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ON HÖLDER CONTINUITY OF SOLUTIONS OF DOUBLY NONLINEAR PARABOLIC EQUATIONS WITH WEIGHT

ГЕЛЬДЕРОВІСТЬ РОЗВ'ЯЗКІВ НЕЛІНІЙНИХ ПАРАБОЛІЧНИХ РІВНЯНЬ З ПОДВІЙНОЮ НЕЛІНІЙНІСТЮ (ВАГОВИЙ ВИПАДОК)

We prove the Hölder regularity of bounded weak solutions of doubly nonlinear degenerate parabolic equations with measurable coefficients.

Доведено регулярність за Гельдером обмежених слабких розв'язків вироджених параболічних рівнянь з подвійною нелінійністю та вимірними коефіцієнтами.

1. Introduction. In this paper, we consider the Hölder continuity of solutions of the equation

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,t,u,\nabla u) = a_0(x,t,u,\nabla u), \tag{1}$$

$$(x,t) \in \Omega_T \equiv \Omega \times (0,T), \qquad 0 < T < \infty .$$

We assume that $a_i(x,t,u,\xi)\colon \Omega_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i = 0, 1, \ldots, N$, are Caratheodory functions and, with some positive constants c_0, c_1, and c_2, the following inequalities hold:

$$\sum_{i=1}^{N} a_i(x,t,u,\xi)\,\xi_i \;\geq\; c_0\, v(x)\,|u|^{\beta}|\xi|^{m} + \varphi_0(x,t),$$

$$|a_i(x,t,u,\xi)| \;\leq\; c_1\, v(x)\,|u|^{\beta}|\xi|^{m-1} + \varphi_1(x,t), \qquad i = 1, 2, \ldots, N,$$

$$|a_0(x,t,u,\xi)| \;\leq\; c_2\, v(x)\,|u|^{\beta}|\xi|^{m} + \varphi_2(x,t),$$

where $m \geq 2$, $\beta > 0$, $\varphi_i(x,t)$, $i = 0, 1, 2$, are given nonnegative functions defined in Ω_T, and $v(x) \in A_{1+m/N}$. For the definition and main properties of the classes A_p, see, e.g., [1]. The first results about the Hölder continuity of solutions of equation (1) with $m = 2$, $\beta = 0$, and $v(x) = 1$ were obtained in [2], where the classes B_2 were defined and the imbedding $B_2 \hookrightarrow C^{\alpha,\alpha/2}(\Omega_T)$ was proved.

In the case where $m \neq 2$, $\beta = 0$, and $v(x) = 1$, the classes B_m were introduced for solutions of equation (1) and the imbedding $B_m \hookrightarrow C^{\alpha,\alpha/m}(\Omega_T)$ was proved in [3, 4]. The case where $m \neq 2$, $\beta \neq 0$, and $v(x) = 1$ was considered in [5, 6].

For $m = 2$, $\beta = 0$, and $v(x) \in A_{1+2/N}$, the Harnack inequality was proved in [7, 8]. In [9], the cases $m \neq 2$, $\beta = 0$, $v(x) \in A_{1+m/N}$ and $m = 2$, $\beta \neq 0$, $v(x) \in A_{1+2/N}$ were considered.

In the present paper, we consider the case where $m \neq 2$, $\beta \neq 0$, and $v(x) \in A_{1+m/N}$.

2. Definitions and auxiliary propositions. As in [1], we say that $v(x) \in D_{\mu}$ if the following inequality holds for arbitrary $B_{\rho}(x_0)$ and $B_s(x_0)$, $0 < s \leq \rho$:

$$\frac{v(B_\rho(x_0))}{v(B_s(x_0))} \leq c\left(\frac{\rho}{s}\right)^{N\mu},$$

where c is a positive constant independent of x_0, ρ, and s. Here $B_\rho(x_0)$ is ball of radius ρ centered at x_0 and $v(B_\rho(x_0)) = \int_{B_\rho(x_0)} v(x)\,dx$.

It is known that $A_p \hookrightarrow D_p$, and there exists $\mu < p$ such that $A_p \hookrightarrow D_\mu$.

Let $W_m^{1,0}(\Omega_T, v)$ be a Banach space endowed with the norm

$$\|u\|^m_{W_m^{1,0}(\Omega_T, v)} = \int_{\Omega_T} \left\{[1 + v(x)]\,|u(x,t)|^m + v(x)\left|\frac{\partial u(x,t)}{\partial x}\right|^m\right\} dx\,dt.$$

We also use functions from the space $V(\Omega_T, v)$ endowed with the norm

$$\|u\|_{V(\Omega_T,v)} = \left\{\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |u|^2\,dx\right\}^{1/2} + \left\{\int_{\Omega_T} v(x)\,|u|^\beta \left|\frac{\partial u(x,t)}{\partial x}\right|^m dx\,dt\right\}^{1/m}.$$

We also define the space $V_m(\Omega_T, v)$ with the norm

$$\|u\|^m_{V_m(\Omega_T, v)} = \operatorname*{ess\,sup}_{0<t<T} \int_\Omega |u|^m\,dx + \int_{\Omega_T} v(x)\,|u|^\beta \left|\frac{\partial u(x,t)}{\partial x}\right|^m dx\,dt.$$

We say that a function $u(x,t) \in V(\Omega_T, v)$ is a solution of equation (1) if, for all $\varphi \in \overset{\circ}{W}{}_m^{1,0}(\Omega_T, v)$ and $\varphi_t \in L_2(\Omega_T)$, the following identity is satisfied for any t_1 and t_2 such that $0 < t_1 < t_2 \leq T$:

$$\int_\Omega u(x,t)\,\varphi(x,t)\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \Big\{-u(x,t)\,\varphi_t(x,t) + \sum_{i=1}^N a_i(x,t,u(x,t),u_x(x,t))\,\frac{\partial\varphi}{\partial x_i}(x,t) + a_0(x,t,u(x,t),u_x(x,t))\,\varphi(x,t)\Big\}\,dx\,dt = 0.$$

If $v(x) \in A_{1+m/N}$, then, by using the results of [1], one can easily see that

$$\frac{1}{\operatorname{mes} B_\rho(x_0)}\int_0^T\int_{B_\rho(x_0)} |u(x,t)|^q\,dx\,dt + \frac{1}{v(B_\rho(x_0))}\int_0^T\int_{B_\rho(x_0)} v(x)\,|u(x,t)|^q\,dx\,dt \leq$$

$$\leq c\left\{\operatorname*{ess\,sup}_{0<t<T}\frac{1}{\operatorname{mes} B_\rho(x_0)}\int_{B_\rho(x_0)} |u(x,t)|^m\,dx\right\}^{(q-m)/m} \times$$

$$\times\left\{\operatorname*{ess\,sup}_{0<t<T}\frac{T}{\operatorname{mes} B_\rho(x_0)}\int_{B_\rho(x_0)} |u(x,t)|^m\,dx + \frac{\rho^m}{v(B_\rho(x_0))}\int_0^T\int_{B_\rho(x_0)} v(x)\left|\frac{\partial u}{\partial x}\right|^m dx\,dt\right\}, \tag{2}$$

where c is a constant independent of T, ρ, $q > m$, and

$$u(x,t) \in \overset{\circ}{V}_m(B_\rho(x_0)\times(0,T), v).$$

Further, for simplicity of exposition, we consider the case where the estimate

$$\operatorname*{ess\,sup}_{x \in B_\rho(x_0)} v(x) \le c \frac{v(B_\rho(x_0))}{\rho^N} \tag{3}$$

with a constant c independent of ρ holds for arbitrary $B_\rho(x_0)$.

In the general case, we will make only necessary remarks in the course of the proof (see [9]).

If (3) is satisfied, then we only need the following inequality:

$$\frac{1}{\operatorname{mes} B_\rho(x_0)} \int_0^T \int_{B_\rho(x_0)} |u(x,t)|^{m(m+N-N(\mu-1))/N} dx\,dt \le$$

$$\le c \left\{ \operatorname*{ess\,sup}_{0<t<T} \frac{1}{\operatorname{mes} B_\rho(x_0)} \int_{B_\rho(x_0)} |u(x,t)|^m dx \right\}^{(m-N(\mu-1))/N} \times$$

$$\times \left\{ \frac{\rho^m}{v(B_\rho(x_0))} \int_0^T \int_{B_\rho(x_0)} v(x) \left| \frac{\partial u}{\partial x} \right|^m dx\,dt \right\}, \tag{4}$$

where c is a constant independent of $T, \rho, \mu < 1 + \dfrac{m}{N}$, and $u(x,t) \in \overset{\circ}{V}_m(B_\rho(x_0) \times \times (0,T), v)$.

We assume that $v(x)^{-(\hat{q}-1)/\hat{q}} \varphi_0(x,t)$, $v(x)^{-1/(m-1)-(\hat{q}-1)/\hat{q}} [\varphi_1(x,t)]^{m/(m-1)}$, $v(x)^{-(\hat{q}-1)/\hat{q}} \varphi_2(x,t) \in L_{\hat{q}}(\Omega_T)$, where

$$\hat{q} = \frac{m + N - N(\mu-1)}{m(1 - \kappa_1 - N(\mu-1)/m)}, \qquad 0 < \kappa_1 < 1 - \frac{N}{m}(\mu-1).$$

***Remark* 1.** If $v(x) \equiv |x|^\gamma$, then we have $v(x) \in A_{1+m/N}$ if $-N < \gamma < m$; moreover, inequality (3) holds if $0 \le \gamma < m$.

***Remark* 2.** If $a_0(x,t,u,\xi)$ satisfies the inequality

$$|a_0(x,t,u,\xi)| \le c_2 v(x) |\xi|^{m-1} + \varphi_2(x,t),$$

then it is easy to see that

$$\operatorname*{ess\,sup}_{(x,t) \in \Omega_T} |u(x,t)| \le M < \infty.$$

Below, we shall use the following well-known inequality, which can be found, e.g., in [2]:

$$(l-k) \operatorname{mes} A^+_{l,R} \le c \frac{R^{N+1}}{\operatorname{mes}\{B_R \setminus A^+_{k,R}\}} \int_{A^+_{k,R} \setminus A^+_{l,R}} \left| \frac{\partial u}{\partial x} \right| dx, \tag{5}$$

where $u \in W^1_1(B_R)$ is an arbitrary function and $l, k \in \mathbb{R}$, $l > k$.

Here, $A^+_{k,R} = \{x \in B_R : (u-k)_+ > 0\}$.

3. Definitions of classes $B_{m,\beta}(\Omega_T, M, \gamma, q, \delta, \kappa, v)$, $m \ge q$, and main result. Let Ω be an open bounded set in $\mathbb{R}^N$ and let $\Omega_T \equiv \Omega \times (0,T)$. If $(x_0, t_0) \in \Omega_T$, we define $B_R = \{x \in \Omega : |x - x_0| < R\}$ and $Q(R,\rho) \equiv B_R \times \{t_0 - \rho, t_0\}$, $\rho > 0$. We consider R and ρ so small that $Q(R,\rho) \subset \Omega_T$.

For a bounded measurable function u defined in $Q(R,\rho)$, we consider the functions $(u-k)_\pm$, $k \in \mathbb{R}$, and $H_\pm \in \mathbb{R}^+$ such that

$$\| (u-k)_\pm \|_{\infty, Q(R,\rho)} \le H_\pm \le \delta,$$

where $\delta < \delta_0$ is a given positive number.

We also define

$$\psi(H_\pm, (u-k)_\pm, \nu) = \left[\ln \left\{ \frac{H_\pm}{H_\pm - (u-k)_\pm + \nu} \right\} \right]_+, \qquad \nu < \min\{H_\pm, 1\}.$$

We say that a function $u(x,t) \in V(\Omega_T, \upsilon) \cap L_\infty(\Omega_T)$ belongs to $B_{m,\beta}(\Omega_T, M, \gamma, q, \delta, \kappa, \upsilon)$ if the following inequalities hold:

$$\operatorname*{ess\,sup}_{t_0-\rho \le t \le t_0} \int_{B_R} (u-k)_\pm^2 \, \xi^m(x,t)\, dx + \int_{Q(R,\rho)} \upsilon(x) |u|^\beta \left| \frac{\partial}{\partial x}(u-k)_\pm \right|^m \xi^m \, dx\, dt \le$$

$$\le \int_{B_R} (u-k)_\pm^2 \, \xi^m(x, t_0-\rho)\, dx + \gamma \left\{ \int_{Q(R,\rho)} (u-k)_\pm^2 \, \xi^{m-1} \left| \frac{\partial \xi}{\partial t} \right| dx\, dt + \right.$$

$$\left. + \int_{Q(R,\rho)} \upsilon(x) |u|^\beta (u-k)_\pm^m \left| \frac{\partial \xi}{\partial x} \right|^m dx\, dt + \left[\int_{t_0-\rho}^{t_0} \upsilon(A_{k,r}^\pm)\, dt \right]^{m(1+\kappa)/q} \right\}, \tag{6}$$

$$\operatorname*{ess\,sup}_{t_0-\rho \le t \le t_0} \int_{B_R} \psi^2(H_\pm, (u-k)_\pm, \nu)\, \xi^m(x)\, dx \le$$

$$\le \int_{B_R \times \{t_0-\rho\}} \psi^2(H_\pm, (u-k)_\pm, \nu)\, \xi^m(x)\, dx +$$

$$+ \gamma \int_{Q(R,\rho)} \upsilon(x) |u|^\beta \psi(H_\pm, (u-k)_\pm, \nu) |\psi_u(H_\pm, (u-k)_\pm, \nu)|^{2-m} \left| \frac{\partial \xi}{\partial x} \right|^m dx\, dt +$$

$$+ \frac{\gamma}{\nu^2} \left(1 + \ln \frac{H_\pm}{\nu} \right) \left\{ \int_{t_0-\rho}^{t_0} \upsilon(A_{k,R}^\pm)\, dx \right\}^{m(1+\kappa)/q}, \tag{7}$$

where

$$A_{k,R}^\pm(t) = \{ x \in B_R : (u-k)_\pm > 0 \},$$

$$\upsilon(A_{k,R}^\pm) = \int_{A_{k,R}^\pm} \upsilon(x)\, dx .$$

The parameters in (6) and (7) are as follows:

(i) $\delta \le \delta_0$ and γ are some positive numbers;

(ii) $\| (u-k)_\pm \|_{\infty, Q(R,\rho)} \le \delta$;

(iii) $\kappa = \dfrac{m}{N} \kappa_1$, $q = \dfrac{\hat{q}}{\hat{q}-1} m(1+\kappa) = m \dfrac{m + N - N(\mu-1)}{N}$.

We also use the integral identity

$$\int_{t_1}^{t_2} \int_\Omega \left\{ \frac{\partial u_h}{\partial t} \varphi + \sum_{i=1}^N [a_i(x,t,u,u_x)]_{\bar{h}} \frac{\partial \varphi}{\partial x_i} + [a_0(x,t,u,u_x)]_{\bar{h}} \varphi \right\} dx\, dt = 0, \tag{8}$$

where $\varphi \in \overset{\circ}{W}{}_m^{1,0}(\Omega_T, \upsilon)$ and $h < t_1 < t_2 < T - h$.

Here,

$$g_h(x,t) = \frac{1}{h}\int\limits_t^{t+h} g(x,\tau)\,d\tau\,,$$

$$g_{\bar h}(x,t) = \frac{1}{h}\int\limits_{t-h}^{t} g(x,\tau)\,d\tau\,.$$

***Remark* 3.** If inequality (3) is not satisfied, then we assume that there exists $0 < \kappa < 1$ such that, for $q = \frac{\hat q}{\hat q - 1}\, m(1+\kappa)$, inequality (2) is satisfied.

Inequalities (6), (7) can be proved analogously to [3] by substitution of the functions $\varphi = (u-k)_\pm \xi^m(x,t)$ and $\varphi = [\psi^2(H_\pm, (u-k)_\pm, \nu)]\,\xi^m(x)$ in the integral identity (8).

Here, $\xi(x,t)$ is a cut-off function for the cylinder $Q(R,\rho)$, and $\xi(x)$ is a cut-off function for the ball B_R.

4. Interior regularity. Let us fix a point $(x_0, t_0) \in \Omega_T$ and define

$$\kappa_0 = \frac{\kappa(m+N) + N(\mu-1)}{m+N-N(\mu-1)}, \tag{9}$$

$$Q_R^{N\kappa_0} \equiv B_R \times \left\{ t_0 - R^{m - N\kappa_0(m-2+\beta)/(m+\beta)} \frac{\operatorname{mes} B_R}{\upsilon(B_R)}, t_0 \right\}.$$

Let

$$\mu_+ = \operatorname*{ess\,sup}_{Q_R^{N\kappa_0}} u\,, \qquad \mu_- = \operatorname*{ess\,inf}_{Q_R^{N\kappa_0}} u\,,$$

and let ω and M be defined according to the formulas $\omega = \mu_+ - \mu_- = \operatorname*{ess\,osc}_{Q_R^{N\kappa_0}} u$ and $M = \max\left\{ \|u\|_{\infty, Q_R^{N\kappa_0}}, \omega \right\}$.

Analogously to [9], we can find a sequence $\alpha_n = \alpha^{-n}$, $n = 0, 1, 2, \ldots$, such that $\alpha_n \downarrow 0$ and

$$\alpha_{n+1}^m \frac{\operatorname{mes} B_{\alpha_{n+1},R}}{\upsilon(B_{\alpha_{n+1},R})} \le \alpha_n^m \frac{\operatorname{mes} B_{\alpha_n,R}}{\upsilon(B_{\alpha_n,R})}. \tag{10}$$

Let s^* be a positive number, which will be determined in what follows, and let

$$\theta = \left(\frac{\alpha^{s^*}}{\omega}\right)^{m-2} \frac{1}{M^\beta}.$$

Consider the cylinder

$$Q_R^\theta \equiv B_R \times \left\{ t_0 - \theta R^m \frac{\operatorname{mes} B_R}{\upsilon(B_R)}, t_0 \right\}.$$

If $\left(\frac{\omega}{\alpha^{s^*}}\right)^{m-2} M^\beta \ge R^{N\kappa_0(m-2+\beta)/(m+\beta)}$, then $Q_R^\theta \subset Q_R^{N\kappa_0}$.

We also consider the cylinder

$$\overline{Q}_R^\eta \equiv B_R \times \left\{ \bar t - \eta R^m \frac{\operatorname{mes} B_R}{\upsilon(B_R)}, \bar t \right\},$$

where $\bar{t} \le t_0$, $\bar{t} - \eta R^m \dfrac{\operatorname{mes} B_R}{v(B_R)} \ge t_0 - \theta R^m \dfrac{\operatorname{mes} B_R}{v(B_R)}$,

$$\eta = \left(\frac{\alpha^{s_0}}{\omega}\right)^{m-2} \frac{1}{M^{\beta}}, \qquad s_0 < s^*,$$

and s_0 is the smallest positive number that satisfies the inequality

$$2\|u\|_{\infty, Q_R^{N\kappa_0}} \alpha^{-s_0} \le \delta.$$

Lemma 1. *There exists a number* $\nu_0 \in (0, 1)$ *independent of* ω, R, *and* s^* *and such that if for some cylinder* $\overline{Q}_R^{\eta}$

$$\operatorname{mes}\left\{(x, t) \in \overline{Q}_R^{\eta}:\ u(x, t) < \mu_- + \frac{\omega}{\alpha^{s_0}}\right\} \le \nu_0 \operatorname{mes} \overline{Q}_R^{\eta}, \tag{11}$$

then either

$$R^{N\kappa_0/(m+\beta)} \ge \left(\frac{\omega}{\alpha^{s_0}}\right)^{1+\frac{m}{m+\beta}\left(\frac{1+\kappa}{q}-\frac{1}{m}\right)_+(m-2+\beta)} \tag{12}$$

or

$$u(x, t) \ge \mu_- + \frac{\omega}{\alpha^{s_0+4}}, \tag{13}$$

for almost every $(x, t) \in \overline{Q}_{R/\alpha}^{\eta}$.

Proof. Assume that (12) is not satisfied. Then $Q_R^{\theta} \subset Q_R^{N\kappa_0}$ and we let

$$R_n = \frac{R}{\alpha} + \frac{R}{\alpha^n}, \qquad \overline{R}_n = \frac{R}{\alpha} + \frac{R}{2}\frac{1+\alpha}{\alpha^{n+1}}, \quad n = 1, 2, \ldots .$$

Let $\xi(x, t) \in C^{\infty}(Q_R^{N\kappa_0})$, $\xi(x, t) \equiv 1$ in $\overline{Q}_{\overline{R}_n}^{\eta}$, $\xi(x, t) = 0$ for $t = \bar{t} - \eta R_n^m \dfrac{\operatorname{mes} B_{R_n}}{v(B_{R_n})}$, and

$$\left|\frac{\partial \xi}{\partial x}\right| \le \frac{2}{\alpha - 1}\frac{\alpha^{n+1}}{R}, \qquad \left|\frac{\partial \xi}{\partial t}\right| \le c\,\frac{\alpha^{n - nN(\mu-1)}}{R^m}\frac{v(B_R)}{\operatorname{mes} B_R}\left(\frac{\omega}{\alpha^{s_0}}\right)^{m-2} M^{\beta}.$$

For $n = 1, 2, \ldots$, we take

$$\kappa_n = \mu_- + \frac{\omega}{\alpha^{s_0+1}} + \frac{\omega}{\alpha^{s_0+n}} \quad \text{if} \quad \mu_- + \frac{\omega}{\alpha^{s_0+1}} \ge \frac{\omega}{\alpha^{s_0+n}},$$

$$\kappa_n = \mu_- + \frac{\omega}{\alpha^{s_0+4}} + \frac{\omega}{\alpha^{s_0+n+4}} \quad \text{if} \quad \mu_- + \frac{\omega}{\alpha^{s_0+1}} < \frac{\omega}{\alpha^{s_0+n}}.$$

Let $\mu_- + \dfrac{\omega}{\alpha^{s_0+1}} \ge \dfrac{\omega}{\alpha^{s_0+n}}$, then, analogously to [6, p. 159–161], we can obtain

$$\operatorname*{ess\,sup}_{\bar{t} - \eta \overline{R}_n^m \frac{\operatorname{mes} B_{\overline{R}_n}}{v(B_{\overline{R}_n})} \le t < \bar{t}} \|(w-k)_-\|_{m, B_{\overline{R}_n}}^m + \frac{\gamma(s_0)}{\eta}\left\| v^{1/m}\frac{\partial}{\partial x}(w - k_n)\right\|_{m, \overline{Q}_{\overline{R}_n}^{\eta}}^m \le$$

$$\le c\alpha^{mn}\left(\frac{\omega}{\alpha^{s_0}}\right)^m \frac{v(B_R)}{R^m \operatorname{mes} B_R}\left\{\frac{1}{\eta}\int_{\bar{t} - \eta R_n^m \frac{\operatorname{mes} B_{R_n}}{v(B_{R_n})}}^{\bar{t}} \operatorname{mes} A_{k_n, R_n}^-(t)\, dt\right\} +$$

$$+ \; c\alpha^{mn}\left(\frac{1}{\eta}\frac{\upsilon(B_R)}{\operatorname{mes} B_R}\int\limits_{\bar t-\eta \bar R_n^m \frac{\operatorname{mes} B_{R_n}}{\upsilon(B_{R_n})}}^{\bar t} \operatorname{mes} A^-_{\bar k_n, R_n}(t)\,dt\right)^{m(1+\kappa)/q} \eta^{m(1+\kappa)/q}\left(\frac{\omega}{\alpha^{s_0}}\right)^{m-2}, \tag{14}$$

where $\gamma(s_0)$ is a constant depending on s_0 and

$$w = \begin{cases} u & \text{if } u > \dfrac{\omega}{\alpha^{s_0+4}}, \\ \dfrac{\omega}{\alpha^{s_0+4}} & \text{if } u \le \dfrac{\omega}{\alpha^{s_0+4}}. \end{cases}$$

Now, we introduce the new variable $z = \dfrac{t-\bar t}{\eta}$ and define

$$Q_n \equiv B_{R_n} \times \left\{-R_n^m \frac{\operatorname{mes} B_{R_n}}{\upsilon(B_{R_n})}, 0\right\}, \qquad \bar Q_n \equiv B_{\bar R_n} \times \left\{-\bar R_n^m \frac{\operatorname{mes} B_{\bar R_n}}{\upsilon(B_{\bar R_n})}, 0\right\},$$

$$A_n(z) = \{x \in B_{R_n} : u(x,z) < k_n\},$$

$$|A_n| = \int\limits_{-R_n^m \frac{\operatorname{mes} B_{R_n}}{\upsilon(B_{R_n})}}^{0} \operatorname{mes} A_n(z)\,dz\,.$$

Then, from (14), we obtain

$$\|(w-k_n)_-\|^m_{V_m(\bar Q_n,\upsilon)} \le c\alpha^{mn}\left(\frac{\omega}{\alpha^{s_0}}\right)^m \frac{\upsilon(B_R)}{R^m \operatorname{mes} B_R}|A_n| + \\ + \; c\alpha^{mn}\left(\frac{\upsilon(B_R)}{\operatorname{mes} B_R}|A_n|\right)^{m(1+\kappa)/q} \eta^{m(1+\kappa)/q}\left(\frac{\omega}{\alpha^{s_0}}\right)^{m-2}. \tag{15}$$

From (15), using inequality (4), we get

$$|A_{n+1}| \le c\alpha^{2mn}\left[\frac{R^m(\operatorname{mes} B_R)^2}{\upsilon(B_R)}\right]^{m/q}|A_n|^{(q-m)/q}\left\{\frac{\upsilon(B_R)}{R^m(\operatorname{mes} B_R)^2}|A_n| + \\ + \left(\frac{\alpha^{s_0}}{\omega}\right)^2 \frac{1}{\operatorname{mes} B_R}\left(\frac{\upsilon(B_R)}{\operatorname{mes} B_R}|A_n|\right)^{m(1+\kappa)/q}\eta^{m(1+\kappa)/q}\right\}. \tag{16}$$

Let us define

$$Y_n = \frac{\upsilon(B_R)}{R^m(\operatorname{mes} B_R)^2}|A_n|,$$

$$Z_n = (\operatorname{mes} B_R)^{(1+\kappa)/(1+\kappa_0)}\left(\frac{\upsilon(B_R)}{\operatorname{mes} B_R}|A_n|\right)^{m/q}.$$

Note that if (12) is violated, then

$$\left(\frac{\alpha^{s_0}}{\omega}\right)^2 \eta^{m(1+\kappa)/q} R^{N\kappa_0} \le 1\,.$$

Therefore, from (16), we obtain

$$Y_{n+1} \le c\alpha^{2mn}\{Y_n^{1+(q-m)/q} + Y_n^{(q-m)/q} Z_n^{1+\kappa}\}. \tag{17}$$

Bearing in mind that, by virtue of (9) and the equality $q = m\dfrac{m + N - N(\mu - 1)}{N}$, we have

$$(m+N)\frac{m}{q} = N\frac{1+\kappa_0}{1+\kappa},$$

analogously to [2, p. 77] we obtain

$$Z_{n+1} \le c\alpha^{2mn}\{Y_n + Z_n^{1+\kappa}\}. \tag{18}$$

Consequently, from (17) and (18) with some easy modifications of [2, p. 96], we get (13). If $\mu_- + \dfrac{\omega}{\alpha^{s_0+1}} < \dfrac{\omega}{\alpha^{s_0+n}}$, we can obtain (13) by analogy with [6] without using the truncated function ω.

***Remark* 4.** If inequality (3) is not satisfied, then we set

$$\begin{aligned} Y_n &= \frac{v(B_R)}{R^m(\operatorname{mes} B_R)^2}|A_n| + \frac{v(A_n)}{R^m \operatorname{mes} B_R}, \\ Z_n &= (\operatorname{mes} B_R)^{(1+\kappa)/(1+\kappa_0)}(v(A_n))^{m/q}, \\ \tilde{\kappa}_0 &= \frac{m+N}{N}\frac{m}{q}(1+\kappa) - 1, \end{aligned} \tag{19}$$

and use the following inequality, which characterizes the A_p-weight, for $A_1 \subset Q_1$ and some $\varepsilon > 0$:

$$\frac{v(A_1)}{v(Q_1)} \le \left(\frac{\operatorname{mes} A_1}{\operatorname{mes} Q_1}\right)^{\varepsilon}.$$

Then we can deduce inequalities analogous to (17) and (18). We now assume that the conditions of Lemma 1 are satisfied for some cylinder

$$Q \equiv B_{R/\alpha} \times \left\{\bar{t} - \eta\left(\frac{R}{\alpha}\right)^m \frac{\operatorname{mes} B_{R/\alpha}}{v(B_{R/\alpha})}, t_0\right\}.$$

The length of this cylinder is at least $\eta\left(\dfrac{R}{\alpha}\right)^m \dfrac{\operatorname{mes} B_{R/\alpha}}{v(B_{R/\alpha})}$ and at most

$$\eta\left(\frac{R}{\alpha}\right)^m \frac{\operatorname{mes} B_{R/\alpha}}{v(B_{R/\alpha})} + (\theta - \eta)R^m\frac{\operatorname{mes} B_R}{v(B_R)} \le \theta R^m \frac{\operatorname{mes} B_R}{v(B_R)};$$

here, we have used inequality (10). If we take $\rho = R/\alpha^2$, then we can write

$$Q \equiv Q_{\alpha\rho}^{\tilde{\theta}} = B_{\alpha\rho} \times \left\{t_0 - \tilde{\theta}(\alpha\rho)^m \frac{\operatorname{mes} B_{\alpha\rho}}{v(B_{\alpha\rho})}, t_0\right\},$$

where

$$\tilde{\theta} = \left(\frac{\alpha^{\bar{s}_0}}{\omega}\right)^{m-2}\frac{1}{M^{\beta}}, \qquad s_0 \le \bar{s} \le s^*,$$

and the constant c depends only on α, μ, m, and N.

Lemma 2. *Suppose that*

$$H_- = \left\|\left(u - \left(\mu_- + \frac{\omega}{\alpha^{s_0+4}}\right)\right)_-\right\|_{\infty,Q} > \frac{\omega}{\alpha^{s_0+5}}.$$

Then, for every $\nu_1 \in (0, 1)$, *there exists a positive number* $s_1 = s_1(\nu_1, \gamma, \kappa, \delta, q)$ *such that either*

$$R^{N\kappa_0/(m+\beta)} \geq \left(\frac{\omega}{\alpha^{s_1}}\right)^{1+\frac{m}{m+\beta}\left(\frac{1+\kappa}{q}-\frac{1}{m}\right)_+(m-2+\beta)} \tag{20}$$

or

$$\text{mes}\left\{x \in B_\rho:\ u(x,t) < \mu_- + \frac{\omega}{\alpha^{s_1}}\right\} < \nu_1 \,\text{mes}\, B_\rho \tag{21}$$

for all $t \in \left[t_0 - \tilde{\theta}(\alpha\rho)^m \dfrac{\text{mes}\, B_{\alpha\rho}}{\upsilon(B_{\alpha\rho})}, t_0\right]$.

Proof. Analogously to [3], with some modifications of [6], we can obtain

$$\int\limits_{B_\rho\times\{t\}} \psi^2\left(H_-, \left(u - \left(\mu_- + \frac{\omega}{\alpha^{s_0+4}}\right)\right)_-, \frac{\omega}{\alpha^{s_0+n+4}}\right) dx \leq$$

$$\leq \frac{c}{\rho^m}(n-1)\frac{\upsilon(B_\rho)}{\text{mes}\, B_\rho}\left(\frac{\omega}{\alpha^{s_0}}\right)^{m-2} M^\beta \,\text{mes}\, Q_{\alpha\rho}^{\tilde{\theta}} +$$

$$+ cn\left(\frac{\alpha^{s_0+n+4}}{\omega}\right)^2\left(1 + \ln H_- \frac{\alpha^{s_0+n+4}}{\omega}\right)\tilde{\theta}^{m(1+\kappa)/q} R^{N\kappa_0}\,\text{mes}\, B_\rho, \tag{22}$$

where κ_0 is defined by (9). Let us take $s_1 = s_0 + n + 4$. If (20) is violated, then the right-hand side of (22) is bounded from above by

$$n\, c(s^*)\,\text{mes}\, B_\rho.$$

On this set, since $H_- > \dfrac{\omega}{\alpha^{s_0+5}}$, we have

$$\psi^2\left(H_-, \left(u - \left(\mu_- + \frac{\omega}{\alpha^{s_0+4}}\right)\right)_-, \frac{\omega}{\alpha^{s_0+n+4}}\right) \geq \ln^2 \frac{\omega/\alpha^{s_0+5}}{\omega/\alpha^{s_0+n+4}} = (n-1)^2 \ln^2 \alpha.$$

Therefore, for all $t \in \left[t_0 - \tilde{\theta}(\alpha\rho)^m \dfrac{\text{mes}\, B_{\alpha\rho}}{\upsilon(B_{\alpha\rho})}, t_0\right]$, we have

$$\text{mes}\, A^-_{\mu_- + \omega/\alpha^{s_0+n+4}} \leq c(s^*)\frac{n}{(n-1)^2}\,\text{mes}\, B_\rho. \tag{23}$$

If we take n sufficiently large, then we obtain (21) from (23).

Remark 5. If inequality (3) is not satisfied, then we can obtain the inequality

$$\int\limits_{B_\rho\times\{t\}} \psi^2\left(H_-, \left(u - \left(\mu_- + \frac{\omega}{\alpha^{s_0+2}}\right)\right)_-, \frac{\omega}{\alpha^{s_0+n+4}}\right) dx \leq$$

$$\leq \frac{c}{\rho^m}(n-1)\left(\frac{\omega}{\alpha^{s_0}}\right)^{m-2} M^\beta \upsilon(Q_{\alpha\rho}^{\tilde{\theta}}) + cn\left(\frac{\alpha^{s_0+n+4}}{\omega}\right)^2 \tilde{\theta}^{m(1+\kappa)/q} R^{N\kappa_0}\,\text{mes}\, B_\rho, \tag{24}$$

where $\tilde{\kappa}_0$ is defined by (19). By analogy with Lemma 2, using (24) and taking into account that

$$\upsilon(Q_{\alpha\rho}^{\tilde{\theta}}) = \int\limits_{t_0 - \tilde{\theta}(\alpha\rho)^m \frac{\text{mes}\, B_{\alpha\rho}}{\upsilon(B_{\alpha\rho})}}^{t_0} dt \int\limits_{B_{\alpha\rho}} \upsilon(x)\, dx \leq c\tilde{\theta}\rho^m \,\text{mes}\, B_\rho,$$

we can now prove estimate (23). Hence, in the same way as in [3], with slight modifications of the procedure used in [9], we can obtain the following theorem.

Theorem 1. *There exists positive numbers* $\nu_0 \in (0, 1)$ *and* s *independent of* ω *and* R *and such that if for some cylinder* $\overline{Q}_R^\eta$ *we have*

$$\operatorname{mes}\left\{(x, t) \in \overline{Q}_R^\eta:\ u(x, t) < \mu_- + \frac{\omega}{\alpha^{s_0}}\right\} \le \nu_0 \operatorname{mes} \overline{Q}_R^\eta,$$

then either

$$\omega < \alpha^s R^{N\kappa_0\xi/(m+\beta)}, \qquad \xi = \left[1 + \frac{m}{m+\beta}\left(\frac{1+\kappa}{q} - \frac{1}{m}\right)_+ (m-2+\beta)\right]^{-1}$$

or

$$\operatorname*{ess\,osc}_{Q^\eta_{R/\alpha^s}} u \le \omega\left(1 - \frac{1}{\alpha^s}\right).$$

Now assume that assumption (11) of Lemma 1 is violated, i.e., for every cylinder $\overline{Q}_R^\eta$, we have

$$\operatorname{mes}\left\{(x, t) \in \overline{Q}_R^\eta:\ u(x, t) < \mu_- + \frac{\omega}{\alpha^{s_0}}\right\} \ge \nu_0 \operatorname{mes} \overline{Q}_R^\eta. \tag{25}$$

Since we obviously have

$$\mu^+ - \frac{\omega}{\alpha^{s_0}} \ge \mu^- + \frac{\omega}{\alpha^{s_0}},$$

we can rewrite (25) as follows:

$$\operatorname{mes}\left\{(x, t) \in \overline{Q}_R^\eta:\ u(x, t) > \mu_+ - \frac{\omega}{\alpha^{s_0}}\right\} \ge (1-\nu_0) \operatorname{mes} \overline{Q}_R^\eta. \tag{26}$$

The lemma below can be proved similary to [3].

Lemma 3. *Let* $\overline{Q}_R^\eta \subset Q_R^\theta$ *be fixed and let inequality* (26) *be satisfied. Then there exists* $t^* \in \left[\bar{t} - \eta R^m \dfrac{\operatorname{mes} B_R}{\upsilon(B_R)}, \bar{t} - \dfrac{\nu_0}{2}\eta R^m \dfrac{\operatorname{mes} B_R}{\upsilon(B_R)}\right]$ *such that*

$$\operatorname{mes}\left\{A^+_{\mu_+ - \omega/\alpha^{s_0}, R}(t^*)\right\} \le \frac{1-\nu_0}{1-\nu_0/2} \operatorname{mes} B_R.$$

We define

$$H_+ = \left\| \left(u - \left(\mu_+ - \frac{\omega}{\alpha^{s_0}}\right)\right)_+ \right\|_{\infty, Q_R^\theta}.$$

Lemma 4. *Let* $\overline{Q}_R^\eta \subset Q_R^\theta$ *be fixed and let the inequality* $H_+ > \dfrac{\omega}{\alpha^{s_0+1}}$ *be satisfied. Then there exists a positive number* l *independent of* ω *and* R *and such that either*

$$R^{N\kappa_0/(m+\beta)} \ge \left(\frac{\omega}{\alpha^{s_0+l}}\right)^{1+\frac{m}{m+\beta}\left(\frac{1+\kappa}{q} - \frac{1}{m}\right)_+ (m-2+\beta)} \tag{27}$$

or

$$\operatorname{mes}\left\{A^+_{\mu_+ - \omega/\alpha^{s_0+l}, R}\right\} \le \left[1 - \left(\frac{\nu_0}{2}\right)^2\right] \operatorname{mes} B_R$$

for all $t \in \left[\bar{t} - \dfrac{\nu_0}{2}\eta R^m \dfrac{\operatorname{mes} B_R}{\upsilon(B_R)}, \bar{t}\right]$.

Proof. We consider inequality (7) for

$$Q_R^* \equiv B_R \times [t^*, t], \qquad Q_{R-\sigma R}^* \equiv B_{R-\sigma R} \times [t^*, t].$$

Here, t^* is the number defined in Lemma 3, $\sigma \in (0,1)$, $k = \dfrac{\omega}{\alpha^{s_0}}$, and $\nu = \dfrac{\omega}{\alpha^{s_0+l}}$. We also take the cut-off function $\xi(x)$ for B_R, $\xi(x) \equiv 1$ in $B_{R-\sigma R}$, $\left|\dfrac{\partial \xi}{\partial x}\right| \le (\sigma R)^{-1}$. From (7), we obtain

$$\int\limits_{B_{R-\sigma R}\times\{t\}} \psi^2\left(H_+, \left(u - \left(\mu_+ - \frac{\omega}{\alpha^{s_0}}\right)\right)_+, \frac{\omega}{\alpha^{s_0+l}}\right) dx \le$$

$$\le \int\limits_{B_R\times\{t^*\}} \psi^2\left(H_+, \left(u - \left(\mu_+ - \frac{\omega}{\alpha^{s_0}}\right)\right)_+, \frac{\omega}{\alpha^{s_0+l}}\right) dx \; +$$

$$+ \frac{c}{(\sigma R)^m} \frac{v(B_R)}{\operatorname{mes} B_R} \int\limits_{t^*}^{t} \int\limits_{B_R} |u|^\beta \, \psi\left(H_+, \left(u - \left(\mu_+ - \frac{\omega}{\alpha^{s_0}}\right)\right)_+, \frac{\omega}{\alpha^{s_0+l}}\right) \times$$

$$\times \left| \psi_u\left(H_+, \left(u - \left(\mu_+ - \frac{\omega}{\alpha^{s_0}}\right)\right)_+, \frac{\omega}{\alpha^{s_0+l}}\right)\right|^{2-m} dx\, dt \; +$$

$$+ \; c\left(\frac{\alpha^{s_0+l}}{\omega}\right)^2 \ln \frac{H_+ \alpha^{s_0+m}}{\omega} \left(\frac{v(B_R)}{\operatorname{mes} B_R} \int\limits_{t^*}^{\bar t} \operatorname{mes} A^-_{\mu_+ - \omega/\alpha^{s_0}, R}(t)\, dt \right)^{m(1+\kappa)/q}. \tag{28}$$

We have

$$\psi\left(H_+, \left(u - \left(\mu_+ - \frac{\omega}{\alpha^{s_0}}\right)\right)_+, \frac{\omega}{\alpha^{s_0+l}}\right) \le l \ln \alpha,$$

$$\left|\psi_u\left(H_+, \left(u - \left(\mu_+ - \frac{\omega}{\alpha^{s_0}}\right)\right)_+, \frac{\omega}{\alpha^{s_0+l}}\right)\right|^{2-m} \le l c \left(\frac{\omega}{\alpha^{s_0}}\right)^{m-2},$$

$$\ln \frac{H_+ \alpha^{s_0+l}}{\omega} \le l \ln \alpha.$$

With the use of Lemma 3, the first integral on the right-hand side of (28) is estimated from above by

$$l^2 \ln^2 \alpha \left(\frac{1-\nu_0}{1-\nu_0/2}\right) \operatorname{mes} B_R.$$

Since $\bar t - t^* \le \eta R^m \dfrac{\operatorname{mes} B_R}{v(B_R)} = \left(\dfrac{\alpha^{s_0}}{\omega}\right)^{m-2} \dfrac{1}{M^\beta} R^m \dfrac{\operatorname{mes} B_R}{v(B_R)}$, the second integral on the right-hand side of (28) is estimated from above by

$$\frac{c}{\sigma^m}\, l \operatorname{mes} B_R.$$

The third integral on the right-hand side of (28) is estimated from above by

$$c\, l \operatorname{mes} B_R \left(\frac{\alpha^{s_0+l}}{\omega}\right)^{m+\beta} \left[1 + \frac{m}{m+\beta}\left(\frac{1+\kappa}{q} - \frac{1}{m}\right)_+ (m-2+\beta)\right] R^{N\kappa_0}.$$

If (27) is not satisfied, then the third integral on the right-hand side of (28) is estimated from above by

$$c\, l \operatorname{mes} B_R.$$

From (28), we now obtain

$$\int_{B_{R-\sigma R}\times\{t\}} \psi^2\left(H_+, \left(u - \left(\mu_+ - \frac{\omega}{\alpha^{s_0}}\right)\right)_+, \frac{\omega}{\alpha^{s_0+l}}\right) dx \le$$

$$\le c l^2 \ln^2 \alpha \frac{1-\nu_0}{1-\nu_0/2} \operatorname{mes} B_R + c l \operatorname{mes} B_R,$$

whence, by analogy with [3, p. 516, 517], we can obtain (9).

We now take

$$s_2 = s_0 + l.$$

Since $s^* > s_0$, for $m \ge 2$ we have

$$\left(1 - \frac{\nu_0}{2}\right) \alpha^{s_0(m-2)} < \frac{1}{1+\nu_0/2} \alpha^{s^*(m-2)}.$$

Lemma 5. *Let* $H_+ > \dfrac{\omega}{\alpha^{s_0+1}}$. *Then either*

$$R^{N\kappa_0/(m+\beta)} > \left(\frac{\omega}{\alpha^{s_2}}\right)^{1+\frac{m}{m+\beta}\left(\frac{1+\kappa}{q}-\frac{1}{m}\right)_+(m-2+\beta)}$$

or

$$\operatorname{mes}\left\{A^+_{\mu_+-\omega/\alpha^{s_0+l}, R}\right\} \le \left[1 - \left(\frac{\alpha_0}{2}\right)^2\right] \operatorname{mes} B_R \tag{29}$$

for all $t \in \left[t_0 - \dfrac{\alpha_0}{3} \theta R^m \dfrac{\operatorname{mes} B_R}{v(B_R)}, t_0\right]$.

Proof. The proof is analogous to that of Corollary 4.3 in [3, p. 517].

Consider the cylinder

$$Q_R^\theta(\nu_0) \equiv B_R \times \left\{t_0 - \frac{\nu_0}{3} \theta R^m \frac{\operatorname{mes} B_R}{v(B_R)}, t_0\right\}.$$

Lemma 6. *Suppose that* (29) *holds. Then, for every* $\varepsilon \in (0, 1)$, *there exists a number* s^* *independent of* ω *and* R *and such that either*

$$R^{N\kappa_0/(m+\beta)} \ge \left(\frac{\omega}{\alpha^{s^*}}\right)^{1+\frac{m}{m+\beta}\left(\frac{1+\kappa}{q}-\frac{1}{m}\right)_+(m-2+\beta)} \tag{30}$$

or

$$\operatorname{mes}\left\{(x,t) \in Q_R^\theta(\nu_0): \ u(x,t) > \mu_+ - \frac{\omega}{\alpha^{s^*}}\right\} \le \varepsilon \operatorname{mes} Q_R^\theta(\nu_0). \tag{31}$$

Proof. Using inequality (1) and assuming that inequality (30) is violated, by analogy with the procedure used in [6], we get

$$\int_{Q_R^\theta(\alpha_0)} v(x) \left|\frac{\partial}{\partial x}\left(u - \left(\mu_+ - \frac{\omega}{\alpha^n}\right)\right)_+\right|^m dx\,dt \le$$

$$\le c \frac{v(B_R)}{R^m \operatorname{mes} B_R} \left\{\left(\frac{\omega}{\alpha^n}\right)^m + \left(\frac{\omega}{\alpha^n}\right)^2 \left(\frac{\omega}{\alpha^{s^*}}\right)^{m-2}\right\} \operatorname{mes} Q_R^\theta(\nu_0). \tag{32}$$

We now consider inequality (5) for

$$l = \mu_+ - \frac{\omega}{\alpha^n}, \quad K = \mu_+ - \frac{\omega}{\alpha^{n-1}}.$$

It follows from Lemma 5 with $t \in \left[t_0 - \frac{\nu_0}{3}\theta R^m \frac{\operatorname{mes} B_R}{\upsilon(B_R)}, t_0\right]$ that

$$\operatorname{mes}\{B_R \setminus A_{\mu_+ - \omega/\alpha^n, R}(t)\} \geq \left(\frac{\alpha_0}{2}\right)^2 \operatorname{mes} B_R.$$

Therefore, in this case, for every $t \in \left[t_0 - \frac{\nu_0}{3}\theta R^m \frac{\operatorname{mes} B_R}{\upsilon(B_R)}, t_0\right]$, inequality (5) yields

$$\left(\frac{\omega}{\alpha^n}\right) \operatorname{mes} A_{\mu_+ - \omega/\alpha^n, R}(t) \leq cR \int\limits_{A^+_{k,R}(t) \setminus A^+_{l,R}(t)} \left|\frac{\partial u}{\partial x}\right| dx. \tag{33}$$

With the use of the Hölder inequality, the right-hand side of (33) is estimated from above by

$$R\left(\int\limits_{B_R} \upsilon(x)\left|\frac{\partial u}{\partial x}\right|^m dx\right)^{1/m} \left(\int\limits_{A^+_{k,R}(t) \setminus A^+_{l,R}(t)} [\upsilon(x)]^{-1/(m-1)} dx\right)^{(m-1)/m}. \tag{34}$$

By using (34) and integrating (33) over $\left[t_0 - \frac{\alpha_0}{3}\theta R^m \frac{\operatorname{mes} B_R}{\upsilon(B_R)}, t_0\right]$, we obtain

$$\left(\frac{\omega}{\alpha^n}\right) A_n \leq cR\left(\int\limits_{Q^\theta_R(\nu_0)} \upsilon(x)\left|\frac{\partial}{\partial x}\left(u - \left(\mu_+ - \frac{\omega}{\alpha^{n-1}}\right)\right)_+\right|^m dx\,dt\right)^{1/m} \times$$
$$\times [\tilde{\upsilon}(A_{n-1}) - \tilde{\upsilon}(A_n)]^{(m-1)/m},$$

where

$$A_n = \int\limits_{t_0 - \frac{\nu_0}{3}\theta R^m \frac{\operatorname{mes} B_R}{\upsilon(B_R)}}^{t_0} \operatorname{mes} A^+_{\mu_+ - \omega/\alpha^n, R}(t)\,dt,$$

$$\tilde{\upsilon}(A_n) = \int\limits_{t_0 - \frac{\alpha_0}{3}\theta R^m \frac{\operatorname{mes} B_R}{\upsilon(B_R)}}^{t_0} dt \int\limits_{A^+_{\mu_+ - \omega/\alpha^n, R}}^{t_0} [\upsilon(x)]^{-1/(m-1)} dx.$$

From (34), by using inequality (32), we get

$$A_n^{m/(m-1)} \leq c\left[\frac{\upsilon(B_R)}{\operatorname{mes} B_R}\right]^{1/(m-1)} [\operatorname{mes} Q^\theta_R(\nu_0)]^{1/(m-1)} [\tilde{\upsilon}(A_{n-1}) - \tilde{\upsilon}(A_n)]. \tag{35}$$

We now sum up inequalities (35) for $n = 1, 2, \ldots, s^*$. As a result, we obtain

$$(s^* - s_2 - 1) A_{s^*}^{m/(m-1)} \leq c\left[\frac{\upsilon(B_R)}{\operatorname{mes} B_R}\right]^{1/(m-1)} [\operatorname{mes} Q^\theta_R(\nu_0)]^{1/(m-1)} \leq$$

$$\leq \int\limits_{t_0 - \frac{\alpha_0}{3}\theta R^m \frac{\operatorname{mes} B_R}{\upsilon(B_R)}}^{t_0} dt \int\limits_{B_R} [\upsilon(x)]^{-1/(m-1)} dx. \tag{36}$$

By using the definition of A_p-class, we can obtain the inequality

$$\left[\frac{\upsilon(B_R)}{\operatorname{mes} B_R}\right]^{1/(m-1)} \int_{B_R} [\upsilon(x)]^{-1/(m-1)}\,dx \;\le\; c \operatorname{mes} B_R. \tag{37}$$

Relations (36) and (37) now yield

$$(s^* - s_2 - 1) A_{s^*}^{m/(m-1)} \;\le\; c\,[\operatorname{mes} Q_R^{\theta}(\nu_0)]^{m/(m-1)}.$$

We can obtain (31) if we take s^* so large that

$$\frac{c}{s^* - s_2 - 1} \le \varepsilon.$$

***Remark* 6.** If inequality (3) is violated, Lemmas 4–6 can be proved by analogy.

Following [3], we can now deduce the following result.

Theorem 2. *There exists a positive number s^* independent of ω and R and such that if (26) is satisfied for every cylinder $\overline{Q}_R^{\eta} \subset Q_R^{\theta}$, $\theta = \left(\frac{\alpha^{s^*}}{\omega}\right)^{m-2} \frac{1}{M^{\beta}}$, then either*

$$\omega < \alpha^{s^*+1} R^{N\kappa_0 \xi/(m+\beta)}$$

or

$$\operatorname*{ess\,osc}_{Q^{\theta}_{R/\alpha}(\nu_0)} u \;\le\; \omega\left(1 - \frac{1}{\alpha^{s^*+1}}\right).$$

Theorems 1 and 2 imply the following statement.

Theorem 3. *Let $u(x,t) \in V_{\mathrm{loc}}(\Omega_T, \upsilon) \cap L_{\infty,\mathrm{loc}}(\Omega_T)$ be a solution of equation* (1) *and let $\upsilon(x) \in A_{1+m/N}$. Then $u(x,t)$ is locally Hölder continuous in Ω_T.*

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