UDC 531.36

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STABILITY ANALYSIS OF LINEAR IMPULSIVE DIFFERENTIAL SYSTEMS UNDER STRUCTURAL PERTURBATION

АНАЛІЗ СТІЙКОСТІ ЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ ІМПУЛЬСНИХ СИСТЕМ ІЗ СТРУКТУРНИМИ ЗБУРЕННЯМИ

The stability and asymptotic stability of the solutions of large-scale linear impulsive systems under structural perturbations are investigated. Sufficient conditions for stability and instability are formulated in terms of the fixed signs of special matrices.

Досліджуються стійкість та асимптотична стійкість розв'язків великомасштабної лінійної імпульсної системи при структурних збуреннях. Достатні умови стійкості та нестійкості сформульовані на основі знаковизначеності спеціальних матриць.

1. Introduction. Many of the processes in engineering and technology deal with an overcoming of the "threshold" phenomena. This is expressed in particular, in accomulation by the process of some property with the consequent sudden change of the state. The modelling of such a process when the ordinary differential equations are employed is difficult to some extend and an attempt to encorporate new classes of the systems of equations seems natural. The impulsive systems with structural perturbations belong to a class of this type.

This paper concentrates on the investigation of stability and instability of large scale linear impulsive systems under structural perturbations by the Lyapunov's direct method in terms of matrix-valued functions.

Sufficient conditions for various types of stability and instability are established. A numerical example showing the application of some general results is given.

2. Preliminaries. According to [1 – 3] we consider the linear large scale impulsive system decomposed into s subsystems

$$\begin{gathered}\frac{dx_i}{dt} = A_i x_i + \sum_{j=1}^{s} S_{ij} A_{ij} x_j, \qquad t \neq \tau_k(x), \\ \Delta x_i = J_{ki} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{s} J_{kij} x_j, \qquad t = \tau_k(x), \\ i = 1, 2, \ldots, s, \qquad k = 1, 2, \ldots,\end{gathered} \tag{1}$$

where $x_i \in R^{n_i}$, $\sum_{i=1}^{s} n_i = n$, $x = (x_1^T, x_2^T, \ldots, x_s^T)^T \in R^n$, A_i, J_{ki}, A_{ij}, J_{kij} are constant matrices of the correspondent dimensions, the set $\mathcal{G}_s$ and matrices S, S_i, S_{ij} are defined in Appendix 1, the values $\tau_k(x)$, $k = 1, 2, \ldots$, are ordered by $\tau_k(x) < \tau_{k+1}(x)$ and such that $\tau_k(x) \to +\infty$ as $k \to +\infty$. We shall assume, for simplicity, that the system (1) satisfies all required conditions so that all solutions $x(t) = x(t, t_0, x_0)$ of (1) exist for all $t \geq t_0$.

For the system (1) we construct a matrix-valued function

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* This work was done while the author was visiting the Department of Mathematics, University of Ioannina, in the framwork of the NATO Science Fellowships Programme through the Greek Ministry of National Economy.

$$U(x) = [U_{ij}(x_i, x_j)], \quad i, j = 1, 2, \dots, s, \tag{2}$$

with elements

$$U_{ii}(x_i) = x_i^T B_{ii} x_i, \quad i = 1, 2, \dots, s, \tag{3}$$

and

$$U_{ij}(x_i, x_j) = x_i^T B_{ij} x_j, \quad i \neq j, \quad i, j = 1, 2, \dots, s. \tag{4}$$

Here B_{ii} are constant positive definite matrices, and B_{ij} are constant matrices.

If in Hypothesis 1 from [4] we will accept

$$\varphi_{j1}(\|x_j\|) = \psi_{j1}(\|x_j\|) = \|x_j\|, \quad j = 1, 2, \dots, s,$$

where $\varphi_{j1}(\cdot, \cdot)$ and $\psi_{j1}(\cdot)$ are functions from class K (see [5]), $x_j \in R^{n_j}$, $j = 1, 2, \dots, s$, then we obtain the following.

Hypothesis 1. *Assume that there exist*

(i) *the matrix-valued function* (2) *with the elements* (3) *and* (4);

(ii) *the constants* a_{ji}, b_{ji}, $i, j = 1, 2, \dots, s$, *such that*

$$a_{ii}\|x_i\|^2 \le U_{ii}(x_i) \le b_{ii}\|x_i\|^2 \quad \forall x_i \in \mathcal{N}_{ix}, \quad i = 1, 2, \dots, s,$$

$$a_{ji}\|x_j\|\|x_i\| \le U_{ij}(x_i, x_j) \le b_{ji}\|x_j\|\|x_i\| \quad \forall (x_i, x_j) \in \mathcal{N}_{ix} \times \mathcal{N}_{jx},$$

$$i \neq j, \quad i, j = 1, 2, \dots, s.$$

Lemma 1. *If all conditions of Hypothesis* 1 *are satisfied, then the function*

$$V(x, \eta) = \eta^T U(x) \eta, \quad \eta \in R_+^s, \quad \eta > 0, \tag{5}$$

satisfies the bilateral inequality

$$u^T H^T A H u \le V(x, \eta) \le u^T H^T B H u \quad \forall x \in \mathcal{N}_x = \mathcal{N}_{1x} \times \mathcal{N}_{2x} \times \dots \times \mathcal{N}_{sx}. \tag{6}$$

Here

$$u^T = (\|x_1\|, \|x_2\|, \dots, \|x_s\|) \quad A = [a_{ij}], \quad B = [b_{ij}],$$

$$H = \operatorname{diag}[\eta_1, \eta_2, \dots, \eta_s].$$

The proof of the Lemma 1 is similar to that of Lemma 1 in [6].

Together with the function (5) its total derivative

$$D V(x, \eta) = u^T D U(x) \eta \tag{7}$$

along the solutions $x(t, t_0, x_0)$ of the system (1) is constructed.

Hypothesis 2. *Assume that there exist*

(i) *the matrix-valued function* (2) *with the elements* (3) *and* (4);

(ii) *the constants* $\tilde{p}_j^{(1)}(S)$, $\tilde{p}_j^{(2)}(S)$, $\tilde{p}_{ij}(S)$, $i \neq j$, $i, j = 1, 2, \dots, s$, *such that*

$$\eta_j^2\{(D_{x_j} U_{jj}(x_j))^T A_j x_j\} \le \tilde{p}_j^{(1)}(S)\|x_j\|^2 \quad \forall x_i \in \mathcal{N}_{jx0}, \quad j = 1, 2, \dots, s,$$

$$\sum_{j=1}^{s} \eta_j^2 (D_{x_j} U_{jj}(x_j))^T \sum_{\substack{i=1 \\ i \neq j}}^{s} S_{ij} A_{ij} x_i +$$

$$+ \; 2\sum_{j=1}^{s}\sum_{\substack{i=2\\ i>j}}^{s} \eta_j \eta_i \left\{ (D_{x_j} U_{ji}(x_j, x_i))^T \left(A_j x_j + \sum_{\substack{i=1\\ i\neq j}}^{s} S_{ij} A_{ij} x_i \right) + \right.$$

$$\left. + \; (D_{x_j} U_{ji}(x_j, x_i))^T \left(A_i x_i + \sum_{\substack{j=1\\ j\neq i}}^{s} S_{ij} A_{ij} x_j \right) \right\} \leq$$

$$\leq \sum_{j=1}^{s} \tilde{p}_j^{(2)}(S) \|x_j\|^2 + 2\sum_{j=1}^{s}\sum_{\substack{i=2\\ i>j}}^{s} \tilde{p}_{ji}(S) \|x_j\| \|x_i\|$$

$$\forall\, (x_i, x_j) \in \mathcal{N}_{jx0} \times \mathcal{N}_{ix0} \times \mathcal{G}_s, \quad i \neq j, \quad i, j = 1, 2, \ldots, s.$$

Lemma 2. *If all conditions of Hypothesis* 2 *are satisfied, then for expression* (7) *we get*

$$D\,V(x, \eta) \leq u^T \overline{G}(S) u \qquad \forall\, x \in \mathcal{N}_{x0} \times \mathcal{G}_s, \tag{8}$$

where

$$u^T = (\|x_1\|, \|x_2\|, \ldots, \|x_s\|),$$

$$\overline{G}(S) = [\overline{\sigma}_{ji}(S)], \qquad i, j = 1, 2, \ldots, s,$$

$$\overline{\sigma}_{ji}(S) = \overline{\sigma}_{ij}(S),$$

$$\overline{\sigma}_{jj}(S) = \tilde{p}_j^{(1)}(S) + \tilde{p}_j^{(2)}(S),$$

$$\overline{\sigma}_{ji}(S) = \tilde{p}_{ji}(S), \qquad j \neq i, \quad i, j = 1, 2, \ldots, s.$$

The proof of Lemma 2 is similar to that of Lemma 5.5.1 in [7].

It can be easily verified that for $t \neq \tau_k(x)$, $k = 1, 2, \ldots$, the estimate

$$D\,V(x, \eta) \leq \lambda_M(\overline{G}(S)) \|u\|^2 \qquad \forall\, x \in \mathcal{N}_{x0}, \quad \forall\, S \in \mathcal{G}_s, \tag{9}$$

is true. Here $\lambda_M(\cdot)$ is the maximal eigenvalues of $(\cdot)$. If $\eta^T = (1, 1, \ldots, 1) \in R_+^s$ then from (6) we get

$$\lambda_m(A) \| u \|^2 \leq V(x, \eta) \leq \lambda_M(B) \| u \|^2 \tag{10}$$

and for $\lambda_m(A) > 0$ we get

$$\lambda_M^{-1}(B)\, V(x, \eta) \leq \| u \|^2 \leq \lambda_m^{-1}(A)\, V(x, \eta). \tag{11}$$

Therefore the estimate (9) can be represented as

$$D\,V(x, \eta) \leq \begin{cases} \lambda_M(\overline{G}(S))\, \lambda_m^{-1}(A)\, V(x, \eta) & \text{for} \quad \lambda_M(\overline{G}(S)) > 0; \\ \lambda_M(\overline{G}(S))\, \lambda_M^{-1}(B)\, V(x, \eta) & \text{for} \quad \lambda_M(\overline{G}(S)) < 0. \end{cases}$$

Lemma 3. *If for the system* (1) *the condition* (i) *of Hypothesis* 1 *is satisfied, then for the function* (5) *when* $t = \tau_k(x)$, $k = 1, 2, \ldots$, *the inequalities*

$$V(x + J_k(x), \eta) - V(x, \eta) \leq u_k^T \overline{C} u_k; \tag{12}$$

and

$$V(x + J_k(x), \eta) \le u_k^T \overline{C}^* u_k, \tag{13}$$

are satisfied, where

$$u_k^T = (\|x_1(\tau_k(x))\|, \|x_2(\tau_k(x))\|, \ldots, \|x_s(\tau_k(x))\|),$$

$$J_k(x) = J_{ki} x_i + \sum_{\substack{j=1 \\ j \ne i}}^{s} J_{kij} x_j,$$

$$\overline{C} = [\overline{c}_{ij}], \quad \overline{c}_{ij} = \overline{c}_{ji}, \quad i, j = 1, 2, \ldots, s,$$

$$\overline{C}^* = [\overline{c}_{ij}^*], \quad \overline{c}_{ij}^* = \overline{c}_{ji}^*, \quad i, j = 1, 2, \ldots; s,$$

$$\overline{c}_{ii} = \lambda_M(C_{ii}), \quad \overline{c}_{ij} = \lambda_M^{1/2}(C_{ij} C_{ij}^T), \quad i \ne j, \quad i, j = 1, 2, \ldots, s;$$

$\lambda_M^{1/2}(\cdot)$ *is a norm of matrix* $(\cdot)$,

$$\overline{c}_{ii}^* = \lambda_M(C_{ii}^*), \quad \overline{c}_{ij}^* = \lambda_M^{1/2}(C_{ij}^* C_{ij}^{*T}), \quad i \ne j, \quad i, j = 1, 2, \ldots, s;$$

and

$$C_{ii} = J_{ki}^T B_{ii} + B_{ii} J_{ki} + J_{ki}^T B_{ii} J_{ki} + \sum_{\substack{j=1 \\ j \ne i}}^{s} J_{kij}^T B_{jj} J_{kji} +$$

$$+ \sum_{\substack{j=1 \\ j \ne i}}^{s} (B_{ij} J_{kji} + J_{kji}^T B_{ij}) + \sum_{\substack{j=1 \\ j \ne i}}^{s} (J_{ki}^T B_{ij} J_{kji} + J_{kji}^T B_{ij} J_{ki}) +$$

$$+ \sum_{\substack{l=1 \\ l \ne i}}^{s} \sum_{\substack{j=1 \\ j \ne i}}^{s} (J_{kli}^T B_{lj} J_{kji} + J_{kji}^T B_{lj} J_{kli}), \quad i = 1, 2, \ldots, s;$$

$$C_{ij} = B_{ii} J_{kij} + J_{kij}^T B_{ii} + J_{ki}^T B_{ii} J_{kij} + J_{kij}^T B_{ii} + J_{ki} +$$

$$+ \sum_{\substack{l=1 \\ l \ne i,j}}^{s} (J_{kli}^T B_{ll} J_{klj} + J_{kli}^T B_{ll} J_{kli}) + B_{ij} J_{kj} + J_{ki}^T B_{ij} + J_{ki}^T B_{ij} J_{kj} +$$

$$+ \sum_{\substack{l=1 \\ l \ne i,j}}^{s} (B_{il} J_{klj} + J_{kli}^T B_{lj} + J_{ki}^T B_{il} J_{klj} + J_{kli}^T B_{lj} J_{kj}) +$$

$$+ \sum_{l=1}^{s} \sum_{\substack{l=1 \\ l \ne r}}^{s} J_{kli}^T B_{lr} J_{krj}, \quad j \ne i, \quad i, j = 1, 2, \ldots, s;$$

$$C_{ii}^* = B_{ii} + C_{ii}, \quad C_{ij}^* = B_{ij} + C_{ij}, \quad j \ne i, \quad i, j = 1, 2, \ldots, s.$$

Proof. The proof of this Lemma is an Appendix 2.

***Corollary* 1.** *If all conditions of Lemma* 3 *are satisfied, then for the function* (5) *for* $t = \tau_k(x)$, $k = 1, 2, \ldots$, *the following estimates hold true*:

$$V(x + J_k(x), \eta) - V(x, \eta) \leq \Delta V(x, \eta) \tag{14}$$

where

$$\Delta = \begin{cases} \lambda_M(\overline{C})\,\lambda_M^{-1}(B) & \text{for } \lambda_M(\overline{C}) < 0; \\ \lambda_M(\overline{C})\,\lambda_m^{-1}(A) & \text{for } \lambda_M(\overline{C}) > 0, \end{cases}$$

and

$$V(x + J_k(x), \eta) \leq \Delta^* V(x, \eta), \tag{15}$$

where

$$\Delta^* = \begin{cases} \lambda_M(\overline{C}^*)\,\lambda_M^{-1}(B) & \text{for } \lambda_M(\overline{C}^*) < 0; \\ \lambda_M(\overline{C}^*)\,\lambda_m^{-1}(A) & \text{for } \lambda_M(\overline{C}^*) > 0. \end{cases}$$

The assertions (14) and (15) follow from Lemma 3 and the inequality (11).

Lemma 4. *If $t \neq \tau_k(x)$, $k = 1, 2, \ldots$, then for the total derivative* (7) *of the function* (5) *the following estimate is true*:

$$D V(x, \eta) \geq u^T \underline{G}(S)\, u \qquad \forall\, (x \neq 0) \in R^n \quad \text{and} \quad \forall\, S \in \mathcal{G}_s, \tag{16}$$

where

$$\underline{G}(S) = [\underline{\sigma}_{ij}(S)], \qquad \underline{\sigma}_{ij} = \underline{\sigma}_{ji}, \quad i, j = 1, 2, \ldots, s,$$

$$\underline{\sigma}_{ii} = \underline{\rho}_{1i}(S) + \underline{\rho}_{2i}(S), \qquad i = 1, 2, \ldots, s, \quad S \in \mathcal{G}_s,$$

$$\underline{\sigma}_{ij} = \frac{1}{2}\left(\underline{\rho}_{1ij}(S) + \underline{\rho}_{1ji}(S) + \underline{\rho}_{2ij}(S) + \underline{\rho}_{2ji}(S) + \underline{\rho}_{3ij}(S) + \underline{\rho}_{3ji}(S)\right),$$

$$i, j = 1, 2, \ldots, s, \qquad i \neq j.$$

$\underline{\rho}_{1i}(S)$ and $\underline{\rho}_{2i}(S)$ are minimal eigenvalues of matrices

$$\eta_i^2 (B_{ii} A_i + A_i^T B_{ii}), \qquad i = 1, 2, \ldots, s;$$

$$\sum_{j=1}^{i-1} \eta_i \eta_j (B_{ji}^T S_{ij} A_{ji} + (S_{ji} A_{ji})^T B_{ji}) +$$

$$+ \sum_{j=i+1}^{i-1} \eta_i \eta_j (B_{ij} S_{ji} + (S_{ji} A_{ji})^T B_{ij}), \qquad j \neq i, \quad i, j = 1, 2, \ldots, s,$$

where vector u^T is defined as in Lemma 1, *and $\underline{\rho}_{rij}$, $r = 1, 2, 3$; $i, j = 1, 2, \ldots, s$, are computed.*

Proof. The proof of this lemma is similar to that of Lemma 2.

Let $\eta^T = (1, 1, \ldots, 1) \in R_+^s$. Then in view of (16) and (11) the inequality

$$D V(x, \eta) \geq \lambda_m(\underline{G}(S))\, \|u\|^2 \qquad \forall\, S \in \mathcal{G}_s, \tag{17}$$

can be rewritten in the form

$$D V(x, \eta) = \begin{cases} \lambda_m(\underline{G}(S))\,\lambda_m^{-1}(A)\, V(x, \eta) & \text{for } \lambda_m(\underline{G}(S)) < 0; \\ \lambda_m(\underline{G}(S))\,\lambda_M^{-1}(B)\, V(x, \eta) & \text{for } \lambda_m(\underline{G}(S)) > 0. \end{cases}$$

Lemma 5. *Let* $t = \tau_k(x)$, $k = 1, 2, \ldots$, *then for the function* (5) *and the system* (1)

$$V(x + J_k(x), \eta) - V(x, \eta) \geq u_k^T \underline{C} u_k, \quad k = 1, 2, \ldots,$$

and

$$V(x + J_k(x), \eta) \geq u_k^T \underline{C}^* u_k, \quad k = 1, 2, \ldots,$$

where

$$\underline{C} = [\underline{c}_{ij}], \quad \underline{c}_{ij} = \underline{c}_{ji}, \quad i, j = 1, 2, \ldots, s,$$

$$\underline{C}^* = [\underline{c}_{ij}^*], \quad \underline{c}_{ij}^* = \underline{c}_{ji}^*, \quad i, j = 1, 2, \ldots, s,$$

$$\underline{c}_{ii} = \lambda_m(C_{ii}), \quad \underline{c}_{ij} = -\overline{c}_{ij}, \quad i \neq j, \quad i, j = 1, 2, \ldots, s,$$

$$\underline{c}_{ii}^* = \lambda_m(C_{ii}^*), \quad \underline{c}_{ij}^* = -\overline{c}_{ij}^*, \quad i \neq j, \quad i, j = 1, 2, \ldots, s;$$

and u^T, $J_k(x)$, $\underline{c}_{ij}$, $\underline{c}_{ij}^*$, $\underline{c}_{ii}$, $\underline{c}_{ii}^*$ *are defined as in Lemma* 3.

Proof. The proof of this lemma is similar to that of Lemma 3.

Corollary 2. *If all conditions of Lemma 5 are satisfied, then for the function* (5) *and the system* (1) *for* $t = \tau_k(x)$, $k = 1, 2, \ldots$,

$$V(x + J_k(x), \eta) - V(x, \eta) \geq \underline{\Delta} V(x, \eta)$$

where

$$\underline{\Delta} = \begin{cases} \lambda_m(\underline{C})\, \lambda_m^{-1}(A) & \text{for } \lambda_m(\underline{C}) < 0; \\ \lambda_m(\underline{C})\, \lambda_M^{-1}(B) & \text{for } \lambda_m(\underline{C}) > 0, \end{cases}$$

and

$$V(x + J_k(x), \eta) \geq \underline{\Delta}^* V(x, \eta),$$

where

$$\underline{\Delta}^* = \begin{cases} \lambda_m(\underline{C}^*)\, \lambda_m^{-1}(A) & \text{for } \lambda_m(\underline{C}^*) < 0; \\ \lambda_m(\underline{C}^*)\, \lambda_M^{-1}(B) & \text{for } \lambda_m(\underline{C}^*) > 0. \end{cases}$$

Proof. The proof follows from Lemma 5 and (11).

3. Results on stability under structural perturbation. The preliminary analysis allows us to formulate theorems on stability and asymptotical stability of the zero solution of (1) on $\mathcal{G}_s$.

Here we give the definitions of stability and instability under structural perturbation of impulsive system (1).

Definition 1. *The zero solution of* (1) *is*

a) *stable in the whole on* $\mathcal{G}_s$ *if and only if it is stable in the whole for each* $S \in \mathcal{G}_s$ *in the sense of Lyapunov;*

b) *asymptotically stable in the whole on* $\mathcal{G}_s$ *if it is asymptotically stable in the whole for each* $S \in \mathcal{G}_s$ *in the sense of Lyapunov;*

c) *unstable on* $\mathcal{G}_s$ *if there exists at least one* $S \in \mathcal{G}_s$ *for which the zero solution of* (1) *is unstable in the sense of Lyapunov.*

For the definition of stability in the sense of Lyapunov see [1, 2, 5, 8].

Theorem 1. *Let the system* (1) *be such that matrix-valued function* (2) *is constructed with the elements* (3) *and* (4), *and*

(i) *the matrix* A *in* (6) *is positive definite, i.e.* $\lambda_m(A) > 0$;

(ii) *there exists a matrix* Q *such that for the matrix* $\overline{G}(S)$ *the estimate*

$$\overline{G}(S) \leq Q \qquad \forall S \in \mathcal{G}_s$$

is satisfied component-wise;

(iii) *the matrix* Q *is negative semidefinite or equal to zero, i.e.* $\lambda_m(Q) \leq 0$.

Then the zero solution of the system (1) *is stable in the whole on* $\mathcal{G}_s$.

If instead of the condition (iii) *the following condition is satisfied*

(iv) *the matrix* $\overline{C}$ *in* (12) *is negative definite, i.e.* $\lambda_M(\overline{C}) < 0$,

then the zero solution of the system (1) *is asymptotically stable in the whole on* $\mathcal{G}_s$.

The proof of the Theorem 1 is similar to that of Theorem 1 in [4].

Theorem 2. *Let the system* (1) *be such that matrix-valued function* (2) *is constructed with the elements* (3) *and* (4), *and*

(i) *the matrix* A *in* (6) *is positive definite, i.e.* $\lambda_m(A) > 0$;

(ii) *there exists a negative definite matrix* $Q^- \in R^{s\times s}$ *such that*

$$\overline{G}(S) \leq Q^- \qquad \forall S \in \mathcal{G}_s;$$

(iii) $\lambda_M(\overline{C}^*) > 0$;

(iv) *the functions* $\tau_k(x)$, $k = 1, 2, \ldots$, *satisfy the inequality*

$$\tau_{k+1}(x) - \tau_k(x) = \theta, \qquad \theta > 0.$$

If

$$-\frac{\lambda_M(B)}{\lambda_M(Q^-)} \ln \frac{\lambda_M(\overline{C}^*)}{\lambda_m(A)} \leq \theta, \tag{18}$$

then the zero solution of the system (1) *is stable in the whole on* $\mathcal{G}_s$.

If instead of (18) *the condition*

$$-\frac{\lambda_M(B)}{\lambda_M(Q^-)} \ln \frac{\lambda_M(\overline{C}^*)}{\lambda_m(A)} \leq \theta - \gamma \tag{19}$$

holds for some $\gamma > 0$, *then the zero solution of the system* (1) *is asymptotically stable in the whole on* $\mathcal{G}_s$.

Proof. The assertion of Theorem 2 follows from Theorem 2 in [4].

Theorem 3. *Let the system* (1) *be such that matrix-valued function* (2) *is constructed with the elements* (3) *and* (4), *and*

(i) *the matrix* A *in* (6) *is positive definite, i.e.* $\lambda_m(A) > 0$;

(ii) *there exists a matrix* $Q^+ \in R^{s\times s}$ *for which*

a) $\overline{G}(S) \leq Q^+ \quad \forall S \in \mathcal{G}_s$,

b) $\lambda_M(Q^+) > 0$;

(iii) $\lambda_M(\overline{C}^*) > 0$;

(iv) *the functions* $\tau_k(x)$ *satisfy for some* $\theta_1 > 0$ *and for all* $k = 1, 2, \ldots$ *the inequality*

$$\tau_k(x) - \tau_{k-1}(x) \le \theta_1.$$

If in addition the condition

$$\frac{\lambda_m(A)}{\lambda_M(Q^+)} \ln \frac{\lambda_m(A)}{\lambda_M(\overline{C}^*)} \ge \theta_1 \tag{20}$$

is satisfied, then the zero solution of the system (1) *is stable in the whole on* $\mathcal{G}_s$.

If instead of (20) *the inequality*

$$\frac{\lambda_m(A)}{\lambda_M(Q^+)} \ln \frac{\lambda_m(A)}{\lambda_M(\overline{C}^*)} \ge \theta_1 + \gamma$$

holds for some $\gamma > 0$, *then the zero solution of the system* (1) *is asymptotically stable in the whole on* $\mathcal{G}_s$.

Proof. The assertion of this theorem follows from Theorem 3 in [4].

4. Results on instability under structural perturbations. Consider the system (1). Let for this system the matrix-valued function (2) with elements (3) and (4) be constructed and the function (5) which satisfies (6) is introduced.

Theorem 4. *Let the system* (1) *be such that matrix-valued function* (2) *is constructed with the elements* (3) *and* (4), *and*

(i) *the matrix* A *in* (6) *is positive definite, i.e.* $\lambda_m(A) > 0$;

(ii) *there exists a positive semidefinite or equal to zero matrix* $\underline{Q}$ *such that for the matrix* $\underline{G}(S)$ *the estimate*

$$\underline{G}(S) \ge \underline{Q} \qquad \forall\, S \in \mathcal{G}_s$$

is fulfilled element-wise;

(iii) *the matrix* $\underline{C}$ *is positive definite, i.e.* $\lambda_m(\underline{C}) > 0$.

Then the zero solution of the system (1) *is unstable on* $\mathcal{G}_s$.

The proof of the theorem is similar to that of Theorem 4 in [4].

Theorem 5. *Let the system* (1) *be such that matrix-valued function* (2) *be constructed with elements* (3) *and* (4), *and*

(i) *the matrix* A *in* (6) *is positive definite, i.e.* $\lambda_m(A) > 0$;

(ii) *there exists a matrix* $\underline{Q}^- \in R^{s\times s}$ *such that*

a) $\underline{G}(S) \ge \underline{Q}^- \quad \forall\, S \in \mathcal{G}_s$,

b) $\lambda_m(\underline{Q}^-) < 0$;

(iii) *the matrix* $\underline{C}^*$ *is positive definite, i.e.* $\lambda_m(\underline{C}^*) > 0$;

(iv) *for some constant* $\theta_1 > 0$ *the values* $\tau_k(x)$, $k = 1, 2, \ldots$, *satisfy the inequality*

$$\tau_k(x) - \tau_{k-1}(x) \le \theta_1, \quad \textit{for all} \quad k = 1, 2, \ldots .$$

If for some $\gamma > 0$ *the inequality*

$$-\frac{\lambda_m(A)}{\lambda_M(\underline{Q}^-)}\ln\frac{\lambda_m(\overline{C}^*)}{\lambda_M(B)} \geq \theta_1+\gamma$$

holds, then the zero solution of the system (1) *is unstable on* $\mathcal{G}_s$.

Proof. The validity of this theorem follows from Theorem 5 in [4].

Theorem 6. *Let the system* (1) *be such that matrix-valued function* (2) *be constructed with elements* (3) *and* (4), *and*

(i) *the matrix* A *in* (6) *is positive definite, i.e.* $\lambda_m(A) > 0$;

(ii) *there exists a matrix* $\underline{Q}^+ \in R^{s\times s}$ *such that*

a) $\underline{G}(S) \geq \underline{Q}^+ \quad \forall S\in \mathcal{G}_s$,

b) $\lambda_m(\underline{Q}^+) > 0$;

(iii) $\lambda_m(\underline{C}^*) > 0$, *i.e. the matrix* $\underline{C}^*$ *is positive definite;*

(iv) *for some constant* $\theta>0$ *the values* $\tau_k(x)$, $k = 1, 2, \ldots$, *satisfy the correlation*

$$\tau_{k+1}(x)-\tau_k(x) = \theta > 0.$$

If for some $\gamma > 0$

$$\frac{\lambda_m(B)}{\lambda_M(\underline{Q}^+)}\ln\frac{\lambda_M(B)}{\lambda_m(\overline{C}^*)} \leq \theta-\gamma,$$

then the zero solution of the system (1) *is unstable on* $\mathcal{G}_s$.

Proof. The assertion of this theorem is similar to the proof of Theorem 6 in [4].

Example. Let the system (1) be a fourth order system decomposed into two subsystems of the second order which are defined by the matrices:

$$A_1 = \begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix},\quad A_2 = \begin{pmatrix}-2 & 1\\ -1 & -2\end{pmatrix},\quad A_{12}=A_{21}=I_2, \tag{21}$$

$$J_{ki} = \operatorname{diag}\{-1,-1\},\quad i=1,2;\quad J_{k12}=J_{k21}=10^{-2}I_2,$$

$$\mathcal{G}_s = \{S\colon S=\operatorname{diag}\{S_1,S_2\}\},\quad S_i=[S_{i1},S_{i2}],$$

$$S_{ii}=I_2,\quad S_{ij}=s_{ij}I_2,\quad 0\leq s_{ij}\leq 1,\quad i\neq j,\quad i,j=1,2,$$

where $I_2 = \operatorname{diag}\{1,1\}$.

For this example the elements (3) and (4) of the matrix-valued function (2) are constructed in the form

$$U_{ii}(x_i) = x_i^T I_2 x_i,\quad i=1,2,$$

$$U_{12}(x_1,x_2) = U_{21}(x_1,x_2) = x_1^T 10^{-1} I_2 x_2.$$

It is clear that they satisfy the estimates

$$\|x_i\|^2 \leq U_{ii}(x_i)\quad \forall x_i\in R^{n_i},\quad i=1,2,$$

$$-0{,}1\,\|x_1\|\,\|x_2\| \leq U_{12}(x_1,x_2) \leq 0{,}1\,\|x_1\|\,\|x_2\|.$$

For $\eta^T = (1,1)\in R^2$ the matrices

$$A = \begin{pmatrix} 1 & -0,1 \\ -0,1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0,1 \\ 0,1 & 1 \end{pmatrix};$$

are positive definite, i.e.

$$\lambda_m(A) = 0,9, \quad \lambda_M(B) = 1,1.$$

For this choice of the elements of the matrix-valued function we have

$$\overline{\sigma}_{11}(S) = -2 + 0,2 s_{21} \le -1,8,$$

$$\overline{\sigma}_{22}(S) = -4 + 0,2 s_{12} \le -3,8,$$

$$\overline{\sigma}_{12}(S) = \frac{1}{2}\Big(s_{12} + s_{21} + \sqrt{(-0,2 + s_{21})^2 + 0,01} + $$

$$+ \sqrt{(-0,1 + s_{12})^2 + 0,01} + 0,432\Big) \le 2,1,$$

$$\overline{c}_{ii} = -0,968, \quad i = 1, 2; \quad \overline{c}_{12} = 0,099.$$

The matrices

$$\overline{G}(S) \le Q = \begin{pmatrix} -1,8 & 2,1 \\ 2,1 & -3,8 \end{pmatrix}$$

and

$$\overline{C} = \begin{pmatrix} -0,968 & 0,099 \\ 0,099 & -0,968 \end{pmatrix}$$

are negative definite which is confirmed by the estimate

$$\lambda_M(Q) = -0,474 < 0; \quad \lambda_M(\overline{C}) = -0,867 < 0.$$

Thus, all conditions of the Theorem 1 are satisfied and the zero solution of the system (1) with matrices (21) is structurally asymptotically stable in the whole on $\mathcal{G}_s$.

Appendix 1. In order to describe the structurally variable large scale system (1) let the following notation be introduced. The structural parameter $s_{ij}\colon [0, \infty) \to \{0, 1\}$ is a binary valued function of t, or $s_{ij}\colon [0, \infty) \to [0, 1]$, and represents the (i, j)-th element of the structural matrix S_i of the i-th interconnected subsystem S_i,

$$S_i = [s_{i1} I_i, s_{i2} I_i, \ldots, s_{in} I_i], \quad I_1 = \operatorname{diag}\{1, 1, \ldots, 1\} \in R^{n_i} \times R^{n_i}.$$

Notice that it may be, but need not be, required that $s_{ij}(t) = 1$ implies $s_{ik}(t) = 0$ for all $k \ne j$.

Let

$$S = \operatorname{diag}[S_1, S_2, \ldots, S_s], \quad 0_{ij} \in R^{n_i} \times R^{n_i}, \quad i \ne j.$$

The matrix $S(t)$ describes all structural variations of the system (1) and will be called the structural matrix of the system (1). The set of all possible $S(t)$ will be denoted by $\mathcal{G}_s$ and referred to us the structural set of the system (1):

$$\mathcal{G}_s = \{S\colon S = \operatorname{diag}[S_1, S_2, \ldots, S_s], \ S_i = [s_{i1} I_i, s_{i2} I_i, \ldots, s_{in} I_i], \ i, j \in \{0, 1\}\}.$$

For a detailed discussion of this notion, see [1] and references in this monograph.

Appendix 2. The proof of Lemma 3.

First we consider the assertion a). For all $t = \tau_k(x)$, $k = 1, 2, \ldots$, for the function (5) and the system (1), we have

$$V(x + J_k(x), \eta) - V(x, \eta) = \bar{c}_{ii} \sum_{i=1}^{s} U_{ii} \left(x_i + J_{ki} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{s} J_{kij} x_j \right) +$$

$$+ 2 \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j > i}}^{s} U_{ij} \left(x_i + J_{ki} x_i + \sum_{\substack{l=1 \\ l \neq i}}^{s} J_{kil} x_l ,\; x_j + J_{kj} x_j + \sum_{\substack{l=1 \\ l \neq i}}^{s} J_{kjl} x_l \right) -$$

$$- \sum_{i=1}^{s} U_{ii}(x_i) - 2 \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j > i}}^{s} U_{ij}(x_i, x_j) =$$

$$= \sum_{i=1}^{s} \left(x_i + J_{ki} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{s} J_{kij} x_j \right)^T B_{ii} \left(x_i + J_{ki} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{s} J_{kij} x_j \right) +$$

$$+ 2 \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j > i}}^{s} \left(x_i + J_{ki} x_i + \sum_{\substack{l=1 \\ l \neq i}}^{s} J_{kil} x_l \right)^T B_{ij} \left(x_j + J_{kj} x_j + \sum_{\substack{l=1 \\ l \neq j}}^{s} J_{kjl} x_l \right) -$$

$$- \sum_{i=1}^{s} x_i^T B_{ii} x_i - 2 \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j > i}}^{s} x_i^T B_{ij} x_j =$$

$$= \sum_{i=1}^{s} x_i^T \left(B_{ii} J_{ki} + J_{ki}^T B_{ii} + J_{ki}^T B_{ii} J_{ki} + \sum_{\substack{j=1 \\ j \neq i}}^{s} J_{kji}^T B_{ii} J_{kji} \right) x_i +$$

$$+ 2 \sum_{i=1}^{s} \sum_{j=1}^{s} x_i^T \left(B_{ii} J_{kij} + J_{kij}^T B_{ii} + J_{ki}^T B_{ii} J_{kij} + J_{kij}^T B_{ii} J_{ki} + \right.$$

$$\left. + \sum_{\substack{l=1 \\ l \neq i,j}}^{s} \left(J_{kli}^T B_{ll} J_{klj} + J_{klj}^T B_{ll} J_{kli} \right) \right) x_j +$$

$$+ \sum_{i=1}^{s} x_i^T \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{s} \left(B_{ij} + J_{kij} + J_{kij}^T B_{ij} \right) + \sum_{\substack{j=1 \\ j \neq i}}^{s} \left(J_{ki}^T B_{ij} J_{kji} + J_{kji}^T B_{ij} J_{ki} \right) + \right.$$

$$+ \sum_{\substack{l=1 \\ l \neq i}}^{s} \sum_{\substack{j=1 \\ j \neq i}}^{s} (J_{kli}^T B_{lj} J_{kji} + J_{kji}^T B_{lj} J_{kli}) \Bigg\} x_i +$$

$$+ 2 \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j>i}}^{s} x_i^T \Bigg\{ B_{ij} J_{kj} + J_{ki}^T B_{ij} + J_{ki}^T B_{ij} J_{kj} +$$

$$+ \sum_{\substack{l=1 \\ l \neq i,j}}^{s} (B_{il} J_{klj} + J_{kli}^T B_{lj} + J_{ki}^T B_{il} J_{klj} + J_{kli}^T B_{lj} J_{kj}) + \sum_{l=1}^{s} \sum_{\substack{r=1 \\ l \neq r}}^{s} J_{li}^T B_{ir} J_{rj} \Bigg\} x_j =$$

$$= \sum_{i=1}^{s} x_i^T C_{ii} x_i + 2 \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j>i}}^{s} x_i^T C_{ij} x_j \leq$$

$$\leq \sum_{i=1}^{s} \lambda_M(C_{ii}) \|x_i\|^2 \; 2 \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j>i}}^{s} \lambda_M^{1/2}(C_{ij} C_{ij}^T) \|x_i\| \, \|x_j\| = u_k^T C u_k ,$$

$$k = 1, 2, \ldots .$$

Inequality b) is proved in the same way.

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Received 09.04.97