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SEMI-NONOSCILLATION INTERVALS AND SIGN-CONSTANCY OF GREEN'S FUNCTIONS OF TWO-POINT IMPULSIVE BOUNDARY-VALUE PROBLEMS*

ІНТЕРВАЛИ МАЙЖЕ ВІДСУТНОСТІ КОЛИВАНЬ ТА ЗБЕРЕЖЕННЯ ЗНАКА ФУНКЦІЙ ГРІНА ДВОТОЧКОВИХ ІМПУЛЬСНИХ КРАЙОВИХ ЗАДАЧ

We consider the second order impulsive differential equation with delays

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = f(t), \quad t \in [0,\omega],$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \ldots, r,$$

where $\gamma_k > 0$, $\delta_k > 0$ for $k = 1, 2, \ldots, r$. In this paper, we obtain the conditions of semi-nonoscillation for the corresponding homogeneous equation on the interval $[0,\omega]$. Using these results, we formulate theorems on sign-constancy of Green's functions for two-point impulsive boundary-value problems in terms of differential inequalities.

Розглядається імпульсне диференціальне рівняння другого порядку із запізненням

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = f(t), \quad t \in [0,\omega],$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \ldots, r,$$

де $\gamma_k > 0$, $\delta_k > 0$ для $k = 1, 2, \ldots, r$. Знайдено умови майже відсутності коливань для відповідного однорідного рівняння на інтервалі $[0,\omega]$. За допомогою цих результатів сформульовано теореми про збереження знака функцій Гріна двоточкових імпульсних крайових задач у термінах диференціальних нерівностей.

1. Introduction. Impulsive differential equations have attracted an attention of a number of recognized mathematicians and have applications in many spheres of science from physics, biology, medicine to economical studies. The following well-known books can be noted in this context [20, 23 – 25]. In the books [4, 5], the concept of the general theory of functional differential equations was presented. On the basis of this concept nonoscillation for the first order functional differential equations was considered in [7], where positivity of the Cauchy and Green's functions of the periodic problem was firstly studied. A concept of nonoscillation for the first order differential equations is also considered in the book [1]. Sign properties of the first order impulsive initial, one-point and periodic boundary-value problems were studied in [13], and for nonlocal problems in [11, 12]. The positivity of solutions and sign-constancy of Green's function of one- and two-point impulsive boundary-value problems for second order functional differential impulsive equations was studied in [2, 8, 9, 14 – 17, 21, 22].

Let us consider the impulsive equation

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$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = f(t), \quad t \in [0,\omega], \tag{1.1}$$

$$\begin{gathered} x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \ldots, r, \\ 0 = t_0 < t_1 < t_2 < \ldots < t_r < t_{r+1} = \omega, \end{gathered} \tag{1.2}$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0, \tag{1.3}$$

where $f,\ a_j,\ b_j : [0,\omega] \to \mathbb{R}$ are summable functions and $\tau_j,\ \theta_j : [0,\omega] \to [0,+\infty)$ are measurable functions for $j = 1, 2, \ldots, p,$ p and r are natural numbers, γ_k and δ_k are real positive numbers.

Let $D(t_1, t_2, \ldots, t_r)$ be a space of functions $x\colon [0,\omega] \to \mathbb{R}$ such that their derivative $x'(t)$ is absolutely continuous on every interval $t \in [t_i, t_{i+1}),\ i = 0, 1, \ldots, r,\ x'' \in L_\infty,$ we assume also that there exist the finite limits $x(t_i - 0) = \lim_{t \to t_i^-} x(t)$ and $x'(t_i - 0) = \lim_{t \to t_i^-} x'(t)$ and condition (1.2) is satisfied at points $t_i,\ i = 0, 1, \ldots, r.$ As a solution x we understand a function $x \in D(t_1, t_2, \ldots, t_r)$ satisfying (1.1)–(1.3).

It should be noted that the space of solutions of the homogeneous equation

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = 0, \quad t \in [0,\omega], \tag{1.4}$$

$$\begin{gathered} x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \ldots, r, \\ 0 = t_0 < t_1 < t_2 < \ldots < t_r < t_{r+1} = \omega, \end{gathered} \tag{1.5}$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0, \tag{1.6}$$

is two-dimensional. The fundamental system of solutions of impulsive differential equation (1.4)–(1.6) consists of two linearly independent solutions x_1 and x_2. Like in the case of ordinary differential equations its Wronskian

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}$$

could be one of the classical objects in this theory. Various tests, where the Wronskian does not vanish for the nonimpulsive equation were obtained in [3, 6, 19].

2. Preliminaries. Wronskian is one of the classical objects in the theory of differential equations. Properties of Wronskian lead to important conclusions on behavior of solutions of delay equations [6]. The following theorem claims that nonvanishing Wronskian ensures validity of Sturm separation theorem (between two adjacent zeros of any solution there is one and only one zero of every other nontrivial linearly independent solution) for delay impulsive equation [10]

$$x''(t) + b_1 x(t - \theta_1) = 0, \tag{2.1}$$

with the conditions (1.2), (1.3).

Theorem 2.1. *Let* $p = 1,\ b_1(t) \geq 0,\ t - \theta_1(t)$ *be nondecreasing*, $\gamma_i > 0$ *and* $\delta_i > 0$ *for every* $i = \overline{1, r}$. *If the Wronskian* $W(t)$ *of the fundamental system of* (1.4)–(1.6) *does not have zeros*, *then Sturm's separation theorem is valid*.

In contrast to this assertion, in this paper, the case $a_j(t) \geq 0,\ b_j(t) \leq 0,\ j = 1, \ldots, p,\ t \in [0, \omega]$, is considered and the number of terms can be $p > 1$. We focus on the connection between nonvanishing Wronskian, the property of semi-nonoscillation and the sign-constancy of Green's functions for impulsive boundary-value problems.

Below the following definition will be used.

Definition 2.1. *We call* $[0, \omega]$ *a semi-nonoscillation interval of* (1.4)–(1.6), *if every nontrivial solution having zero of derivative does not have zero on this interval*.

For equation (1.1)–(1.3) we consider the following variants of boundary conditions:

$$x(0) = 0, \quad x(\omega) = 0, \tag{2.2}$$

$$x'(0) = 0, \quad x(\omega) = 0, \tag{2.3}$$

$$x(0) = 0, \quad x'(\omega) = 0, \tag{2.4}$$

$$x'(0) = 0, \quad x'(\omega) = 0. \tag{2.5}$$

General solution of the equation (1.1)–(1.3) can be represented in the form [7]

$$x(t) = \nu_1(t)x(0) + C(t, 0)x'(0) + \int_0^t C(t, s)f(s)ds, \tag{2.6}$$

where $\nu_1(t)$ is a solution of the homogeneous equation (1.4)–(1.6) with the initial conditions $x(0) = 1,\ x'(0) = 0$; $C(t, s)$, called the Cauchy function of the equation (1.4)–(1.6), is the solution of the equation

$$(L_s x)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t - \theta_j(t)) = 0, \quad t \in [s, \omega],$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = m, \ldots, r,$$

$$0 = t_0 < t_1 < t_2 < \ldots < t_r < t_{r+1} = \omega.$$

Here, m is a number such that $t_{m-1} < s \leq t_m$,

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < s,$$

satisfying the initial conditions $C(s, s) = 0,\ C'_t(s, s) = 1$ and $C(t, s) = 0$ for $t < s$.

If the boundary-value problem (1.1)–(1.3), (2.1)–(2.4) is uniquely solvable, then its solution can be represented as

$$x(t) = \int\limits_0^\omega G_i(t,s)f(s)ds, \quad i = \overline{1,4},$$

where $G_i(t,s)$ is Green's function of the problem (1.1)–(1.3), (2.1)–(2.4), respectively [8].

Using general representation of the solution (2.6), the following formulas for Green's functions can be obtained:

$$G_1(t,s) = C(t,s) - C(t,0)\frac{C(\omega,s)}{C(\omega,0)}, \tag{2.7}$$

$$G_2(t,s) = C(t,s) - C(\omega,s)\frac{\nu_1(t)}{\nu_1(\omega)}, \tag{2.8}$$

$$G_3(t,s) = C(t,s) - C(t,0)\frac{C'_t(\omega,s)}{C'_t(\omega,0)}, \tag{2.9}$$

$$G_4(t,s) = C(t,s) - C'_t(\omega,s)\frac{\nu_1(t)}{\nu'_1(\omega)}. \tag{2.10}$$

Denote $G^\xi(t,s)$ the Green's function of the problem (1.1)–(1.3) with boundary conditions

$$x(\xi) = 0, \quad x'(\xi) = 0. \tag{2.11}$$

In the paper [8], the following theorem has been proven for the problems (1.1)–(1.3), (2.1)–(2.4).

Lemma 2.1. *Assume that the following conditions are fulfilled*:

(1) $b_j(t) \le 0,\ j = 1,\ldots,p,\ t \in [0,\omega]$;

(2) *the Cauchy function* $C_1(t,s)$ *of the first order impulsive equation*

$$\begin{aligned} &y'(t) + \sum_{j=1}^{p} a_j(t)y(t-\tau_j(t)) = 0, \quad t \in [0,\omega], \\ &y(t_k) = \delta_k y(t_k - 0), \quad k = 1,2,\ldots,r, \\ &y(\zeta) = 0, \quad \zeta < 0, \end{aligned} \tag{2.12}$$

is positive for $0 \le s \le t \le \omega$;

(3) *Green's function* $G^\xi(t,s)$ *of the problem* (1.1)–(1.3), (2.11) *is nonnegative for* $t,s \in [0,\xi]$ *for every* $0 < \xi < \omega$;

(4) $[0,\omega]$ *is a semi-nonoscilation interval of* $(Lx)(t) = 0$.

Then Green's functions $G_i(t,s),\ i = \overline{1,3}$, *are nonpositive for* $t,s \in [0,\omega]$ *and under the additional condition* $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t-\theta_j(t)) \not\equiv 0,\ t \in [0,\omega]$, *where*

$$\chi_{[0,\omega]}(t) = \begin{cases} 1, & t \in [0,\omega], \\ 0, & t \notin [0,\omega], \end{cases} \tag{2.13}$$

$G_4(t,s) \le 0$ *for* $t,s \in [0,\omega]$.

3. Conditions of semi-nonoscillation. Let us formulate a theorem on semi-nonoscillation of the interval $[0, \omega]$.

Theorem 3.1. *Assume that the following conditions are fulfilled*:

(1) $a_j(t) \geq 0,\ b_j(t) \leq 0,\ j = 1, \ldots, p,\ t \in [0, \omega]$;

(2) *the Wronskian* $W(t)$ *of the fundamental system of solutions of homogeneous equation* (1.4)–(1.6) *satisfies the inequality* $W(t) \neq 0,\ t \in [0, \omega]$;

(3) *the Cauchy function* $C_1(t, s)$ *of the first order equation* (2.12) *is positive for* $0 \leq s \leq t \leq \omega$. *Then the interval* $[0, \omega]$ *is a semi-nonoscillation interval of* (1.4)–(1.6).

Remark **3.1.** It looks that conditions 2 and 3 are hard for verification. In Section 4, we explain how these conditions could be verified in almost all cases which come from various applications. The first conditions on positivity of the Cauchy function of the equation (2.12) are obtained in [7].

Proof. First of all, let us prove that if the conditions 1 and 3 of Theorem 3.1 are fulfilled, then there exists a positive solution $x(t)$ of the equation (1.4)–(1.6), satisfying the conditions $x(0) = \alpha > 0,\ x'(0) > 0$.

It is clear that in this case, $x(t)$ satisfies the equation

$$x''(t) + \sum_{j=1}^{p} a_j(t) x'(t - \tau_j(t)) = \phi(t), \quad t \in [0, \omega], \tag{3.1}$$

where $\phi(t) = -\sum_{j=1}^{p} b_j(t) x(t - \theta_j(t))$ for $t \in [0, \omega]$.

Let us denote $y(t) = x'(t)$. Then we can write an equation for $y(t)$ in the form

$$\begin{aligned} & y'(t) + \sum_{j=1}^{p} a_j(t) y(t - \tau_j(t)) = \phi(t), \quad t \in [0, \omega], \\ & y(t_k) = \delta_k y(t_k - 0), \quad k = 1, 2, \ldots, r, \\ & y(\zeta) = 0, \quad \zeta < 0. \end{aligned} \tag{3.2}$$

Since $y(t)$ satisfies a condition $y(0) > 0$, it is clear that there exists an interval $[0, \mu)$ such that $y(t) > 0$ for $t \in [0, \mu)$, and, consequently, $x(t) > 0$ and $x(t)$ increases at all the points $t \neq t_k$ for $t \in [0, \mu)$.

In this case, $\sum_{j=1}^{p} b_j(t) x(t - \theta_j(t)) \leq 0$, so it means that $\phi(t) \geq 0$ for $t \in [0, \mu)$. Then, according to representation of solutions [7],

$$y(t) = y(0)\, C_1(t, 0) + \int_0^t C_1(t, s) \phi(s) ds,$$

where $C_1(t, s)$ is the Cauchy function of (3.2), and we obtain $y(t) > 0$ for $t \in [0, \omega]$. It is clear that $x'(t) > 0$ and $x(t) > 0$ for $t \in (0, \omega]$.

Now we start to prove that $[0, \omega]$ is a semi-nonoscillation interval of (1.4)–(1.6). Let us assume the contrary that [0, ω] is not a semi-nonoscillation interval. It means that one of the conditions is fulfilled:

$$x_1'(\eta) = 0, \quad x_1(\xi) = 0, \quad \eta < \xi,$$

$$x_1(\eta) = 0, \quad x_1'(\xi) = 0, \quad \eta < \xi.$$

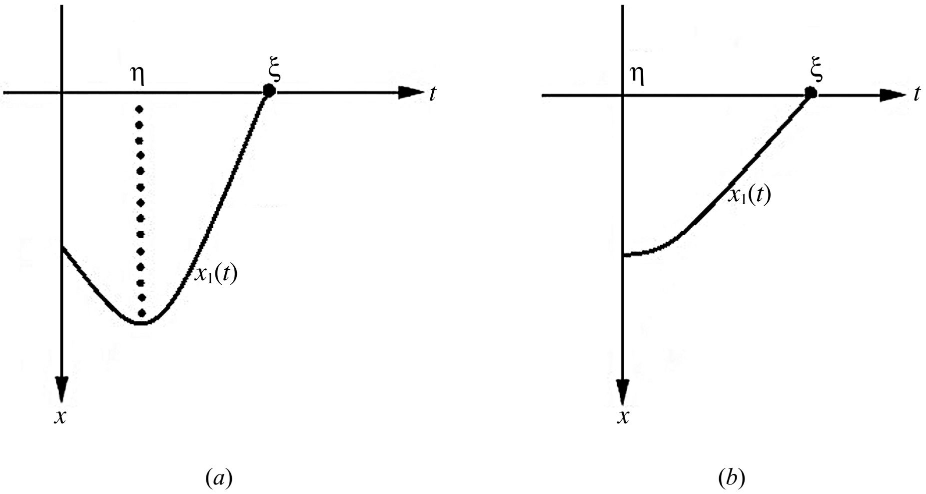


Fig. 1. Cases $x_1' < 0$ (*a*) and $x_1' = 0$ (*b*).

We will prove that both of them are impossible.

1. Let us suppose that there exists a nontrivial solution $x_1(t)$ of the problem (1.4)–(1.6) such that $x_1(\xi) = 0,$ and $x_1(t)$ has zero of the derivative in the interval $(0, \xi)$. Without loss of generality we can assume that $x_1(0) < 0,$ $x_1'(\eta) = 0,$ $x_1'(t) < 0$ for $t \in [0, \eta)$ and $x_1'(t) > 0$ for $t \in (\eta, \omega]$.

There are 2 possible situations for $x_1(t)$ that can be considered (see Fig. 1):

(a) $x_1'(0) < 0$ (see Fig. 1 (*a*)).

It is easy to show that in such situation the solution $x_1(t)$ will be negative and decreasing at all the points $t \neq t_k$, so it is impossible to achieve the boundary condition $x_1(\xi) = 0$.

Indeed, let us denote $y(t) = x_1'(t)$. Then we can write an equation for $y(t)$ in the form (3.2).

Since $y(t)$ satisfies the condition $y(0) < 0$, it is clear that there exists an interval $[0, \mu)$ such that $y(t) < 0$ for $t \in [0, \mu)$, and, consequently, $x_1(t) < 0$ and $x_1(t)$ decreases at all the points $t \neq t_k$ for $t \in [0, \mu)$.

In this case, $\sum_{j=1}^{p} b_j(t) x_1(t - \theta_j(t)) \geq 0$, so it means that $\phi(t) \leq 0$ for $t \in [0, \mu)$. Then, according to representation of solutions [7],

$$y(t) = y(0)\, C_1(t, 0) + \int\limits_0^t C_1(t, s)\phi(s) ds,$$

where $C_1(t, s)$ is the Cauchy function of (3.2), and we obtain $y(t) < 0$ for $t \in [0, \omega]$. Now it is clear that $x_1'(t) < 0$ and $x_1(t) < 0$ for $t \in (0, \omega]$.

We got a contradiction with the assumption that $x_1(\xi) = 0$.

(b) $x_1'(0) = 0$ (see Fig. 1 (*b*)).

It is easy to show that in such situation the solution $x_1(t)$ will be negative and decreasing at all the points $t \neq t_k$, so it will not be able to satisfy the boundary conditions $x_1(\xi) = 0$.

Since $x_1(0) < 0$, it is clear that there exists an interval $[0, \mu)$ such that $x_1(t) < 0$ for $t \in [0, \mu)$.

Let us denote $y(t) = x_1'(t)$. Then we can write an equation for $y(t)$ in the form (3.2).

In this case, $\sum_{j=1}^{p} b_j(t) x_1(t - \theta_j(t)) \geq 0$, so it means that $\phi(t) \leq 0$ for $t \in [0, \mu)$. Then, according to the representation of solutions [7],

$$y(t) = \int_0^t C_1(t,s)\phi(s)ds,$$

where $C_1(t,s)$ is the Cauchy function of (3.2), and we obtain $y(t) < 0$ for $t \in [0,\omega]$. Now it is clear that $x_1'(t) < 0$ and $x_1(t) < 0$ for $t \in (0,\omega]$.

We got a contradiction with the assumption that $x_1(\xi) = 0$.

2. Let us suppose that there exists a nontrivial solution $x_1(t)$ of the problem (1.4)–(1.6) such that $x_1(\xi) = \beta > 0,\ x_1'(\xi) = 0$, and $x_1(t)$ changes its sign in the interval $(0,\xi)$. Without the loss of generality we can assume that $x_1(0) < 0,\ x_1(\eta) = 0,\ x_1(t) < 0$ for $t \in [0,\eta)$ and $x_1(t) > 0$ for $t \in (\eta,\omega]$.

There are 3 possible cases for $x_1(t)$ that can be considered (see Fig. 2):

(a) $x_1'(0) < 0$ (see Fig. 2 (*a*)).

This completely corresponds to the situation 1 (*a*). We have a contradiction with the boundary condition $x_1(\xi) = \beta > 0$.

(b) $x_1'(0) = 0$ (see Fig. 2 (*b*)).

This completely corresponds to the situation 1 (*b*). We have a contradiction with the boundary condition $x_1(\xi) = \beta > 0$.

(c) $x_1'(0) > 0$ (see Fig. 2 (*c*)).

The solution $x_1(t)$ is increasing, so it can satisfy the boundary conditions $x_1(\xi) = \beta > 0$, $x_1'(\xi) = 0$.

Denote a solution $z(t)$ such that

$$z(t) = \frac{x(\xi)}{x_1(\xi)} x_1(t) - x(t).$$

It is clear that

$$z(0) = \frac{x(\xi)}{x_1(\xi)} x_1(0) - x(0) < 0,$$

$$z(\xi) = \frac{x(\xi)}{x_1(\xi)} x_1(\xi) - x(\xi) = 0,$$

and

$$z'(\xi) = \frac{x(\xi)}{x_1(\xi)} x_1'(\xi) - x'(\xi) = -x'(\xi) < 0.$$

In other words, the solution $z(t)$ starts from a negative value at the point $t = 0$ and comes to zero (from above) at the point $t = \xi$. Taking into account the fact that $\gamma_k > 0$ and $\delta_k > 0,\ k = \overline{1,r}$, we see that the solutions cannot change sign at the points of impulses. This means that there exists a point $\mu \in (0,\xi)$ such that $z(\mu) = 0$. Thus, the solution $z(t)$ has at least 2 zeros in the interval $(0,\omega)$.

We have proven that the solution $z(t)$ has at least 2 zeros in the interval $[0,\omega]$. On the other hand, we know that $x(t)$, which is another linearly independent solution of (1.4)–(1.6), is positive and increasing for $t \in (0,\omega]$. Thus, $x(t)$ has no zeros on (0, ω] in contradiction to Lemma 2.1 (see [10]).

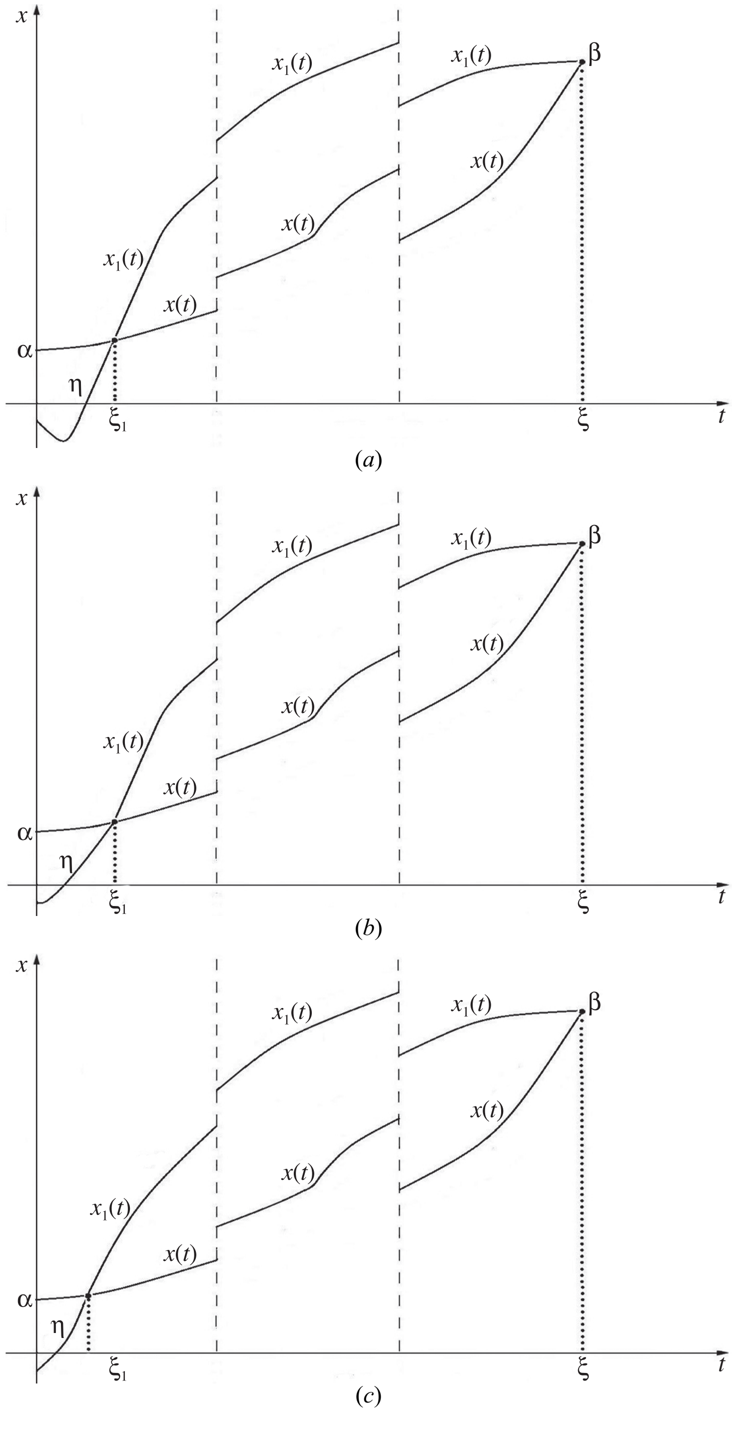


Fig. 2. Examples of solution $x_1(t)$: cases $x_1' < 0$ (*a*), $x_1' = 0$ (*b*) and $x_1' > 0$ (*c*).

These contradictions show that every nontrivial solution of (1.4) – (1.6), having zero of derivative on the interval $(0, \omega)$, does not have zero itself on this interval. So, according to the Definition 2.1, the interval $[0, \omega]$ is a semi-nonoscillation interval.

Theorem 3.1 is proved.

In Theorem 3.1 we have assumed that the Cauchy function $C_1(t,s)$ of the first order impulsive equation (1.4)–(1.6) is positive. In the lemma below, we will formulate the results of [7] on the conditions of positivity of Cauchy function of the first order impulsive differential equation

$$\begin{aligned} &y'(t) + a_1(t)\,y(t-\tau_1(t)) = 0, \quad t \in [0,\omega], \\ &y(t_k) = \delta_k y(t_k - 0), \quad k = 1, 2, \ldots, r, \\ &y(\zeta) = 0, \quad \zeta < 0. \end{aligned} \tag{3.3}$$

Lemma 3.1. *Let $\delta_k < 1$ for $k = 1, \ldots, r$ and the following condition be fulfilled*:

$$\frac{1 + \ln B(t)}{e} \geq \int\limits_{m(t)}^{t} a_+(s)ds, \tag{3.4}$$

where $B(t) = \prod_{k\in D_t} \delta_k$, $D_t = \{i : t_i \in [t-\tau_1(t), t]\}$, $a_+(t) = \max\{a_1(t), 0\}$ and $m(t) = \max\{t - \tau_1(t), 0\}$. Then Cauchy function of the first order impulsive differential equation (3.3) *is positive.*

Using Lemma 3.1, we can reformulate Theorem 3.1 as follows.

***Corollary* 3.1.** *Assume that the following conditions are fulfilled*:

(1) $p = 1$, $a_1(t) \geq 0$, $b_1(t) \leq 0$, $t \in [0,\omega]$;

(2) *the Wronskian $W(t)$ of the fundamental system of solutions of homogeneous equation* (1.4)–(1.6) *satisfies the inequality $W(t) \neq 0$, $t \in [0,\omega]$*;

(3) $\delta_k < 1$ *for $k = 1, \ldots, r$ and the condition* (3.4) *is fulfilled.*

Then the interval $[0,\omega]$ is a semi-nonoscillation interval of (1.4)–(1.6).

In the case of $p > 1$ the following sufficient condition proven in [7] can be used.

Lemma 3.2. *Let $a_j(t) \geq 0$ for $j = 1, \ldots, p$, $\delta_k < 1$ for $k = 1, \ldots, r$, and*

$$\int\limits_0^{\omega} \sum_{j=1}^{p} a_j(s)ds < \prod_{k=1}^{r} \delta_k. \tag{3.5}$$

Then Cauchy function of the first order impulsive differential equation (2.12) *is positive.*

***Remark* 3.2.** In Lemma 3.2, the interval $[0,\omega]$ has to be small enough. In Lemma 3.1, the delay has to be small enough.

Using Lemma 3.2, we can reformulate Theorem 3.1 as follows.

***Corollary* 3.2.** *Assume that the following conditions are fulfilled*:

(1) $a_j(t) \geq 0$, $b_j(t) \leq 0$, $j = 1, \ldots, p$, $t \in [0,\omega]$;

(2) *the Wronskian $W(t)$ of the fundamental system of solutions of homogeneous equation* (1.4)–(1.6) *satisfies the inequality $W(t) \neq 0$, $t \in [0,\omega]$*;

(3) $\delta_k < 1$ *for $k = 1, \ldots, r$ and the condition* (3.5) *is fulfilled.*

Then the interval $[0,\omega]$ is a semi-nonoscillation interval of (1.4)–(1.6).

Consider the equation

$$
\begin{aligned}
&y'(t) + a\, y(h(t)) = 0, \quad t \in [0,\omega],\\
&y(t_k) = \delta_k y(t_k - 0), \quad k = 1,2,\ldots,r,\\
&y(\zeta) = 0, \quad \zeta < 0,
\end{aligned}
\tag{3.6}
$$

where a is a positive constant and deviations are piecewise constant functions $h(t) = t_k,\ t \in \in [t_k, t_{k+1})$. Solving this equation on the intervals $t \in [t_k, t_{k+1})$, we can obtain the following fact.

Lemma 3.3. *If* $a > 0,\ h(t) = t_k,\ t \in [t_k, t_{k+1})$, *then the solution* $y(t)$ *of* (3.6) *has the following representation*:

$$
y(t) = \prod_{j=1}^{i} \delta_j \prod_{j=1}^{i} \big(1 - a(t_j - t_{j-1})\big)\big(1 - a(t - t_i)\big), \quad t \in [t_i, t_{i+1}).
$$

***Remark* 3.3.** It is clear that Cauchy function $C_1(t,s)$ for the equation (3.6) satisfies the inequality $C_1(t,s) > 0$, if

$$
a \max_{j=1,\ldots,r+1} (t_j - t_{j-1}) < 1. \tag{3.7}
$$

Using Lemma 3.3 and Remark 3.3, we can reformulate Theorem 3.1 in the following way.

***Corollary* 3.3.** *Assume that the following conditions are fulfilled*:

(1) $p = 1,\ a_1(t) = a \geq 0,\ b_1(t) \leq 0,\ t - \tau_1(t) = t_k,\ t \in [t_k, t_{k+1}),\ t \in [0,\omega]$;

(2) *the Wronskian* $W(t)$ *of the fundamental system of solutions of homogeneous equation* (1.4)–(1.6) *satisfies the inequality* $W(t) \neq 0,\ t \in [0,\omega]$;

(3) $\delta_k < 1$ *for* $k = 1,\ldots,r$ *and the condition* (3.7) *is fulfilled*.

Then the interval $[0,\omega]$ *is a semi-nonoscillation interval of* (1.4)–(1.6).

4. Semi-nonoscillation and sign-constancy of Green's functions. In this section, we formulate our main results.

Lemma 4.1. $W(\xi) \neq 0$ *if and only if the boundary-value problem* (1.1)–(1.3) *with boundary conditions* $x(\xi) = 0,\ x'(\xi) = 0$ *is uniquely solvable*.

Proof. The Wronskian $W(\xi)$ is a determinant of the system

$$
\begin{aligned}
&c_1\nu_1(\xi) + c_2 C(\xi,0) = 0,\\
&c_1\nu_1'(\xi) + c_2 C_t'(\xi,0) = 0.
\end{aligned}
\tag{4.1}
$$

Every solution of the homogeneous equation (1.4)–(1.6) has the form $x(t) = c_1\nu_1(t) + c_2 C(t,0)$, then a nontrivial solution of (1.1)–(1.3) with boundary conditions $x(\xi) = 0,\ x'(\xi) = 0$ exists if and only if a nontrivial solution $\{c_1, c_2\}$ of the system (4.1) exists. This is equivalent to $W(\xi) = 0$.

It is known from the general theory of functional differential equations [4] that the boundary-value problem (1.1)–(1.3) with boundary conditions $x(\xi) = 0,\ x'(\xi) = 0$ is uniquely solvable (i.e., Green's function $G^{\xi}(t,s)$ exists), if and only if the system (4.1) has only the trivial solution. This means that $W(\xi) \neq 0$.

Lemma 4.1 is proved.

Theorem 4.1. *If* $a_j(t) \geq 0,\ b_j(t) \leq 0$, *then the following assertions are equivalent*:

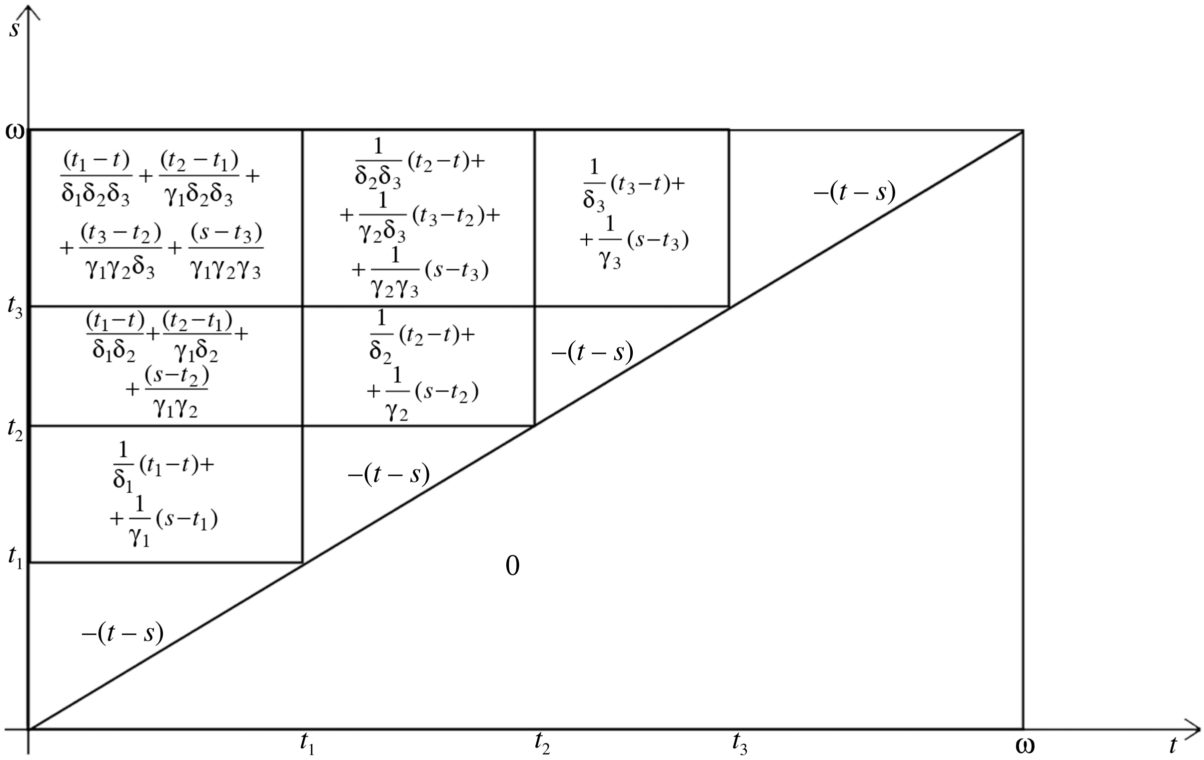


Fig. 3. The Green's function of impulsive equation (4.2) – (4.4) with boundary conditions $x(\omega) = 0,\ x'(\omega) = 0$.

(a) *the Wronskian* $W(t)$ *of the fundamental system of solutions of a homogeneous equation* (1.4) – (1.6) *satisfies* $W(t) \neq 0,\ t \in [0, \omega]$;

(b) *Green's function* $G^{\xi}(t,s)$ *is nonnegative for* $t, s \in [0, \xi]$ *for every* $0 < \xi < \omega$.

Proof. Firstly, let us prove the implication (b) $\Rightarrow$ (a). Its proof is based on Lemma 4.1.

According to Lemma 4.1, it follows from the fact of the existence of Green's function $G^{\xi}(t,s)$ for every $0 < \xi < \omega$, that the Wronskian $W(t) \neq 0,\ t \in [0, \omega]$.

Now let us prove the implication (a) $\Rightarrow$ (b). If the Wronskian $W(t) \neq 0,\ t \in [0, \omega]$, then, according to Lemma 4.1, Green's function $G^{\xi}(t,s)$ exists for every $0 < \xi < \omega$.

Consider the auxiliary boundary-value problem

$$x''(t) = z(t), \quad t \in [0, \omega], \tag{4.2}$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \ldots, r, \tag{4.3}$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0, \tag{4.4}$$

with the boundary conditions (2.11) for every $0 < \xi < \omega$. Let us denote Green's function for boundary-value problem (4.2) – (4.4), (2.11) as $G_{0\xi}(t,s)$. It is nonnegative (see Fig. 3).

We can rewrite the problem (1.1) – (1.3), (2.11) as follows:

$$z(t) + \sum_{j=1}^{p} a_j(t) \int_0^{\xi} G'_{0\xi}\left(t - \tau_j(t), s\right) \chi_{[0,\xi]}\left(t - \tau_j(t)\right) z(s)ds +$$

$$+\sum_{j=1}^{p} b_j(t)\int_0^{\xi} G_{0\xi}\left(t-\theta_j(t),s\right)\chi_{[0,\xi]}\left(t-\theta_j(t)\right)z(s)ds = f(t), \tag{4.5}$$

where the characteristic function $\chi_{[0,\xi]}(t)$ is defined by (2.13).

Denote

$$(K_{0\xi}z)(s) = \int_0^{\xi}\Bigg[-\sum_{j=1}^{p} a_j(t)G'_{0\xi}\left(t-\tau_j(t),s\right)\chi_{[0,\xi]}\left(t-\tau_j(t)\right) -$$

$$-\sum_{j=1}^{p} b_j(t)G_{0\xi}\left(t-\theta_j(t),s\right)\chi_{[0,\xi]}\left(t-\theta_j(t)\right)\Bigg]z(s)ds.$$

For $a_j(t)\ge 0,\ b_j(t)\le 0,$ the operator $K_{0\xi}$ is positive. According to Lemma 4.1, if the Wronskian $W(t)\ne 0,\ t\in[0,\omega]$, then the problem (1.1)–(1.3) with condition (2.11) is uniquely solvable for every $f\in L_\infty$ and Green's function $G^{\xi}(t,s)$ exists for every $0<\xi<\omega$. The fact of the unique solvability for every $0<\xi<\omega$ implies that the spectral radius $\rho(K_{0\xi})<1$ (see [18]). So, the solution of (4.5) can be represented in the form

$$z = (I-K_{0\xi})^{-1}f = \left[\sum_{j=0}^{\infty} K_{0\xi}^{j}\right] f,$$

where $I\colon L_\infty[0,\xi]\to L_\infty[0,\xi]$ is a unit operator acting in the space of essentially bounded functions $f\colon[0,\xi]\to\mathbb{R}$.

The solution $x(t)$ of the boundary-value problem (1.1)–(1.3), (2.11) can be written in the form

$$x = \left(G_{0\xi}\sum_{j=0}^{\infty} K_{0\xi}^{j}\right) f,$$

where $G_{0\xi}\sum_{j=0}^{\infty} K_{0\xi}^{j}$ is Green's operator for (1.1)–(1.3) with boundary conditions (2.11). From nonnegativity of Green's function $G_{0\xi}(t,s)$ and positivity of operator $K_{0\xi}$, it follows that Green's function $G^{\xi}(t,s)$ for (1.1)–(1.3), (2.11) is nonnegative for $(t,s)\in[0,\xi]\times[0,\xi]$ for every $0<\xi<\omega$.

Theorem 4.1 is proved.

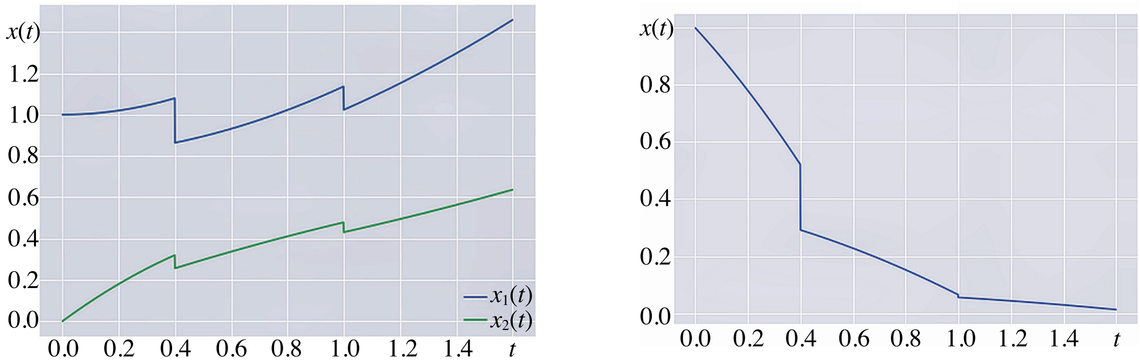
We use the semi-nonoscillation intervals in the proof of the following assertion on nonpositivity of Green's functions (2.7)–(2.10).

Theorem 4.2. *Assume that the following conditions are fulfilled*:

(1) $a_j(t)\ge 0,\ b_j(t)\le 0,\ j=1,\ldots,p,\ t\in[0,\omega]$;

(2) *the Wronskian* $W(t)$ *of the fundamental system of solutions of the homogeneous equation* $(Lx)(t)=0$, (1.2)–(1.3) *satisfies the inequality* $W(t)\ne 0,\ t\in[0,\omega]$;

(3) *the Cauchy function* $C_1(t,s)$ *of the first order equation* (2.12) *is positive for* $0\le s\le t\le\omega$. *Then Green's functions* $G_i(t,s),\ i=\overline{1,3}$, *are nonpositive for* $t,s\in[0,\omega]$ *and under the additional condition* $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t-\theta_j(t))\not\equiv 0,\ t\in[0,\omega]$, *Green's function* $G_4(t,s)$ *is nonpositive for* $t,s\in[0,\omega]$.



(*a*) (*b*)

Fig. 4. Calculating the Wronskian.

Proof. All the conditions of Theorem 3.1 are fulfilled. According to Theorem 3.1, the interval $[0, \omega]$ is a semi-nonoscillation one. It was proven in Theorem 4.1 that the inequality $W(t) \neq 0$ for $t \in [0, \omega]$, i.e., the condition 2 of Theorem 4.2, is equivalent to the condition 3 of Lemma 2.1. Now we see that all the conditions of Lemma 2.1 are fulfilled. According to Lemma 2.1, Green's functions $G_i(t, s)$, $i = \overline{1, 4}$, are nonpositive.

Theorem 4.1 is proved.

***Example* 4.1.** Let us consider the differential equation

$$x''(t)+x'(h(t)) - x(h(t)) = f(t), \quad t \in [0, 1.6],$$
$$t_1 = 0.4, \quad \gamma_1 = 0.8, \quad \delta_1 = 0.7, \tag{4.6}$$
$$t_2 = 1, \quad \gamma_2 = 0.9, \quad \delta_2 = 0.95.$$

Let us assume that the delays have the following forms:

$$h(t) = t_k, \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2.$$

According to Lemma 3.3 and Remark 3.3, the Cauchy function of the first order impulsive equation

$$y'(t) + y(h(t)) = 0, \quad t \in [0, 1.6],$$
$$t_1 = 0.4, \quad \gamma_1 = 0.8, \quad \delta_1 = 0.7,$$
$$t_2 = 1, \quad \gamma_2 = 0.9, \quad \delta_2 = 0.95,$$

satisfies the inequality $C_1(t, s) \geq 0$, if

$$\max_{k=1,\ldots,3} (t_k - t_{k-1}) < 1,$$

where $t_0 = 0$, $t_3 = 1.6$.

Thus, in our example, the condition (3.7) holds. Let us solve impulsive equation (4.6) with the conditions $x_1(0) = 1$, $x_1'(0) = 0$ and $x_2(0) = 0$, $x_2'(0) = 1$. We obtain

$$x_1(t) = \begin{cases} 0, & t < 0, \\ 0.5t^2 + 1, & t \in [0, 0.4), \\ 0.292(t-0.4)^2 + 0.28t + 0.752, & t \in [0.4, 1), \\ 0.212(t-1.0)^2 + 0.599t + 0.425, & t \in [1, 1.6), \end{cases}$$

$$x_2(t) = \begin{cases} 0, & t < 0, \\ -0.5t^2 + t, & t \in [0, 0.4), \\ -0.082(t-0.4)^2 + 0.42t + 0.088, & t \in [0.4, 1), \\ 0.063(t-1.0)^2 + 0.306t + 0.125, & t \in [1, 1.6), \end{cases}$$

see Fig. 4 (*a*).

Calculating the Wronskian, we obtain

$$W(t) = \begin{cases} 0, & t < 0, \\ -0.5t^2 - t + 1, & t \in [0, 0.4), \\ -0.146t^2 - 0.175t + 0.384, & t \in [0.4, 1), \\ -0.027t^2 + 0.082, & t \in [1, 1.6), \end{cases} \tag{4.7}$$

see Fig. 4 (*b*).

Thus, all the conditions of the Theorem 4.2 are fulfilled. So, according to Theorem 4.2, Green's functions $G_i(t,s),\ i = \overline{1,4}$, are nonpositive.

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