x_2) = (q_1 + p_1(t-\tau), p_1, q_1 + p_2(t-\tau), p_2)$$

with parallel differences of positions and momenta and, thus, of lower dimension than the dimension of $D_{-t}^{a}|_{a=1} = D_{-t}^{1}$ and with the Jacobian of the transformation $(x_1, x_2) \to (\tau, \eta, q_1, p_1, p_2)$ equal to zero. The integrand considered as a function of $0 \le \tau \le t$, $\eta \in S_{-}^2$, $q_1 \in R^3$, $p_1 \in R^3$, $p_2 \in R^3$ is a continuous function with compact support and, thus, functional (3.9) exists.

Now define the derivative of functional (3.10) with respect to time at $t = 0$. From (3.9), we obtain

$$\frac{d}{dt}(S_2(t)f_2 - S_2^0(t)f_2, \varphi_2)\big|_{t=0} = \frac{1}{4\pi}\int_{S_-^2} d\eta \int dq_1\, dp_1\, dp_2\, |\eta\,(p_1 - p_2)| \times$$

$$\times\ [f_2(q_1,\ p_1^*, q_2,\ p_2^*) - f_2(q_1, p_1, q_2, p_2)]\,\varphi_2(q_1, p_1, q_2, p_2) =$$

$$= \frac{1}{4\pi}\int_{S_-^2} d\eta \int dq_1\, dq_2\, dp_1\, dp_2\, |\eta \cdot (p_1 - p_2)|\,\delta(q_1 - q_2) \times$$

$$\times\ [f_2(q_1,\ p_1^*, q_2,\ p_2^*) - f_2(q_1, p_1, q_2, p_2)]\,\varphi_2(q_1, p_1, q_2, p_2) =$$

$$= \int_{S^2} d\eta \int dq_1\, dq_2\, dp_1\, dp_2\, \chi(\eta)\,\Theta(-\eta \cdot (p_1 - p_2))|\eta \cdot (p_1 - p_2)|\,\delta(q_1 - q_2) \times$$

$$\times\ [f_2(q_1,\ p_1^*, q_2,\ p_2^*) - f_2(q_1, p_1, q_2, p_2)]\,\varphi_2(q_1, p_1, q_2, p_2),$$

$$\chi(\eta) = \frac{1}{4\pi} \tag{3.12}$$

for arbitrary test functions φ_2. It follows from (3.12) that, for fixed η,

$$\frac{d}{dt}(S_2(t)f_2(x_1, x_2) - S_2^0(t)f_2(x_1, x_2))\big|_{t=0} =$$

$$= \Theta(-\eta \cdot (p_1 - p_2))|\eta \cdot (p_1 - p_2)|\,\delta(q_1 - q_2) \times$$

$$\times \; [f_2(q_1,\, p_1^*, q_2,\, p_2^*) - f_2(q_1, p_1, q_2, p_2)]. \tag{3.13}$$

From (3.13), we have

$$\frac{d}{dt} S_2(t) f_2(x_1, x_2)\big|_{t=0} =$$

$$= \sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i} f_2(x_1, x_2) + \Theta(-\eta \cdot (p_1 - p_2)) |\eta \cdot (p_1 - p_2)| \delta(q_1 - q_2) \times$$

$$\times \; [f_2(q_1,\, p_1^*, q_2,\, p_2^*) - f_2(q_1, p_1, q_2, p_2)]. \tag{3.14}$$

We have the boundary condition according to which, in the first term on the right-hand side of (3.14), one must replace $p_1,\ p_2$ by $p_1^*,\ p_2^*$ for $q_1 = q_2,\ \eta \in S_-^2$. Comparing (3.5) with (3.12), we obtain

$$\lim_{a \to 0} \frac{1}{4\pi a^2} \frac{d}{dt} (S_2^a(t) f_2 - S_2^0(t) f_2, \varphi_2)\big|_{t=0} = \frac{d}{dt} (S_2(t) f_2 - S_2^0(t) f_2, \varphi_2)\big|_{t=0},$$

$$\lim_{a \to 0} \frac{1}{4\pi a^2} \frac{d}{dt} (S_2^a(t) f_2(x_1, x_2) - S_2^0(t) f_2(x_1, x_2))\big|_{t=0} =$$

$$= M \frac{d}{dt} (S_2(t) f_2(x_1, x_2) - S_2^0(t) f_2(x_1, x_2))\big|_{t=0}. \tag{3.15}$$

Consider the derivative of functional (3.9), (3.10) with respect to time $t \neq 0$. We have

$$\frac{d}{dt} (S_2(t) f_2 - S_2^0(t) f_2, \varphi_2) =$$

$$= \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta \cdot (p_1 - p_2)| \delta(q_1 + p_1 \tau - q_2 - p_1 \tau) \times$$

$$\times \left[\left(p_1^* \frac{\partial}{\partial q_1} + p_2^* \frac{\partial}{\partial q_2} \right) f_2(q_1 + p_1 \tau + p_1^*(t - \tau),\, p_1^*, q_2 + p_2 \tau + p_2^*(t - \tau),\, p_2^*) - \right.$$

$$\left. - \left(p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2} \right) f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2) \right] \varphi_2(q_1, p_1, q_2, p_2) +$$

$$+ \frac{1}{4\pi} \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta \cdot (p_1 - p_2)| \delta(q_1 + p_1 t - q_2 - p_1 t) \times$$

$$\times \; [f_2(q_1 + p_1 t,\, p_1^*, q_2 + p_2 t,\, p_2^*) - f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2)] \times$$

$$\times \varphi_2(q_1, p_1, q_2, p_2). \tag{3.16}$$

This formula means that

$$\frac{d}{dt} S_2(t) f_2(x_1, x_2) = \left(p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2} \right) S_2^0(t) f_2(x_1, x_2)$$

if $q_1 - q_2 \neq (p_2 - p_1)\tau$ for all $0 \leq \tau \leq t$, i.e. for $(x_1, x_2) \notin D_{-t}$.

If $q_1 - q_2 = (p_2 - p_1)\tau$ for some $0 \leq \tau \leq t$, i.e. $(x_1, x_2) \in D_{-t}$, then (3.16) implies

$$\frac{d}{dt} S_2(t) f_2(x_1, x_2) = \left(p_1^* \frac{\partial}{\partial q_1} + p_2^* \frac{\partial}{\partial q_2} \right) \times$$

$$\times f_2(q_1 + p_1\tau + p_1^*(t-\tau),\ p_1^*, q_2 + p_2\tau + p_2^*(t-\tau),\ p_2^*) +$$

$$+ \ [f_2(q_1 + p_1\tau,\ p_1^*, q_2 + p_2\tau,\ p_2^*) - f_2(q_1 + p_1\tau, p_1, q_2 + p_2\tau, p_2)]\,\delta(\tau - t).$$

The results obtained above can be considered as the proof of the following theorem:

Theorem. *The average of the operator of the system of two hard spheres over the sphere* $q_1 - q_2 - a\eta = 0$ *converges as* $a \to 0$ *to the evolution operator of the stochastic dynamics of two-point particles* (3.10). *The average of the infinitesimal operator of the system of two hard spheres over the sphere* $q_1 - q_2 - a\eta = 0$ *converges as* $a \to 0$ *to the infinitesimal operator of the stochastic dynamics of two-point particles* (3.15), (3.16). *In both cases, the convergence is in the weak sense (in the sense of generalized functions).*

All result can be extended to the operators $S_2^a(-t)$, $S_2(-t)$, $S_2^0(-t)$, $t \le 0$. It is sufficient to replace the domains D_{-t}^a, D_{-t} by D_t^a, D_t, and the operators $S_2^a(t)$, $S_2(t)$, $S_2^0(t)$ by $S_2^a(-t)$, $S_2(-t)$, $S_2^0(-t)$.

We obtain

$$\frac{d}{dt} S_2^a(-t) f_2(x_1, x_2)\big|_{t=0} =$$

$$= -\sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i} f_2(x_1, x_2) + a^2 \int_{S^2} d\eta\, \delta(q_1 - q_2 - a\eta)\,|\eta\cdot(p_1 - p_2)| \times$$

$$\times \Theta(\eta\cdot(p_1 - p_2))\,[f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1, p_1, q_2, p_2)],$$

$$\frac{d}{dt} S_2(-t) f_2(x_1, x_2)\big|_{t=0} = \tag{3.17}$$

$$= -\sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i} f_2(x_1, x_2) + \delta(q_1 - q_2)\,|\eta\cdot(p_1 - p_2)| \times$$

$$\times \Theta(\eta\cdot(p_1 - p_2))\,[f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1, p_1, q_2, p_2)].$$

In the operators $-\sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i}$, we should replace the momenta p_1, p_2 by p_1^*, p_2^* if $q_1 - q_2 - a\eta = 0$, $\eta \in S_+^2$, for hard spheres or if $q_1 = q_2$, $\eta \in S_+^2$, for stochastic particles.

4. General case of N**-particle systems**. The Hamiltonian dynamics of N hard spheres is defined as follows: hard spheres move as free particles until two of them collide, and their momenta after a collision change according to (2.1). Denote by $X^a(t,x)$ the trajectory of N hard spheres with $x = (x_1, \ldots, x_N)$, $X^a(t,x) = (X_1^a(t,x), \ldots, X_N^a(t,x))$, $X_i^a(t,x) = (Q_i^a(t,x), P_i^a(t,x))$, $X^a(t,x)|_{t=0} = x$.

The stochastic dynamics of N-point-particles is defined as follows: Particles move as free ones until two of them collide and their momenta after a collision change according to (2.1) but with a random vector η. Only pair collisions are considered. Denote by $X(t,x)$ the trajectory of N stochastic particles,

$$X(t,x) = (X_1(t,x), \ldots, X_N(t,x)),$$

$$X_i(t,x) = (Q_i(t,x), P_i(t,x)), \quad X(t,x)|_{t=0} = x.$$

All arguments presented in the previous subsections can be applied to N-particle systems of hard spheres and stochastic particles, more exactly to every pair of colliding particles. We are interested in relations between the operators of evolution of N-particle systems of hard spheres and stochastic particles.

Consider the following functional for the case of N hard spheres:

$$(S_N^a(t)f_N - S_N^0(t)f_N, \varphi_N) =$$

$$= \frac{1}{2}\int [S_N^a(t)f_N(x_1, \ldots, x_N) - S_N^0(t)f_N(x_1, \ldots, x_N)]\, \varphi_N(x_1, \ldots, x_N)\, dx_1, \ldots, dx_N. \tag{4.1}$$

We suppose that the test function φ_N is equal to zero in some neighborhood of the intersection of two and more hypersurfaces $|q_i - q_j| = a,\ (i, j) \in (1, \ldots, N)$, and the function f_N is continuous.

Denote by $D_{-t}^a = D_{-t}^{a;N}$ the domain of the initial data in the phase space, such that spheres interact on the time interval $[0, t],\ t > 0$ or, that is the same, the collection of points of the trajectories $X^a(-\tau, x)$ with $q_i - q_j - a\eta_{ij} = 0$ for arbitrary pairs $(i, j) \subset \subset (1, \ldots, N)$ and $0 \le \tau \le t,\ t > 0,\ |\eta_{ij}| = 1$. It is obvious that, for initial data from D_{-t}^a, hard spheres interact in "forward" evolution on the interval $[0, t]$, the function $[S_N^a(t) - S_N^0(t)]f_N(x_1, \ldots, x_N) \neq 0$ for $x \in D_{-t}^a$, and this function is zero for $x \notin D_{-t}^a$ because hard spheres do not interact for such initial data and $X^a(t, x) = X^0(t, x)$. Thus, we have

$$(S_N^a(t)f_N - S_N^0(t)f_N, \varphi_N) =$$

$$= \frac{1}{2}\int\limits_{D_{-t}^a} [S_N^a(t)f_N(x_1, \ldots, x_N) - S_N^0(t)f_N(x_1, \ldots, x_N)]\, \varphi_N(x_1, \ldots, x_N)\, dx_1, \ldots, dx_N. \tag{4.2}$$

(For forbidden initial data, we put $X^a(t, x) = X^0(t, x)$.)

In order to define the infinitesimal operator of $S_N^a(t)$, we restrict ourselves to an infinitesimal t. Then functional (4.1), (4.2) can be represented in the following form:

$$(S_N^a(t)f_N - S_N^0(t)f_N, \varphi_N) =$$

$$= \sum_{i<j=1}^{N} \int_0^t d\tau \int\limits_{S_-^2} d\eta_{ij} \int dq_i\, dp_i\, dp_j\, dx_1 \ldots \overset{i}{\vee} \ldots \overset{j}{\vee} \ldots dx_N\, a^2 |\eta_{ij} \cdot (p_i - p_j)| \times$$

$$\times [f_N(q_1 + p_1(t-\tau), p_1, \ldots, q_i + p_i^*(t-\tau),\ p_i^*, \ldots, q_i - a\eta_{ij} +$$

$$+ p_j^*(t-\tau),\ p_j^*, \ldots, q_N + p_N(t-\tau), p_N) -$$

$$- f_N(q_1 + p_1(t-\tau), p_1, \ldots, q_i + p_i(t-\tau), p_i, \ldots, q_i - a\eta_{ij} +$$

$$+ p_j(t-\tau), p_j, \ldots, q_N + p_N(t-\tau), p_N] \times$$

$$\times \varphi_N(q_1 - p_1\tau, p_1, \ldots, q_i - p_i\tau, p_i, \ldots, q_i - a\eta_{ij} - p_j\tau, p_j, \ldots, q_N - p_N\tau, p_N). \tag{4.3}$$

Here, the sign $\overset{i}{\vee} \ldots \overset{j}{\vee}$ means that dx_i and dx_j are omitted. It is obvious that

$$\frac{d}{dt}(S_N^a(t)f_N - S_N^0(t)f_N, \varphi_N)\big|_{t=0} =$$

$$= \sum_{i<j=1}^{N} \int_{S_-^2} d\eta_{ij} \int dq_i\, dp_i\, dp_j\, dx_1 \ldots \overset{i}{\vee} \ldots \overset{j}{\vee} \ldots dx_N\, a^2 |\eta_{ij}\cdot(p_i - p_j)| \times$$

$$\times\ [f_N(q_1, p_1, \ldots, q_i, p_i^*, \ldots, q_i - a\eta_{ij}, p_j^*, \ldots, q_N, p_N) -$$

$$- f_N(q_1, p_1, \ldots, q_i, p_i, \ldots, q_i - a\eta_{ij}, p_j, \ldots, q_N, p_N] \times$$

$$\times \varphi_N(q_1 p_1, \ldots, q_i, p_i, \ldots, q_i - a\eta_{ij}, p_j, \ldots, q_N, p_N). \tag{4.4}$$

For (4.4), we obtain

$$\frac{d}{dt} S_N^a(t) f_N(x_1, \ldots, x_N)\big|_{t=0} = \sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(x_1, \ldots, x_N) +$$

$$+ \sum_{i<j=1}^{N} \int_{S_-^2} d\eta_{ij}\, a^2 \delta(q_i - q_j - a\eta_{ij}) |\eta_{ij}\cdot(p_i - p_j)| \times$$

$$\times\ [f_N(x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_N) - f_N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N)],$$

$$x_i^* = (q_i, p_i^*), \qquad x_j^* = (q_j, p_j^*). \tag{4.5}$$

In free Poisson bracket, one should replace (p_i, p_j) by (p_i^*, p_j^*) if $q_i - q_j - a\eta_{ij} = 0$ (boundary condition). Formula (4.5) can be represented in the following identical form:

$$\frac{d}{dt} S_N^a(t) f_N(x_1, \ldots, x_N)\big|_{t=0} = \sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(x_1, \ldots, x_N) +$$

$$+ \sum_{i<j=1}^{N} a^2 |\eta_{ij}\cdot(p_i - p_j)| \Theta(-\eta_{ij}\cdot(p_i - p_j))\, \delta(|q_i - q_j| - a) \times$$

$$\times\ [f_N(x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_N) - f_N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N)],$$

$$\eta_{ij} = \frac{q_i - q_j}{|q_i - q_j|}. \tag{4.6}$$

Now consider the average of functional (4.3) over the sphere $|q_i - q_j| = a$

$$\frac{1}{4\pi a^2}(S_N^a(t)f_N - S_N^0(t)f_N, \varphi_N). \tag{4.7}$$

It is obvious that, for continuous f_N and φ_N, there exists a limit of functional (4.7) as $a \to 0$:

$$\lim_{a\to 0} \frac{1}{4\pi a^2}(S_N^a(t)f_N - S_N^0(t)f_N, \varphi_N) =$$

$$= \sum_{i<j=1}^{N} \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta_{ij} \int dq_i\, dp_i\, dp_j\, dx_1 \ldots \overset{i}{\vee} \ldots \overset{j}{\vee} \ldots dx_N\, |\eta_{ij}\cdot(p_i - p_j)| \times$$

$$\times\ [f_N(q_1 + p_1(t - \tau), p_1, \ldots, q_i + p_i^*(t - \tau), p_i^*, \ldots,$$

$$q_i + p_j^*(t-\tau), p_j^*, \ldots, q_N + p_N(t-\tau), p_N) -$$

$$- f_N(q_1 + p_1(t-\tau), p_1, \ldots, q_i + p_i(t-\tau), p_i, \ldots, q_i +$$

$$+ p_j(t-\tau), p_j, \ldots, q_N + p_N(t-\tau), p_N] \times$$

$$\times \varphi_N(q_1 - p_1\tau, p_1, \ldots, q_i - p_i\tau, p_i, \ldots, q_i - p_j\tau, p_j, \ldots, q_N - p_N\tau, p_N). \tag{4.8}$$

According to the definition of the stochastic dynamics and by analogy with the two-particle system, functional (4.8) is equal to the following functional for the stochastic dynamics:

$$\lim_{a\to 0} \frac{1}{4\pi a^2} (S_N^a(t)f_N - S_N^0(t)f_N, \varphi_N) = (S_N(t)f_N - S_N^0(t)f_N, \varphi_N), \tag{4.9}$$

where $S_N(t)$ is the evolution operator of N stochastic particles.

Using (4.8) we define the following functional for N stochastic particles

$$(S_N(t)f_N, \varphi_N) =$$

$$= \int dq_1\, dp_1 \ldots dq_N\, dp_N\, f_N(q_1 + p_1 t, p_1, \ldots, q_N + p_N t, p_N)\varphi_N(q_1, p_1, \ldots, q_N, p_N) +$$

$$+ \sum_{i<j=1}^{N} \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta_{ij} \int dq_1\, dp_1 \ldots dq_N\, dp_N\, |\eta_{ij}\cdot(p_i - p_j)|\,\delta(q_i + p_i\tau - q_j - p_j\tau) \times$$

$$\times [f_N(q_1 + p_1 t, p_1, \ldots, q_i + p_i\tau + p_i^*(t-\tau), p_i^*, \ldots, q_j + p_j\tau +$$

$$+ p_j^*(t-\tau), p_j^*, \ldots, q_N + p_N t, p_N) -$$

$$- f_N(q_1 + p_1 t, p_1, \ldots, q_i + p_i t, p_i, \ldots, q_j + p_j t, p_j, \ldots, q_N + p_N t, p_N)] \times$$

$$\times \varphi_N(q_1, p_1, \ldots, q_i, p_i, \ldots, q_j, p_j, \ldots, q_N, p_N) =$$

$$= (S_N^0(t)f_N, \varphi_N) + (S_N(t)f_N - S_N^0(t)f_N, \varphi_N).$$

The functional $(S_N(t)f_N, \varphi_N)$ is the average of the observable φ_N over state $S_N(t)f_N$. It consists from two terms: the first one is the average for the free dynamics, the second one takes into account the hypersurfaces $q_i + p_i\tau = q_j + p_j\tau$, $0 \le \tau \le t$ where the stochastic particles interact. In this circumstance consists the principal difference between the traditional and the stochastic statistical mechanics: in the traditional statistical mechanics sets of lower dimension than phase space are neglected.

The following formula is true for the derivative of functional (4.9) with respect to time:

$$\frac{d}{dt}(S_N(t)f_N - S_N^0(t)f_N, \varphi_N)\big|_{t=0} =$$

$$= \sum_{i<j=1}^{N} \frac{1}{4\pi} \int_{S_-^2} d\eta_{ij} \int dq_i\, dp_i\, dp_j\, dx_1 \ldots \overset{i}{\vee} \ldots \overset{j}{\vee} \ldots dx_N\, |\eta_{ij}\cdot(p_i - p_j)| \times$$

$$\times [f_N(q_1, p_1, \ldots, q_i, p_i^*, \ldots, q_i, p_j^*, \ldots, q_N, p_N) -$$

$$- f_N(q_1, p_1, \ldots, q_i, p_i, \ldots, q_i, p_j, \ldots, q_N, p_N] \times$$

$$\times \varphi_N(q_1 p_1, \ldots, q_i, p_i, \ldots, q_i, p_j, \ldots, q_N, p_N). \tag{4.10}$$

For fixed vectors η_{ij}, formula (4.10)yields

$$\frac{d}{dt} S_N(t) f_N(x_1, \ldots, x_N)\Big|_{t=0} = \sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(x_1, \ldots, x_N) +$$

$$+ \sum_{i<j=1}^{N} |\eta_{ij} \cdot (p_i - p_j)| \Theta(-\eta_{ij} \cdot (p_i - p_j)) \delta(q_i - q_j) \times$$

$$\times \left[f_N(q_1, p_1, \ldots, q_i, p_i^*, \ldots, q_j, p_j^*, \ldots, q_N, p_N) - \right.$$

$$\left. - f_N(q_1, p_1, \ldots, q_i, p_i, \ldots, q_j, p_j, \ldots, q_N, p_N) \right]. \qquad (4.11)$$

We have the boundary conditions, and according to them one should replace (p_i, p_j) by (p_i^*, p_j^*) if $q_i = q_j$ in the Poisson bracket.

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