UDC 517.9

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STABILITY ANALYSIS WITH RESPECT TO TWO MEASURES OF IMPULSIVE SYSTEMS UNDER STRUCTURAL PERTURBATIONS

АНАЛІЗ СТІЙКОСТІ ІМПУЛЬСНИХ СИСТЕМ ВІДНОСНО ДВОХ МІР ПРИ СТРУКТУРНИХ ЗБУРЕННЯХ

The asymptotic stability with respect to two measures of impulsive systems under structural perturbations is investigated. Conditions of asymptotic (ρ_0, ρ)-stability of the system in terms of the fixed signs of some special matrices are established.

Досліджується асимптотична стійкість за двома мірами імпульсних систем при структурних збуреннях. Встановлено умови асимптотичної (ρ_0, ρ)-стійкості системи в термінах знаковизначеності спеціальних матриць.

1. Introduction. The systems of impulsive differential equations under structural perturbations attract the attention of experts because they model some realistic phenomena in mechanics, biology, control theory and other branches of natural sciences.

The aim of the present paper is to establish sufficient stability conditions for impulsive systems under structural perturbations in terms of two measures.

The paper consist of six sections.

In sections $2-4$ the stability problem on impulsive systems under structural perturbations in two measures is introduced and necessary definitions on matrix-valued Lyapunov functions are presented. Conditions for positive definiteness and decreascent of matrix-valued functions with respect to a measure are established.

In section 5 results on (ρ_0, ρ)-stability and asymptotic (ρ_0, ρ)-stability of impulsive systems relatively two measures are established.

2. Impulsive systems under structural perturbations in general. We consider an impulsive system with structural perturbation ([1, 3])

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, P, S) \qquad t \neq \tau_k(x), \\ \Delta x &= I_k(x), \qquad t = \tau_k(x), \\ x(t_0^+) &= x_0, \qquad t_0 \geq 0, \qquad k = 1, 2, \ldots, \end{aligned} \tag{1}$$

where $x \in R^n$; $t \in \mathcal{T}_0 = [t_0, \infty)$, $\tau_k \in C^1(R^n, (0; \infty))$, $\tau_k(x) < \tau_{k+1}(x)$ for all k, $\tau_k(x) \to +\infty$ uniformly with respect to $x \in R^n$.

The matrix $P = (P_1^T, P_2^T, \ldots, P_s^T)^T \in R^{s \times q}$ reflects internal (e.g. parameter) and / or external perturbations. The class of all admissible matrices P is denoted by $\mathcal{P}$,

$$\mathcal{P} = \{ P \colon P_1 \leq P(t) \leq P_2 \ \ \forall t \in R_+ \}. \tag{2}$$

The matrices P_1 and P_2 are completely defined. The set $\mathcal{P}$ may be the singleton $\{0\}$.

The matrix $S = S(t) \in \mathcal{G}_s$ describes all structural variations of the system (1) and will be called the structural matrix of the system (1), where

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* This work was done while the author was visiting the Department of Mathematics, University of Ioannina, in the framework of the NATO Science Fellowships Programme through the Greek Ministry of National Economy.

$$\mathcal{G}_s = \{S\colon S = \mathrm{diag}[S_1, S_2, \ldots, S_s],$$
$$S_i = [s_{i1}I_i, s_{i2}I_i, \ldots, s_{iN}I_i], \quad s_{ij} \in \{0, 1\}, \tag{3}$$
$$I_i = \mathrm{diag}\{1, 1, \ldots, 1\} \in R^{n_i \times n_i}, \quad i = 1, 2, \ldots, s.$$

The set of all possible $S(t)$ will be referred to as the structural set of the system (1).

In the system (1) $f \in \mathcal{F}$, where

$$\mathcal{F} = \{f^1, f^2, \ldots, f^N\}, \quad f^k \in C(R \times R^n \times \mathcal{P} \times \mathcal{G}, R^n), \quad k = 1, 2, \ldots, N. \tag{4}$$

Here $\mathcal{N}$ is a natural number.

The number N, the families $\mathcal{F}$ and (possibly arbitrary) variations of the superscript $k = k(t)$ over the set $\mathcal{N} = \{1, 2, \ldots, N\}$, $k(t) \in \mathcal{N}$ $\forall t \in R$, describe (possible arbitrary) structural variations of the whole impulsive system (1) (see [1] and the bibliography therein). The whole system (1) is structurally invariant if and only if $k(t) \equiv K$, i.e. the set $\mathcal{N}$ is singleton, $\mathcal{N} = \{K\}$. The number N is the number of all possible structures of the whole impulsive system (1). At $t = \tau_k(x)$, $k = 1, 2, \ldots$, the relations

$$\Delta x = x(t+0) - x(t-0), \tag{5}$$

hold true, and $I_k\colon R^n \to R^n$ for all $k = 1, 2, \ldots$. We shall say that hypothesis (A) holds if the following conditions are satisfied:

H_1. The function $f(t, x, P, S)$ is continuous in its domain of definition.

H_2. The functions $I_k(x)$, $k = 1, 2, \ldots$ are continuous in their domains of definition.

H_3. The functions $\tau_k(x)$ are continuous and for $x \in R^n$ the following relations hold

$$0 < \tau_1(x) < \tau_2(x) < \ldots, \quad \lim_{k \to \infty} \tau_k(x) = \infty,$$

uniformly in $x \in R^n$,

$$\inf_{x \in R^n} \tau_{k+1}(x) - \sup_{x \in R^n} \tau_k(x) \geq \theta > 0, \quad k = 1, 2, \ldots.$$

H_4. For each point $(t_0, x_0) \in \mathcal{T}_0 \times R^n$ for all $(P, S) \in \mathcal{P} \times \mathcal{G}_s$ the solution $x(t, t_0, x_0, P, S) = x(t)$ of system (1) is unique and defined in (t_0, ∞).

The solutions $x(t)$ of the impulsive system (1) are piecewise continuous functions with points of discontinuity of the first type in which they are left continuous, that is at the moment $\tau_k(x)$, when the integral curve of the solution $x(t)$ meets the hypersurface

$$\sigma_k = \{(t, x) \in \mathcal{T}_0 \times R^n\colon t = \tau_k(x)\}. \tag{6}$$

We shall say that condition B is satisfied if the following condition holds:

B. The integral curve of each solution of system (1) for each pair $(P, S) \in \mathcal{P} \times \mathcal{G}_s$ meets each of the hypersurfaces $\{\sigma_k\}$ at most once.

Condition B means that for system (1) for each pair $(P, S) \in \mathcal{P} \times \mathcal{G}_s$ the phenomenon called "beating" is not observed.

3. (ρ_0, ρ)-stability concepts under structural perturbations. For our study of (1) it is convenient to introduce the following classes of functions [1, 2, 4]:

Definition 1. *A function*:

(a) *a belongs to the class* K *if* $a: R_+ \to R_+$ *and* $a(u)$ *is strictly increasing in* u *and* $a(0) = 0$;

(b) q *belongs to the class* L *if* $q: R_+ \to R_+$, $q(u)$ *is strictly decreasing in* u *and* $\lim_{u \to \infty} q(u) = 0$;

(c) η *belongs to the class* PC *if* $\eta: R_+ \to R_+$ *is continuous on* $(\tau_{k-1}, \tau_k]$ *and* $\lim_{t \to \tau_k^+} \eta(t) = \eta(\tau_k^+)$;

(d) ζ *belongs to the class* PCK *if* $\zeta: R_+ \to R_+$, $\zeta(\cdot, u) \in PC$ *for each* $u \in \in R_+$ *and* $\zeta(t, \cdot) \in K$ *for each* $t \in R_+$;

(e) ρ *belongs to the class* M *if* $\rho: R_+ \times R^n \to R_+$, $\rho(\cdot, x) \in PC$ *for each* $x \in R^n$ *and* $\rho(t, \cdot) \in C(R^n, R_+)$ *for each* $t \in R_+$ *and* $\inf_{x \to R^n} \rho(t, x) = 0$.

Definition 2. *Suppose that* $\rho_0, \rho \in M$. *We say that*:

(a) *the measure* ρ *is continuous with respect to the measure* ρ_0 *if there exist a* $\sigma > 0$ *and a function* $\zeta \in PCK$ *such that* $\rho(t, x) < \zeta(t, \rho_0(t, x))$ *when* $\rho_0(t, x) < \sigma$;

(b) *the measure* ρ *is uniformly continuous with respect to the measure* ρ_0 *if there exist a* $\delta > 0$ *and a function* $\phi \in K$ *such that* $\rho(t, x) \le \phi(\rho_0(t, x))$ *when* $\rho_0(t, x) < \delta$.

Definition 3. *The impulsive system* (1) *is said to be*:

(a) $(\rho_0; \rho)$*-stable on* $\mathcal{P} \times \mathcal{G}_s$ *if and only if for each pair of* $(P, S) \in \mathcal{P} \times \mathcal{G}_s$ *and for every* $t_0 \in R_+$ *and every* $\varepsilon > 0$ *there exists a* $\delta(t_0, \varepsilon) > 0$ *such that* $\rho(t, x(t)) < < \varepsilon$ *for all* $t \ge t_0$ *when* $\rho_0(t_0^+, x_0) < \delta$, *where* $x(t; t_0, x_0, P, S)$ *is any solution of the system* (1) *for* $(P, S) \in \mathcal{P} \times \mathcal{G}_s$;

(b) *uniformly* (ρ_0, ρ)*-stable on* $\mathcal{P} \times \mathcal{G}_s$ *if and only if the conditions of* (a) *hold, and for every* $\varepsilon > 0$ *the maximal quantity* δ_M *satisfies the inequality*

$$\inf(\delta_M(t, \varepsilon): t \in R_+) > 0;$$

(c) (ρ_0, ρ)*-attracting on* $\mathcal{P} \times \mathcal{G}_s$ *if and only if for each pair of* $(P, S) \in \mathcal{P} \times \mathcal{G}_s$ *and for every* $t_0 \in R_+$, *there exists a* $\Delta(t_0) > 0$ *and for every* $\zeta > 0$ *there exists a* $\tau(t_0, x_0, \zeta) \in R_+$ *such that* $\rho(t, x(t)) < \zeta$ *for all* $t \ge t_0 + \tau$ *when* $\rho_0(t_0^+, x_0) < < \Delta(t_0)$;

(d) *uniformly* (ρ_0, ρ)*-attracting on* $\mathcal{P} \times \mathcal{G}_s$ *if and only if the conditions of* (c) *hold with constants* Δ and τ *independent of* $t_0 \in R_+$;

(e) *asymptotically* (ρ_0, ρ)*-stable on* $\mathcal{P} \times \mathcal{G}_s$ *if and only if the conditions of* (a) *and* (c) *hold*;

(f) *uniformly asymptotically* (ρ_0, ρ)*-stable on* $\mathcal{P} \times \mathcal{G}_s$ *if and only if the conditions of* (a) *and* (d) *hold.*

4. Auxiliary lemmas. In order to prove some auxiliary results which are useful in discussing the (ρ_0, ρ)-stability under structural perturbations of (1), we need to employ a matrix-valued Lyapunov function $U(t, x)$ satisfying some properties. Let $U: R_+ \times R^n \to R^{s \times s}$, $s > 1$, be a matrix-valued function [5].

Definition 4. *We say that the matrix-valued function* $U(t, x)$ *belongs to the class* U_0 *if it satisfies the following four conditions*:

(a) $U(t, x)$ *is continuous on each set* $E_k = \{(t, x) \in R_+ \times R^n: \tau_{k-1}(x) < t < < \tau_k(x)\}$, $k = 1, 2, \ldots$, *and* $E = \bigcup_{k=1}^{\infty} E_k$;

(b) *for each* $x \in R^n$ *and* $k = 1, 2, \ldots$ *the limit relations*

$$\lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in G_k}} U(t,x) = U(t_0-0,x_0), \qquad \lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in G_{k+1}}} U(t,x) = U(t_0+0,x_0),$$

hold and $U(t_0-0,x_0)=U(t_0,x_0)$, *where* $(t_0,x_0)\in G_k$;

(c) $U(t,x)$ *is a locally Lipschitz function with respect to* $x\in G$;

(d) $U(t+0,x+I_k(x))\le U(t,x)$ *for any pair* $(t,x)\in G_k$. *The conditions* (a) – (d) *in Definition* 4 *are satisfied componentwise.*

Definition 5. *A matrix-valued function* $U: R_+\times R^n\to R^{s\times s}$ *with* $U\in U_0$ *is said to be*

(a) ρ*-positive definite if there exists a connected neighborhood* $\mathcal{N}\subseteq R^n$ *of the point* $x=0$, *that is invariant in time, a vector* $\psi\in R_+^s$ *with* $\psi>0$, *a constant* $\delta>0$ *and a function* $\alpha\in K$ *such that the condition* $\rho(t,x)<\delta$ *implies the inequality*

$$\alpha(\rho(t,x)) \le \psi^T U(t,x)\psi \tag{7}$$

for all $(t, x\ne 0,\ \psi\ne 0)\in R_+\times\mathcal{N}\times R_+^s$;

(b) ρ*-decreascent if for the neighborhood* $\mathcal{N}\subseteq R^n$ *and the vector* ψ *indicated in* (a) *there exists a constant* $\delta_1>0$ *and a function* $\beta\in K$ *such that the condition* $\rho(t_0,x)<\delta_1$ *implies the inequality*

$$\psi^T U(t,x)\psi \le \beta(\rho(t,x)) \tag{8}$$

for all $(t, x\ne 0,\ \psi\ne 0)\in R_+\times\mathcal{N}\times R_+^s$;

(c) *weakly* ρ*-decreascent if for the neighborhood* $\mathcal{N}\subseteq R^n$ *and the vector* ψ *indicated in* (a) *there exists a constant* $\delta^*>0$ *and a function* $b\in PCK$ *such that the condition* $\rho(t,x)<\delta^*$ *implies the inequality*

$$\psi^T U(t,x)\psi \le b(t,\rho(t,x)) \tag{9}$$

for all $(t, x\ne 0,\ \psi\ne 0)\in R_+\times\mathcal{N}\times R_+^s$;

(d) *asympotically* ρ*-decreascent if for the neighborhood* $\mathcal{N}\subseteq R^n$ *and the vector* ψ *indicated in* (a) *there exists a constant* $\Delta>0$ *and a function* $\zeta\in KL$ *such that the condition* $\rho(t,x)<\delta$ *implies the inequality*

$$\psi^T U(t,x)\psi \le \zeta(\rho(t,x),t)$$

for all $(t, x\ne 0,\ \psi\ne 0)\in R_+\times\mathcal{N}\times R_+^s$;

(e) ρ*-negative definite if* $(-U(t,x))$ *is* ρ*-positive definite.*

Lemma 1. *A matrix-valued function* $U\in U_0$ *is* ρ*-positive definite if and only if it can be represented in the form*

$$\psi^T U(t,x)\psi = \psi^T U_+(t,x)\psi + \alpha(\rho(t,x)) \tag{10}$$

for all $(t, x\ne 0,\ \psi\ne 0)\in R_+\times\mathcal{N}\times R_+^s$, *where* $U_+(t,x)$ *is a positive semi-definite matrix-valued function and* $\alpha\in K$.

Proof. *Necessity.* Let the matrix-valued function $U(t,x)$ be ρ-positive definite. Then, by Definition 5(a) there are a connected neighborhood $\mathcal{N}\subseteq R^n$ of the point $x=0$, a vector $\psi\in R_+^s$, a constant $\delta>0$, and a function $\alpha\in K$ such that in the domain $\rho(t,x)<\delta$ condition (7) holds.

We introduce the function

$$\psi^T U_+(t,x)\psi = \psi^T U(t,x)\psi - \alpha(\rho(t,x)), \tag{11}$$

which by condition (7), is non-negative (i. e. positive semi-definite). From (11) we obtain the expression (10) for $U(t,x)$.

Sufficiency. Let correlation (10) hold, where $\psi^T U_+(t,x)\psi \geq 0$ for all $(t, x \neq 0, \psi \neq 0) \in R_+ \times \mathcal{N} \times R_+^s$, and $\alpha \in K$. Then it follows from (10) that $\psi^T U(t,x) - \alpha(\rho(t,x)) = \psi^T U(t,x)\psi \geq 0$ for all $(t, x \neq 0, \psi \neq 0) \in R_+ \times \mathcal{N} \times R_+^s$.

Hence condition (7) hold for $U(t,x)$, i.e. $U(t,x)$ is ρ-positive definite.

Lemma 2. *A matrix-valued function $U \in U_0$ is ρ-weakly decreascent (ρ-decreascent) if and only if it can be represented in the form*

$$\psi^T U(t,x)\psi = \psi^T U_-(t,x)\psi + \beta(t, \rho(t,x)), \tag{12}$$

$$\left(\psi^T U(t,x)\psi = \psi^T U_-(t,x)\psi + \gamma(\rho(t,x))\right), \tag{13}$$

for all $(t, x \neq 0, \psi \neq 0) \in R_+ \times \mathcal{N} \times R_+^s$, where $U_-(t,x)$ is a negative semi-definite matrix-valued function, $\beta \in PCK$, and $\gamma \in K$.

The proof is analogous to that of Lemma 1.

5. (ρ_0, ρ)-stability conditions under structural perturbations. In this section we will use the function

$$V(t,x,\psi) = \psi^T U(t,x)\psi, \quad \psi \in R_+^s, \quad \psi > 0. \tag{14}$$

For $(t,x) \in \left(\tau_{k-1}, \tau_k\right] \times R^n$ the right-hand upper derivatives of the function (14) are defined, as usual

$$D^+V(t,x,\psi) = \psi^T D^+U(t,x)\psi, \quad \psi \in R_+^s, \quad \psi > 0, \tag{15}$$

where

$$D^+U(t,x) = \left[D^+U_{ij}(t,x)\right], \quad i,j = 1,2,\ldots,s,$$

and

$$D^+U_{ij}(t,x) = \lim\left\{\left[U_{ij}(t+\theta, x+\theta f(t,x,P,S)) - U_{ij}(t,x)\right]: \theta^{-1}, \theta \to 0^+\right\},$$

for all $i,j = 1,2,\ldots,s$ and $(P,S) \in \mathcal{P} \times \mathcal{G}_s$. In the following exposition we use the notation

$$S(\rho, H) = \left\{(t,x) \in \left(\tau_{k-1}, \tau_k\right] \times R^n : \rho(t,x) < H,\ H = \text{const} > 0\right\}, \quad k = 1,2,\ldots.$$

We give some sufficient conditions for the (ρ_0, ρ)-stability of the system (1) in terms of a matrix-valued Lyapunov function [5].

Theorem 1. *Assume that the following conditions hold*:

(i) *the measures ρ_0 and ρ are in M, and ρ is continuous with respect to ρ_0;*

(ii) *there exist a matrix-valued function $U(t,x) \in U_0$ and a vector $\psi \in R_+^s$ with $\psi > 0$ such that*

(a) *$U(t,x) \in U_0$ is ρ-positive definite on $S(\rho, H)$;*

(b) *the functions $D^+U_{ij}(t,x)$ for all $i,j = 1,2,\ldots,s$ are bounded on $\mathcal{P} \times \mathcal{G}_s$ and there exist a matrix $G(P,S)$ and a vector $w^T = \left(w_1^{1/2}(\|x\|), \ldots, w_s^{1/2}(\|x\|)\right)$ such that the inequality*

$$\psi^T D^+U(t,x)\psi \leq w^T G(P,S) w \tag{16}$$

holds on $E \cap S(\rho, H) \times R_+^s$ for each pair $(P,S) \in \mathcal{P} \times \mathcal{G}_s$;

(c) *the inequality*

$$\psi^T\left[U(\tau_k(x), x + I_k(x)) - U(\tau_k(x), x)\right]\psi \leq u_k^T B u_k$$

holds, where

$$u_k^T = \left(u_1^{1/2}(\tau_k(x), \|x\|), \ldots, u_s^{1/2}(\tau_k(x), \|x\|)\right), \qquad k = 1, 2, \ldots, u_j(\cdot, \cdot) \in PCK,$$

and B *is a constant* $s \times s$ *matrix;*

(iii) *there exists an* $\alpha_0 \in (0, \alpha)$ *such that*

$$\rho(\tau_k(x), x + I_k(x)) < \alpha \quad \text{if} \quad \rho(t, x) < \alpha_0;$$

(iv) *there exists a constant matrix* $Q \in R^{s \times s}$ *such that*

$$G(P, S) \leq Q \quad \forall (P, S) \in \mathcal{P} \times \mathcal{G}_s;$$

(v) *the matrices* Q *and* B *are negative semi-definite or identically equal to zero.*

Then the system (1) *is:*

(a) (ρ_0, ρ)*-stable on* $\mathcal{P} \times \mathcal{G}_s$ *if* $U(t, x)$ *is weakly* ρ*-decreascent;*

(b) *uniformly* (ρ_0, ρ_0)*-stable on* $\mathcal{P} \times \mathcal{G}_s$ *if* $U(t, x)$ *is* ρ_0*-decreascent.*

If the condition (v) *is modified as follows:*

(v′) *the matrices* Q *and* B *are negative definite and* $U(t, x)$ *is* ρ_0*-decreasing, then the system* (1) *is*

(c) *asymptotically* (ρ_0, ρ)*-stable on* $\mathcal{P} \times \mathcal{G}_s$.

Proof. We begin with the assertion about (ρ_0, ρ)-stability on $\mathcal{P} \times \mathcal{G}_s$ of the Theorem 1. Because the matrix-valued function $U(t, x)$ is weakly ρ_0-decreasing, by Definition 5 (c) there exists $\delta^* > 0$ and a function $\beta \in PCK$ such that the condition

$$\rho_0(t, x) < \delta^*$$

implies

$$\psi^T U(t, x)\psi \leq \beta(t, \rho_0(t, x)) \quad \text{for} \quad (t, x, \psi) \in S(\rho, H) \times R_+^s.$$

Condition (ii), (a) of Theorem 1 implies that there exists a function $\alpha \in K$ such that

$$\alpha(\rho(t, x)) \leq \psi^T U(t, x)\psi \quad \text{for} \quad (t, x, \psi) \in S(\rho, H) \times R_+^s.$$

Condition (i) of Theorem 1 implies that there exist $\delta_1 > 0$ and a function $\zeta \in PCK$-class such that

$$\rho(t, x) \leq \zeta(t, \rho_0(t, x)) \tag{17}$$

provided that $\rho_0(t, x) < \delta_1$.

Let $t_0 \in R_+$ and $\varepsilon \in (0, H)$. The assumptions concerning the functions $\zeta, \beta \in PCK$-class imply that there exist $0 < \delta_2 < \delta^*$ and $0 < \delta_3 < \delta_1$ such that

$$\beta(t_0, \delta_2) < \alpha(\varepsilon) \quad \text{and} \quad \zeta(t_0, \delta_3) < H. \tag{18}$$

We take $\delta = \min(\delta_2, \delta_3)$ and consider a solution $x(t, t_0, x_0, P, S)$ of system (1) for which

$$\rho_0(t_0, x_0) < \delta. \tag{19}$$

By virtue of conditions (ii), (b) and (iv) the function $m(t) = \psi^T U(t, x)\psi$ satisfies the inequality

$$D^+ m(t) \leq w^T G(P, S) w \leq w^T Q w \leq \lambda_M(Q) w^T w = \lambda_M(Q) \sum_{j=1}^{s} w_j(\|x\|) \leq$$
$$\leq \lambda_M(Q) w(\|x\|) \;\; \forall (P, S) \in \mathcal{P} \times \mathcal{G}_s \tag{20}$$

on the solution $x(t, t_0, x_0, P, S)$ of the system (1). Here $\lambda_M(Q) \leq 0$ is the maximum eigenvalue of the matrix Q, and $w \in K$ is a function satisfying the inequality

$$w(\|x\|) \geq \sum_{j=1}^{s} w_j(\|x\|).$$

Since B is a negative semi-definite matrix, from conditions (ii), (c) and (iii) we have $\lambda_M(B) \le 0$ and

$$m(\tau_k(x)+0) - m(\tau_k(x)) \le u_k^T B u_k \le$$
$$\le -\lambda_M(B)\, u_k^T u_k = \lambda_M(B) \sum_{j=1}^{s} u_j(\tau_k(x), \|x\|) \le -\lambda_M(B)\overline{u}_k(\tau_k(x), \|x\|), \quad (21)$$

where

$$\overline{u}_k(\tau_k(x), \|x\|) = \sum_{j=1}^{s} u_j(\tau_k(x), \|x\|), \qquad k = 1, 2, \ldots .$$

Consequently, the function $m(t)$ decreases for all $t \ge t_0$. Furthermore, conditions (i) – (v) of the Theorem 1 imply

$$\alpha(\rho(t,x)) \le m(t) \le m(t_0+0) \le \beta(t_0, \rho_0(t_0, x_0)) < \alpha(\varepsilon).$$

Hence it follows that $\rho(t, x(t)) < \varepsilon$ for all $t \ge t_0$ for each pair $(P, S) \in \mathcal{P} \times \mathcal{G}_s$ provided that (19) is satisfied. Thus, the assertion about (ρ_0, ρ)-stability on $\mathcal{P} \times \mathcal{G}_s$ of Theorem 1 is proved. Assertions about uniformly (ρ_0, ρ)-stability on $\mathcal{P} \times \mathcal{G}_s$ and about asymptotically (ρ_0, ρ)-stability on $\mathcal{P} \times \mathcal{G}_s$ is proved in a similar way. Since $U(t, x)$ is a uniformly ρ_0-decreasing function, the parameter δ can be chosen independent of the value of t_0.

***Corollary* 1.** *Assume that*

(i) *the conditions* (ii), (b) *and* (ii), (c) *in Theorem* 1 *are replaced by the conditions*

(b′) $\psi^T D^+ U(t,x)\psi \le 0$ *on* $E \cap S(\rho, H) \times R_+^s$ *for each pair* $(P, S) \in \mathcal{P} \times \mathcal{G}_s$,

(c′) $\psi^T [\, U(\tau_k(x), x + I_k(x)) - U(\tau_k(x), x)\,]\psi \le 0$,

(ii) *the conditions* (iv), (v) *and* (v′) *is dropped.*

Then the assertions (a) *and* (b) *of Theorem* 1 *are preserved.*

***Corollary* 2.** *Assume that the following conditions hold*:

(i) *the conditions* (i), (ii) (b), (iii) *and* (iv) *of Theorem* 1 *are satisfied;*

(ii) *the matrix-valued function* $U(t, x)$ *is weakly* ρ_0*-decreascent;*

(iii) *the inequality*

$$\psi^T [\, U(\tau_k(x), x + I_k(x)) - U(\tau_k(x), x)\,]\psi \le -\lambda_k \alpha\big(\psi^T U(\tau_k(x), x)\psi\big)$$

holds for all $(\tau_k(x), x) \in E_k \cap S(\rho, H)$, *where* $\lambda \ge 0$, $\sum_{k=1}^{\infty} \lambda_k = \infty$, $\alpha \in$ $\in C(R_+, R_+)$, $\alpha(0) = 0$, $\alpha(s) > 0$ *for* $s > 0$;

(iv) *the matrix* Q *is negative semi-definite.*

Then the system (1) *is asymptotically* (ρ_0, ρ)*-stable on* $\mathcal{P} \times \mathcal{G}_s$.

Definition 6 (see [8]). *A measurable function* $\lambda : R_+ \to R_+$ *is said to be integrally positive if* $\int_J \lambda(s)\,ds = \infty$, *for* $J = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i]$, $\alpha_i < \beta_i < \alpha_{i+1}$ *and* $\beta_i - \alpha_i \ge \delta > 0$.

Theorem 2. *Assume that the following conditions hold*:

(i) *the measures* ρ_0 *and* ρ *are in M, and* ρ *is continuous with respect to* ρ_0;

(ii) *there exist a matrix-valued function* $U(t,x) \in U_0$ *and a vector* $\psi \in R^s$, $\psi > 0$, *such that*

(a) $U(t, x)$ *is locally Lipschitzian with respect to* x *on each* E_k, ρ*-positive definite, and* ρ_0*-decreascent;*

(b) $\psi^T D^+ U(t,x)\psi \le -\lambda(t)\, a(\rho(t,x))$ *on* $E \cap S(\rho, H) \times R^s$ *for each pair* (P, S) $\in \mathcal{P} \times \mathcal{G}_s$, *where* $a \in K$ *and* $\lambda(t)$ *is an integrally positive function;*

(iii) $\psi^T U(t, +0, x + I_k(x)) \le \psi^T U(t, x)$, *on* $E_k \cap S(\rho, H)$.

Then the system (1) *is uniformly asymptotically* (ρ_0, ρ)-stable on $\mathcal{P} \times \mathcal{G}_s$.

Proof. The conditions of the Theorem 2 imply that system (1) is uniformly (ρ_0, ρ)-stable on $\mathcal{P} \times \mathcal{G}_s$ (Theorem 1). Consequently, for $\varepsilon^* > 0$ there exists $\delta^* = \delta^*(\varepsilon^*)$ such that the condition $\rho_0(t_0, x_0) < \delta^*$ implies the inequality

$$\rho(t, x(t)) < \varepsilon^* \tag{22}$$

for all $t \ge t_0$, where $x(t) = x(t, t_0, x_0, P, S)$ is an arbitrary solution to system (1) for each pair $P, S) \in \mathcal{P} \times \mathcal{G}_s$.

Choose $0 < \varepsilon < \varepsilon^*$, and let $\delta = \delta(\varepsilon)$. Theorem 2 will be proved if we show that under its conditions there exists $T = T(\varepsilon) > 0$ such that

$$\rho(t^*, x(t^*)) < \delta, \quad \text{for some} \quad t^* \in [t_0, t_0 + T]. \tag{23}$$

Assume that this is not true. Then for the indicated $T(\varepsilon) > 0$ there exists a solution $x(t)$ to system (1) satisfying condition (22) and

$$\rho(t, x(t)) \ge \delta, \quad \text{for} \quad t_0 \le t \le t_0 + T(\varepsilon). \tag{24}$$

By condition (ii), (b) of Theorem 2 there exists a function $\beta \in K$- class such that

$$\int_{t_0}^{\infty} \lambda(s) a(\rho_0(s, x(s)))\, ds \le \beta(\delta) \tag{25}$$

for every solution $x(t)$ of system (1), satisfying condition (23).

Since $\lambda(t)$ is an integrally positive function, it follows that there exists $T > 0$ such that

$$\int_{t_0}^{t_0 + T(\varepsilon)} \lambda(s)\, ds > \frac{\beta(\delta) + 1}{\alpha(\delta)}, \quad t_0 \in R_+. \tag{26}$$

Let the solution $x(t)$ of system (1), satisfy conditions (25) with $T(\varepsilon)$ satisfing inequality (26). From (25) and (26) we find

$$\beta(\delta) \ge \int_{t_0}^{\infty} \lambda(s) a(\rho_0(s, x(s)))\, ds > \int_{t_0}^{t_0 + T(\varepsilon)} \lambda(s)\, ds > \beta(\delta) + 1 \tag{27}$$

and thus we are led to a contradiction. This implies that assumption (34) is not true, and consequently, system (1) is uniformly asymptotically (ρ_0, ρ)-stable on $\mathcal{P} \times \mathcal{G}_s$.

6. Conclusion. We remark that the use of an auxiliary matix-valued function and two measures in the construction of the direct Lyapunov method creates a flexible tool for investigating many dynamical properties of the system (1) due to two circumstances.

First, functions satisfying the following estimates can be taken as the elements of the matrix-valued function $[U_{ij}(t, x)]$:

$$\alpha_{ii} \rho_i^2(t, x_i) \le U_{ii}(t, x) \le \beta_{ii} \rho_i^2(t, x_i),$$

where $x_i \in R^{n_i}$, $\sum_{i=1}^{s} n_i = n$, $i = 1, 2, \ldots, s$;

$$\alpha_{ij} \rho_i(t, x_i) \rho_j(t, x_j) \le U_{ij}(t, x) \le \beta_{ij} \rho_i(t, x_i) \rho_j(t, x_j), \quad i \ne j, \tag{28}$$

where $(x_i, x_j) \in R^{n_i} \times R^{n_j}$, $i, j = 1, 2, \ldots, s$.

Here $\alpha_{ii}, \beta_{ii} > 0$, $(\alpha_{ij}, \beta_{ij})$ are constants for $i \ne j$, the $\rho_i(t, x_i)$, $\rho_j(t, x_j)$ are measures evaluating the change in the subvectors $(x_i \in R^{n_i}, x_j \in R^{n_j})$ and

$$\rho(t, x) = \sum_{i=1}^{s} \rho_i(t, x_i)$$

or

$$\rho(t,x) = \sum_{i=1}^{s} \Delta_i \rho_i(t,x_i) \quad \text{for a positive vector} \quad \Delta > 0$$

or

$$\rho(t,x) = \Xi(\rho_1(t,x_1),\dots,\rho_s(t,x_s)),$$

where $\Xi \in C(R_+^n, R_+)$, $\Xi(u)$ is nondecreasing in u and $\Xi(0)=0$.

Second, the collection of measures ρ_0 and ρ adequately characterizing the different dynamical properties of the system (1) is sufficiently broad (see [4, 3, 8]). Here are some such measures:

(a) $\rho(t,x)=\rho_0(t,x)=\|x\|$, where $\|\cdot\|$ is the Eucliden norm of a vector $x \in R^n$;

(b) $\rho(t,x)=\rho_0(t,x)=\|x-x_0(t)\|$, where $x_0(t)$ is some prescribed motion of the system (1);

(c) $\rho(t,x)=\rho_0(t,x)=d(x,A)$, where $d(x,A)$ is the distance from x to the set $A \subseteq R^n$;

(d) $\rho(t,x)=\rho_0(t,x)=\|x\|+\sigma(t)$, where $\sigma(t)\in L$;

(e) $\rho(t,x)=\|x\|_s$, $1<s\le n$, and $\rho_0(t,x)=\|x\|$;

(f) $\rho(t,x)=d(x,B)$ and $\rho_0(t,x)=d(x,A)$, where $A \subseteq B \subseteq R^n$;

(g) $\rho(t,x)=\|x\|$ and $\rho_0(t,x)=\|x\|+d(x,M)$, where M is the k-dimensional manifold containing the origin;

(h) $\rho(t,x)=\rho_0(t,x)=p(x)$, where p is the k-dimensional vector norm with i-th component $p_i(x)=p_i(x_i)$, $p(x): R^n \to R_+^k$ and with the properties

$$p_i(x_i) \ge 0 \quad \forall x_i \in R^{n_i}, \quad i=1,2,\dots,k;$$

$$p_i(x_i) = 0 \quad \forall x_i \in R^{n_i}, \quad i=1,2,\dots,k;$$

$$p_i(x_i+y_i) \le p_i(x_i)+p_i(y_i) \quad \forall x_i, y_i \in R^{n_i}, \quad i=1,2,\dots,k;$$

$$p_i(\lambda x_i) = |\lambda| p_i(x_i) \quad \forall x_i \in R^{n_i}, \quad i=1,2,\dots,k, \quad \lambda \in R.$$

It is easy to show that the components $p_i(x_i)$ of the vector norm $p(x)$ can be used instead of the measures $\rho_i(t,x)$ in the estimates (28). Recently [6] stability conditions for the impulsive system under structural perturbations are studied by using measures (a) together with the matrix-valued Lyapunov functions method.

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Received 27.10.97