UDC 517.5

A. Stepanets (Inst. Math. Nat. Acad. Sci. Ukraine, Kiev),
Wang Kunyang*, **Zhang Xirong** (Norm. Univ., Pekin, China)

APPROXIMATION OF LOCALLY INTEGRABLE FUNCTIONS ON THE REAL LINE

НАБЛИЖЕННЯ ЛОКАЛЬНО ІНТЕГРОВНИХ ФУНКЦІЙ НА ДІЙСНІЙ ОСІ

We introduce the notion of generalized $\bar{\psi}$-derivatives for functions locally integrable on the real axis and investigate problems of approximation of the classes of functions determined by these derivatives with the use of entire functions of exponential type.

Вводиться поняття узагальнених $\bar{\psi}$-похідних для функцій, локально інтегровних на дійсній осі, і вивчаються задачі наближення класів функцій, що визначаються такими похідними, за допомогою цілих функцій експоненціального типу.

1. Introduction. In the recent paper [1], A. Stepanets investigated the problem of approximation for the functions that are locally integrable on the real line. In fact, Stepanets extended his serious investigation for the periodic functions (see. [2 – 5]) to the case of locally integrable functions. This extension [1] only concerns the so called (ψ, β)-derivatives, which are defined via a single function ψ and a phase translation $\beta\frac{\pi}{2}$. At the same time, in the recent papers of Stepanets [6, 7], the concept of (ψ, β)-derivative was extended by introducing the notion of $\bar{\psi}$-derivative defined via a pair of functions ψ_1 and ψ_2.

The aim of the present paper is to extend (ψ, β)-derivatives to $\bar{\psi}$-derivatives for locally integrable functions on the real line and then, correspondingly, extend the results of [1] to the case of approximation of locally integrable functions that have $\bar{\psi}$-derivatives on the real line.

First, we recall certain definitions introduced in [1] and [5] with suitable modifications.

For a function f measurable on R, we define

$$\|f\|_{\hat{p}} = \sup\left\{\left(\int_0^{2\pi} |f(x+y)|^p\,dy\right)^{1/p} : x \in R\right\} \quad \text{for} \quad 1 \le p < \infty;$$

$$\|f\|_{\hat{\infty}} = \sup\{|f(x)| : x \in R\}.$$

We also define the following spaces of functions for $1 \le p < \infty$

$$\hat{L}_p = \{f \text{ is measurable on } R\colon \|f\|_{\hat{p}} < \infty\}.$$

Sometimes we simply write $\hat{L}$ instead of $\hat{L}_1$.

Definition 1. $\mathfrak{U}$ *denotes the set of functions* ψ *satisfying the following conditions*:

(a) $\psi(u) \ge 0$, $\psi(0) = 0$, ψ *is increasing and continuous on* $[0, 1]$;

(b) ψ *is convex on* $[1, \infty)$ *and* $\lim\limits_{u\to\infty} \psi(u) = 0$;

(c) $\psi'(u) := \psi'(u+0)$ *is bounded on* $[0, \infty)$.

* Supported by NSF of China 1977, 1009.

Definition 2. *We set* $F = \left\{\psi \in \mathcal{U}\colon \int\limits_1^\infty \frac{\psi(t)}{t}\,dt < \infty\right\}.$

In [5] for suitable ψ defined on $[0, \infty)$ and for $\beta \in R$, the transform $\hat{\psi}_\beta$ is defined as

$$\hat{\psi}_\beta(t) = \frac{1}{\pi}\int\limits_0^\infty \psi(v)\cos\left(vt + \beta\frac{\pi}{2}\right)dv \tag{1}$$

whenever this integral makes sense. Furthermore, if $\hat{\psi}_\beta \in L(R)$ (this is the case where $\psi \in \mathcal{U}$), the (ψ, β)-derivative of a function $f \in \hat{L}$ is defined as a function $\varphi \in \hat{L}_1$ satisfying the condition

$$f(x) = A_0 + \lim_{R\to\infty}\int\limits_{-R}^{R}\varphi(x+t)\,\hat{\psi}_\beta(t)\,dt, \tag{2}$$

where A_0 is a constant independent of x. We write $\varphi = f_\beta^\psi$.

For simplicity, we assume that $\psi \in \mathcal{U}$. Let ψ_+ and ψ_- be the even and odd extensions of ψ, respectively. Then by (1) we have

$$\hat{\psi}_\beta = \widehat{\psi_+}\cos\beta\frac{\pi}{2} + i\,\widehat{\psi_-}\sin\beta\frac{\pi}{2}, \tag{3}$$

where $\widehat{\psi_+}$ and $\widehat{\psi_-}$ denote the Fourier transforms of ψ_+ and ψ_- respectively, in the original sense, i. e.,

$$\forall\, h \in L(R) \qquad \hat{h}(t) = \frac{1}{2\pi}\int\limits_R h(x)\,e^{-ixt}\,dx. \tag{4}$$

Now assume that $\psi_1 \in \mathcal{U}$ and $\psi_2 \in \mathcal{U}$. Then ψ_{1+}, ψ_{2+}, and ψ_{1-}, ψ_{2-} are even and odd extensions of ψ_1, ψ_2 respectively. For the pair (ψ_1, ψ_2), we define

$$\psi = \psi_{1+} + i\,\psi_{2-}. \tag{5}$$

The corresponding Fourier transform has the form

$$\hat{\psi} = \widehat{\psi_{1+}} + i\,\widehat{\psi_{2-}}. \tag{6}$$

We should remember definition (5) (and (6)). It will be used throughout the paper.

Definition 3. *Assume that* $\psi_1, \psi_2 \in \mathcal{U}$ *and* $\psi = \psi_{1+} + i\,\psi_{2-}$. *If a function* $f \in \hat{L}$ *can be expressed as*

$$f = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(x)\,dx + \varphi * \hat{\psi}$$

with $\varphi \in \hat{L}$, *then we say that* φ *is the* ψ*-derivative of* f *and denote it as* $\varphi = f^\psi$ or $f = I^\psi(\varphi)$.

Note that the convolution "$*$" in Definition 3 is defined as usual, i.e., for $g, h \in L(R)$, we have

$$(g * h)(x) = \int\limits_R g(x-t)\,h(t)\,dt = (h * g)(x).$$

If $h \in \mathcal{U}$ and $\beta \in R$ and

$$h_1 := h \cos \beta \frac{\pi}{2}, \quad h_2 := -h \sin \beta \frac{\pi}{2},$$

then it follows from (3) that $\hat{h}_\beta = \widehat{h_{1+}} + i\widehat{h_{2-}}$ and, hence, by virtue of (5) and (6), for the pair (h_1, h_2), we have $\hat{h} = \hat{h}_\beta$. Hence, we see that Definition 3 is a generalization of the concept of (ψ, β)-derivative considered in [1].

Let $\mathcal{N}$ be a subclass of $\hat{L}$. We use the notation

$$\hat{L}^\psi \mathcal{N} := \{f \in \hat{L}: \ f^\psi \in \mathcal{N}\}$$

as in [1]. For $\mathcal{N} = \hat{L}$, we also simply write $\hat{L}^\psi$ instead of $\hat{L}^\psi \hat{L}$. Denote (see [5])

$$L_{2\pi}^0 = \left\{ f \in L_{2\pi}: \int_{-\pi}^{\pi} f(t)\,dt = 0 \right\}, \qquad L_{2\pi}^\psi = \{ f \in L_{2\pi}: \ f^{\overline{\psi}} \in L_{2\pi} \}$$

where $f^{\overline{\psi}}$ is the $\overline{\psi}$-derivative of f defined in [5]. In this notation, the following statement is true:

Proposition 1. *Suppose that* $\psi_1, \psi_2 \in \mathcal{U}$, $\psi = \psi_{1+} + i\psi_{2-}$. *Then*

$$\hat{L}^\psi L_{2\pi}^0 = L_{2\pi}^\psi .$$

Proof. If $f \in \hat{L}^\psi L_{2\pi}^0$, then

$$f = A_0 + \varphi * \hat{\psi}$$

with $A_0 \in R$ and $\varphi \in L_{2\pi}^0$. We see that f is 2π-periodic, i.e., $f \in L_{2\pi}$. Furthermore, by virtue of the definition of $f^{\overline{\psi}}$ introduced in [6], we see that $f^{\overline{\psi}}$ coincides with f^ψ defined here. This means $\hat{L}^\psi L_{2\pi}^0 = L_{2\pi}^\psi$.

Next we define for $0 \le c < \sigma$

$$\lambda_{\sigma c}(t) = \begin{cases} 1, & 0 \le |t| \le c, \\ \dfrac{\sigma - |t|}{\sigma - c}, & c < |t| < \sigma, \\ 0, & \sigma \le |t|, \end{cases}$$

and for $\psi_1, \psi_2 \in \mathcal{U}$

$$\lambda_{\sigma c}^*(t) = \begin{cases} \lambda_{\sigma c}(t), & |t| \in [0, c] \cap [\sigma, \infty), \\ 1 - \dfrac{|t| - c}{\sigma - c} \dfrac{\psi(\sigma \operatorname{sign}(t))}{\psi(t)}, & c < |t| < \sigma. \end{cases}$$

We see that these are just the even extensions of the corresponding functions $\lambda_{\sigma c}$ and $\lambda_{\sigma c}^*$ defined in [1]. The we construct the operators $F_{\sigma c}$ and $F_{\sigma c}^*$ as in [1], i.e., as follows:

Definition 4. *Let* $0 \le c < \sigma < \infty$ *and* $\psi_1, \psi_2 \in \mathcal{U}$. *For* $f \in \hat{L}^\psi$,

$$F_{\sigma c}(f) = f^\psi * \widehat{\psi \lambda_{\sigma c}} + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt,$$

$$F^*_{\sigma c}(f) = f^{\psi} * \widehat{\psi \lambda^*_{\sigma c}} + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt .$$

2. The class of entire functions of exponential type. Denote by $\mathscr{E}_\sigma$ the class of entire functions of exponential type σ $(\sigma > 0)$. We refer to [8] for the basic knowledge on $\mathscr{E}_\sigma$. Let

$$W^2_\sigma = \left\{ \varphi \in \mathscr{E}_\sigma : \int_R \frac{|\varphi(t)|^2}{1+t^2}\, dt < \infty \right\}.$$

We now give a generalization of Proposition 2 of [1].

Theorem 1. *Suppose that* $\psi_1, \psi_2 \in \mathfrak{U}$, $f \in \hat{L}^{\psi}$:

(a) *If* $0 < \tau \le c < \sigma$ *and* $f^{\psi} \in W^2_\tau$ *then*

$$F_{\sigma c}(f) = f, \qquad F^*_{\sigma c}(f) = f;$$

(b) *If* $f \in L_{2\pi}$ *and* $\sigma > c \ge 0$, *then*

$$F_{\sigma c}(f)(x) = \sum_{|k| < \sigma} \lambda_{\sigma c}(k)\, c_k(f)\, e^{ikx},$$

$$F^*_{\sigma c}(f)(x) = \sum_{|k| < \sigma} \lambda^*_{\sigma c}(k)\, c_k(f)\, e^{ikx},$$

where

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx, \qquad k \in \mathbb{Z};$$

(c) *If* $\psi' \in L_2(0, a)$ $\forall\, a > 0$, $\sigma > c \ge 0$ *and*

$$\int_R \frac{|f^{\psi}(t)|^2}{1 + |t|^2}\, dt < \infty$$

then $F_{\sigma c}(f) \in W^2_\sigma$, $F^*_{\sigma c}(f) \in W^2_\sigma$.

Proof. By the basic result in Fourier analysis, we have

$$\widehat{\psi \lambda_{\sigma c}} = \hat{\psi} * \widehat{\lambda_{\sigma c}}, \qquad \widehat{\psi \lambda^*_{\sigma c}} = \hat{\psi} * \widehat{\lambda^*_{\sigma c}}.$$

Then we get

$$F_{\sigma c}(f) = A_0 + f^{\psi} * (\hat{\psi} * \widehat{\lambda_{\sigma c}}) = A_0 + (f^{\psi} * \widehat{\lambda_{\sigma c}}) * \hat{\psi},$$

$$F^*_{\sigma c}(f) = A_0 + f^{\psi} * (\hat{\psi} * \widehat{\lambda^*_{\sigma c}}) = A_0 + (f^{\psi} * \widehat{\lambda^*_{\sigma c}}) + \hat{\psi}.$$

Generally for any $\varphi \in W^2_\tau$ and $0 < \tau \le c < \sigma$, we have

$$\varphi * \widehat{\lambda_{\sigma c}} = \varphi * \widehat{\lambda^*_{\sigma c}} = \varphi,$$

which is just a consequence of the Wiener – Paley theorem [8]. So, by the definition of ψ-derivative we get conclusion (a).

In the case where $f \in L_{2\pi}$, we can easily check

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f^{\psi}*\widehat{\lambda_{\sigma c}})(x)\,e^{-ikx}\,dx = \int_R\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{\psi}(x-t)\,e^{-ikx}\,dx\right)\widehat{\lambda_{\sigma c}}(t)\,dt =$$

$$= c_k(f^{\psi})\int_R \widehat{\lambda_{\sigma c}}(t)\,e^{-ixt}\,dt = \lambda_{\sigma c}(-k)\,c_k(f^{\psi}).$$

But $\lambda_{\sigma c}$ is even and, therefore,

$$(f^{\psi}*\widehat{\lambda_{\sigma c}})(x) = \sum_{|k|<\sigma}\lambda_{\sigma c}(k)\,c_k(f^{\psi})\,e^{ikx}.$$

Hence,

$$(f^{\psi}*\widehat{\lambda_{\sigma c}}*\hat{\psi})(x) = \sum_{|k|<\sigma}\lambda_{\sigma c}(k)\,c_k(f^{\psi})\int_R e^{ik(x-t)}\,\hat{\psi}(t)\,dt =$$

$$= \sum_{|k|<\sigma}\lambda_{\sigma c}(k)\,c_k(f^{\psi})\,\psi(-k)\,e^{ikx}.$$

By Definition 3, for $k\neq 0$,

$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_R f^{\psi}(x-t)\,\hat{\psi}(t)\,dt\right)e^{-ikx}\,dx =$$

$$= c_k(f^{\psi})\int_R \hat{\psi}(t)\,e^{-ikt}\,dt = c_k(f^{\psi})\,\psi(-k).$$

Therefore,

$$(f^{\psi}*\widehat{\lambda_{\sigma c}}*\hat{\psi})(x) = \sum_{0<|k|<\sigma}\lambda_{\sigma c}(k)\,c_k(f)\,e^{ikx}$$

which implies that

$$F_{\sigma c}(f)(x) = \sum_{|k|<\sigma}\lambda_{\sigma c}(k)\,c_k(f)\,e^{ikx}.$$

The same argument yields

$$F^*_{\sigma c}(f)(x) = \sum_{|k|<\sigma}\lambda^*_{\sigma c}(k)\,c_k(f)\,e^{ikx}.$$

Conclusion (b) is proved.

In the case of $\psi'\in L_2(0,\sigma)$, we write $\gamma = \psi\lambda_{\sigma c}$ or $\gamma = \psi\lambda^*_{\sigma c}$. Then $\operatorname{supp}\gamma\subset$ $\subset[-\sigma,\sigma]$ and γ is absolutely continuous on $[-\sigma,\sigma]$, $\gamma'\in L_2(-\sigma,\sigma)$. Applying the result of [8, p. 228] we have $g*\hat{\gamma}\in W^2_{\sigma}$ whenever g satisfies

$$\int_R\frac{|g(t)|^2}{1+|t|^2}\,dt < \infty.$$

So we get conclusion (c) and complete the proof.

3. Approximation by $F^*_{\sigma c}$. Assume that $\psi_1,\psi_2\in\mathcal{U}$, $0\le c<\sigma<\infty$, and $f\in$ $\in\hat{L}^{\psi}$. We define

$$\rho^*_{\sigma c}(f) = f - F^*_{\sigma c}(f),\qquad \rho_{\sigma c}(f) = f - F_{\sigma c}(f). \tag{7}$$

We write

$$r_{\sigma c}(t) = (1 - \lambda^*_{\sigma c}(t))\,\psi(t) = \begin{cases} 0, & 0 \le |t| \le c, \\ \dfrac{|t| - c}{\sigma - c}\,\psi(\sigma \operatorname{sign}(t)), & c < |t| < \sigma, \\ \psi(t), & \sigma \le |t|. \end{cases} \tag{8}$$

Then, by Definitions 3 and 4, we get

$$\rho^*_{\sigma c}(f) = f^{\psi} * \widehat{r_{\sigma c}}. \tag{9}$$

By Theorem 1, for any $u \in W^2_\tau$ and $0 < \tau \le c < \sigma$ we have $u * \widehat{r_{\sigma c}} = 0$. Hence,

$$\rho^*_{\sigma c}(f) = (f^{\psi} - u) * \widehat{r_{\sigma c}}.$$

This implies that, for $1 \le p \le \infty$ anf $f \in \hat{L}^{\psi}\hat{L}_p$, we get

$$\|\rho^*_{\sigma c}(f)\|_{\hat{p}} \le E_c(f^{\psi})_{\hat{p}}\,\|\widehat{r_{\sigma c}}\|_1, \tag{10}$$

where

$$E_c(h)_{\hat{p}} = \inf\{\|h - u\|_{\hat{p}} :\ u \in W^2_c\}, \qquad \|\widehat{r_{\sigma c}}\|_1 = \int_R |\widehat{r_{\sigma c}}(t)|\,dt.$$

By the definition of Fourier transform (4), we have

$$\widehat{r_{\sigma c}}(t) = \frac{1}{2\pi}\int_R r_{\sigma c}(s)\,e^{-ist}\,ds = \frac{1}{2\pi}\int_c^{\sigma} \frac{s - c}{\sigma - c}\left(\psi(\sigma)\,e^{-ist} + \psi(-\sigma)\,e^{ist}\right) ds +$$

$$+ \frac{1}{2\pi}\int_{\sigma}^{\infty} \left(\psi(s)\,e^{-ist} + \psi(-s)\,e^{ist}\right) ds.$$

By (5),

$$\psi(s)\,e^{-ist} + \psi(-s)\,e^{ist} = 2\left(\psi_1(s)\cos st + \psi_2(s)\sin st\right).$$

We write

$$R_1(t) = \frac{\psi_1(\sigma)}{\pi}\int_c^{\sigma} \frac{s - c}{\sigma - c}\cos st\,ds + \frac{1}{\pi}\int_{\sigma}^{\infty} \psi_1(s)\cos st\,ds, \tag{11}$$

$$R_2(t) = \frac{\psi_2(\sigma)}{\pi}\int_c^{\sigma} \frac{s - c}{\sigma - c}\sin st\,ds + \frac{1}{\pi}\int_{\sigma}^{\infty} \psi_2(s)\sin st\,ds. \tag{12}$$

We see

$$\widehat{r_{\sigma c}}(t) = R_1(t) + R_2(t). \tag{13}$$

As in [1], we denote by F_0 the following subclass of $\mathcal{U}$:

$$F_0 = \{\psi \in \mathcal{U} : \sup\{\eta'(\psi, t) :\ t \ge 1\} < \infty\},$$

where $\eta(\psi, t) = \psi^{-1}\left(\frac{1}{2}\psi(t)\right)$, $t \ge 1$, is introduced by Stepanets [5].

Similarly, we keep the notation $\mathcal{U}_0$, $\mathcal{U}_c$, and $\mathcal{U}_\infty$ as introduced in [1]. We know from [1] that

$$\mathcal{U}_c \cup \mathcal{U}_\infty \subset F_0.$$

For $\psi_1, \psi_2 \in \mathcal{U}$, by (5), we conclude that

$$|\psi(t)| = (\psi_1^2(t) + \psi_2^2(t))^{1/2}$$

is decreasing on $[0, \infty)$. Hence, we may assume that $|\psi(t)| > 0$ for all $t \in [1, \infty)$.

Next for simlicity we assume that $0 \le c = \sigma - 1$ and write r_σ instead of $r_{\sigma,\sigma-1}$. In this case, by (11) and (12), we have

$$R_1(t) = \frac{2\sin\frac{t}{2}}{\pi t^2}\psi_1(\sigma)\sin\left(\left(\sigma - \frac{1}{2}\right)t\right) - \frac{1}{\pi t}\int_\sigma^\infty \psi_1'(s)\sin st\,ds, \qquad (11')$$

$$R_2(t) = \frac{2\sin\frac{t}{2}}{\pi t^2}\psi_2(\sigma)\cos\left(\left(\sigma - \frac{1}{2}\right)t\right) + \frac{1}{\pi t}\int_\sigma^\infty \psi_2'(s)\cos st\,ds. \qquad (12')$$

3.1. The case $\psi_1, \psi_2 \in F_0$. We define

$$\alpha_j(t) = \frac{2\pi}{\eta(\psi_j, t) - t}, \qquad s = 1, 2, \quad t \ge 1,$$

$$\sup\{\eta(\psi_j, t)\colon t \ge 1\} = K_j, \qquad j = 1, 2.$$

Theorem 2. *Suppose that* $\psi_1, \psi_2 \in F_0$. *If there exist constants* $A \ge B > 0$ *such that*

$$\forall\, t \ge 1 \qquad A \ge \frac{\eta(\psi_1, t) - t}{\eta(\psi_2, t) - t} \ge B, \qquad (14)$$

then

$$\|\hat{r}_\sigma\|_1 = \frac{4}{\pi^2}|\psi(\sigma)|\left(\log^+(\eta(\sigma) - \sigma) + O(1)\right)$$

as $\sigma \to \infty$, *where*

$$\eta(\sigma) = \min(\eta(\psi_1, \sigma), \eta(\psi_2, \sigma)), \qquad \sigma \ge 1.$$

Proof. We write

$$\alpha = \frac{2\pi}{\eta(\sigma) - \sigma}, \qquad \sigma \ge 1.$$

By the monotonicity of ψ_1 and the periodicity of cos, we have

$$\left|\int_\sigma^\infty \psi_1'(s)\cos st\,ds\right| \le \int_\sigma^{\sigma+2\pi/|t|} \psi_1(s)\,ds$$

and, hence,

$$\int_{|t|<\alpha}\left|\int_\sigma^\infty \psi_1(s)\cos st\,ds\right| dt \le 2\int_0^\alpha\left(\int_\sigma^{\sigma+2\pi/\alpha} + \int_{\sigma+2\pi/\alpha}^{\sigma+2\pi/t}\right)\psi_1(s)\,ds\,dt \le$$

$$\le 4\pi\psi_1(\sigma) + 4\int_{2\pi/\alpha}^\infty \frac{\psi_1(\sigma+s)}{s}\,ds.$$

We note that

$$\int_{2\pi/\alpha}^{\infty} \frac{\psi_1(\sigma+s)}{s}\,ds = \left(\int_{\eta(\sigma)-\sigma}^{\eta(\psi_1,\sigma)-\sigma} + \int_{\eta(\psi_1,\sigma)-\sigma}^{\infty}\right)\frac{\psi_1(\sigma+s)}{s}\,ds,$$

where, by assumption (14),

$$\int_{\eta(\sigma)-\sigma}^{\eta(\psi_1,\sigma)-\sigma} \frac{\psi_1(\sigma+s)}{s}\,ds \le \psi_1(\sigma)\log(1+A).$$

On the other hand, it is known from [2, p. 134] that, for $\psi_1 \in F_0$,

$$\int_{\eta(\psi_1,\sigma)-\sigma}^{\infty} \frac{\psi_1(\sigma+s)}{s}\,ds \le \left(1 + \sup\{\eta'(\psi_1,t)\colon t \ge 1\}\right)\psi_1(\sigma).$$

Hence, we have

$$\int_{|t|<\alpha} \left|\int_{\sigma}^{\infty} \psi_1(s)\cos st\,ds\right| dt \le C\,\psi_1(\sigma), \tag{15}$$

where $C > 0$ is a constant independent of $\sigma \ge 1$. By the same argument, we have

$$\int_{|t|<\alpha} \left|\int_{\sigma}^{\infty} \psi_2(s)\sin st\,ds\right| dt \le C\,\psi_2(\sigma). \tag{16}$$

We may assume that $\psi_1(\sigma) > 0$ and write

$$\theta = \arctan\frac{\psi_2(\sigma)}{\psi_1(\sigma)}.$$

We define

$$R_3(t) = \frac{|\psi(\sigma)|}{\pi}\int_{\sigma-1}^{\sigma} (s-\sigma+1)\cos(st-\theta)\,ds. \tag{17}$$

Then, by (11)–(13),

$$\hat{r}_\sigma(t) = R_3(t) + \frac{1}{\pi}\int_{\sigma}^{\infty} \left(\psi_1(s)\cos st + \psi_2(s)\sin st\right) ds. \tag{18}$$

It is obvious that

$$|R_3(t)| \le |\psi(\sigma)|.$$

So, if $\alpha \le 1$, then, by (15)–(18), we have

$$\int_{|t|\le\alpha} |\hat{r}_\sigma(t)|\,dt \le C\,|\psi(\sigma)|. \tag{19}$$

If $1 < \alpha$, then

$$\int_{|t|<\alpha} |\hat{r}_\sigma(t)|\,dt \le C\,|\psi(\sigma)| + \int_{1<|t|<\alpha} |R_3(t)|\,dt.$$

By (17), we get

$$R_3(t) = \frac{|\psi(\sigma)|}{\pi t}\sin(\sigma t - \theta) - \frac{2|\psi(\sigma)|}{\pi t^2}\sin\frac{t}{2}\sin\left[\left(\sigma - \frac{1}{2}\right)t - \theta\right].$$

Thus,

$$\int\limits_{1<|t|<\alpha} |R_3(t)|\,dt \;\le\; \frac{2}{\pi}|\psi(\sigma)| + \frac{|\psi(\sigma)|}{\pi} \int\limits_{1<|t|<\alpha} \left|\frac{\sin(\sigma t - \theta)}{t}\right| dt \;\le$$

$$\le\; \frac{4}{\pi^2}|\psi(\sigma)|\,(\log\alpha + C). \tag{20}$$

In order to consider the case $|t| > \alpha$, we use (11′) and (12′) to get

$$\hat{r}_\sigma(t) \;=\; -|\psi(\sigma)|\frac{2\sin\frac{t}{2}}{\pi t^2}\sin\left(\left(\sigma - \frac{1}{2}\right)t - \theta\right) +$$

$$+\; \frac{1}{\pi t}\int\limits_{\sigma}^{\infty} (-\psi_1'(s)\sin st + \psi_2'(s)\cos st)\,ds\,. \tag{21}$$

By virtue of the monotonicity of ψ_1' and the periodicity of sin, we have

$$\left|\int\limits_{\sigma}^{\infty} \psi_1'(s)\sin st\,ds\right| \;\le\; \psi_1(\sigma) - \psi_1\left(\sigma + \frac{2\pi}{|t|}\right).$$

Hence,

$$\int\limits_{|t|>\alpha} \frac{1}{|t|}\left|\int\limits_{\sigma}^{\infty} \psi_1'(s)\sin st\,ds\right| dt \;\le\; 2\int\limits_{0}^{2\pi/\alpha} \frac{\psi_1(\sigma) - \psi_1(\sigma + t)}{t}\,dt \;\le$$

$$\le\; 2\int\limits_{\sigma}^{\eta(\psi,\sigma)} \frac{\psi_1(\sigma) - \psi_1(t)}{t - \sigma}\,dt\,.$$

It is also known from [2, p. 134] that, for $\psi_1 \in F_0$,

$$\int\limits_{\sigma}^{\eta(\psi,\sigma)} \frac{\psi_1(\sigma) - \psi_1(t)}{t}\,dt \;\le\; \sup\{\eta'(\psi_1, t)\colon\, t \ge 1\}\,\psi_1(\sigma).$$

Therefore,

$$\int\limits_{|t|>\alpha} \frac{1}{|t|}\left|\int\limits_{\sigma}^{\infty} \psi_1'(s)\sin st\,ds\right| dt \;\le\; C\,\psi_1(\sigma).$$

The same argument yields

$$\int\limits_{|t|>\alpha} \frac{1}{|t|}\left|\int\limits_{\sigma}^{\infty} \psi_2'(s)\cos st\,ds\right| dt \;\le\; C\,\psi_2(\sigma).$$

By these estimates, we derive from (21) that

$$\int\limits_{|t|>\alpha} |\hat{r}_\sigma(t)|\,dt \;\le\; C\,|\psi(\sigma)| + \frac{|\psi(\sigma)|}{\pi}\int\limits_{|t|>\alpha} \frac{\left|2\sin\frac{t}{2}\sin\left(\left(\sigma - \frac{1}{2}\right)t - \theta\right)\right|}{t^2}\,dt\,.$$

If $\alpha \ge 1$, then

$$\int\limits_{|t|>\alpha} \left|2\sin\frac{t}{2}\sin\left(\left(\sigma - \frac{1}{2}\right)t - \theta\right)\right| t^{-2}\,dt \;\le\; 4\,.$$

If $\alpha < 1$, then

$$\int\limits_{\alpha<|t|<1} \left| 2\sin\frac{t}{2}\sin\left(\left(\sigma-\frac{1}{2}\right)t-\theta\right)\right| t^{-2}\,dt \le$$

$$\le \int\limits_{\alpha<|t|<1} \left| \frac{\sin\left(\left(\sigma-\frac{1}{2}\right)t-\theta\right)}{t} \right| dt \le \frac{4}{\pi}\log\frac{1}{\alpha} + C.$$

This and (21) show that

$$\int\limits_{|t|>\alpha} |\hat{r}_\sigma(t)|\,dt \le \begin{cases} \left(\frac{4}{\pi^2}\log\frac{1}{\alpha}+C\right)|\psi(\sigma)| & \text{if } \alpha<1, \\ C\,|\psi(\sigma)| & \text{if } \alpha>1. \end{cases} \tag{22}$$

Combining (19), (20), and (22), we complete the proof.

Theorem 3. *If there exist constants* $\varepsilon>0$ *and* $t_0\ge 1$ *such that*

$$\eta(\psi_1,t)-t \ge 2(K_1+\varepsilon)(\eta(\psi_2,t)-t) \tag{23}$$

for all $t\ge t_0$, *then, for* $\sigma\ge 1$, *we have*

$$\|\hat{r}_\sigma\|_1 \le \frac{4}{\pi^2}\psi_1(\sigma)\left(\log^+(\eta(\psi_1,\sigma)-\sigma)+C\right). \tag{24}$$

Proof. Using the method from the proof of Theorem 2, we can derive

$$\|R_j\|_1 \le \frac{4}{\pi^2}\psi_j(\sigma)\left(\log^+(\eta(\psi_j,\sigma)-\sigma)+C\right), \quad j=1,2.$$

Under assumption (23), we apply the properties of the functions in F_0 and get

$$\psi_2(t) \le A_0 t^{-\varepsilon\delta}\psi_1(t) \quad \forall\, t\ge t_0,$$

where

$$\delta := \inf\left\{\frac{\tau}{\eta(\psi_1,\tau)-\tau}:\ \tau\ge 1\right\} > 0, \quad A_0 = \frac{\psi_2(t_0)}{\psi_1(t_0)}t_0^{\varepsilon\delta} > 0.$$

Thus, we have

$$\psi_2(\sigma)\log^+(\eta(\psi_2,\sigma)-\sigma) \le A_0\sigma^{-\varepsilon\delta}\psi_1(\sigma)\log\left(\frac{\eta(\psi_2,\sigma)-\sigma}{\sigma}\sigma\right) \le$$

$$\le C\psi_1(\sigma), \quad \sigma\ge t_0.$$

Theorem 3 is proved.

3.2. The case where $\psi_1\in\mathcal{U}_0$ *and* $\psi_2\in\mathcal{U}_0$. We recall that $\psi_j\in\mathcal{U}_0$ means that

$$\sup\{\mu(\psi_j,t):\ t\ge 1\} = M_j < \infty,$$

$$\mu(\psi_j,t) = \frac{t}{\psi_j^{-1}\left(\frac{1}{2}\psi_j(t)\right)-t}, \quad j=1,2.$$

Furthermore, we know that [5, p. 118]

$$\forall\,\sigma\ge 1 \quad \sigma|\psi_j'(\sigma)| \le (1+M_j)\psi_j(\sigma).$$

Theorem 4. *If* $\psi_1,\psi_2\in\mathcal{U}_0$, *then, for* $\sigma\ge 1$,

$$\|\hat{r}_\sigma\|_1 \le |\psi(\sigma)|\left(\frac{4}{\pi^2}\log\sigma + C\right) + \frac{2}{\pi}\int_\sigma^\infty \frac{\psi_2(t)}{t}\,dt.$$

Proof. Following the technique of [1], we first prove that, for $t \in \left(0, \frac{\pi}{2\sigma}\right)$,

$$\frac{1}{\pi}\int_\sigma^\infty \psi_2(s)\sin st\,ds \ge \psi_2\left(\frac{\pi}{2t}\right)\frac{\cos(\sigma t)}{\pi t} \ge 0. \tag{25}$$

Indeed, for $t \in \left(0, \frac{\pi}{2\sigma}\right)$, we have

$$\frac{1}{\pi}\int_\sigma^\infty \psi_2(s)\sin st\,ds =$$

$$= \frac{1}{\pi t}\left(\psi_2(\sigma)\cos(\sigma t) + \frac{1}{t}\left(\int_{\sigma t}^{\pi/2} + \int_{\pi/2}^\infty\right)\psi_2'\left(\frac{s}{t}\right)\cos s\,ds\right) \ge$$

$$\ge \frac{1}{\pi t}\left(\psi_2(\sigma)\cos(\sigma t) + \frac{1}{t}\int_{\sigma t}^{\pi/2}\psi_2'\left(\frac{s}{t}\right)\cos s\,ds\right) \ge$$

$$\ge \frac{1}{\pi t}\left(\psi_2(\sigma)\cos(\sigma t) + \cos(\sigma t)\frac{1}{t}\int_{\sigma t}^{\pi/2}\psi_2'\left(\frac{s}{t}\right)ds\right) = \psi_2\left(\frac{\pi}{2t}\right)\frac{\cos(\sigma t)}{\pi t}.$$

In order to estimate $\|\hat{r}_\sigma\|$, we start from (18).

It now follows from (18) and (25) that

$$\int_{|t|<\pi/2\sigma}|\hat{r}_\sigma(t)|\,dt \le \frac{\pi}{\sigma}|\psi(\sigma)| + \int_{|t|<\pi/2\sigma}\frac{1}{\pi}\left|\int_\sigma^\infty \psi_1(s)\cos(st)\,ds\right|dt +$$

$$+ \frac{2}{\pi}\int_\sigma^\infty \psi_2(s)\left(\int_0^{\pi/2\sigma}\sin(st)\,dt\right)ds =$$

$$= \frac{2}{\pi}\int_\sigma^\infty \frac{\psi_2(s)}{s}\,ds - \frac{2}{\pi}\int_\sigma^\infty \psi_2(s)\frac{\cos\left(s\frac{\pi}{2\sigma}\right)}{s}\,ds +$$

$$+ \frac{\pi}{\sigma}|\psi(\sigma)| + \int_{|t|<\pi/2\sigma}\frac{1}{\pi}\left|\int_\sigma^\infty \psi_1(s)\cos(st)\,ds\right|dt.$$

It is obvious that

$$\int_\sigma^\infty \psi_2(s)\frac{\cos\left(s\frac{\pi}{2\sigma}\right)}{s}\,ds = O(\psi_2(\sigma)).$$

Integrating by parts, we get

$$\int_\sigma^\infty \psi_1(s)\cos(st)\,ds = -\frac{1}{t}\psi_1(\sigma)\sin(\sigma t) - \frac{1}{t}\int_\sigma^\infty \psi_1'(s)\sin(st)\,ds.$$

Since $-\psi_1'$ is decreasing and positive, for $t > 0$ we have

$$-\int_0^\sigma \psi_1'(\sigma)\sin(st)\,ds - \int_\sigma^\infty \psi_1'(s)\sin(st)\,ds \ge 0.$$

Hence,

$$\int_\sigma^\infty \psi_1'(s)\sin(st)\,ds \le -\frac{1-\cos(\sigma t)}{t}\psi_1'(\sigma).$$

Let

$$E = \left\{t\in(0,\infty):\ \int_\sigma^\infty \psi_1'(s)\sin(st)\,ds > 0\right\}.$$

We see that

$$\int_0^\infty \left|\frac{1}{t}\int_\sigma^\infty \psi_1'(s)\sin(st)\,ds\right| dt =$$

$$= -\int_0^\infty \frac{1}{t}\int_\sigma^\infty \psi_1'(s)\sin(st)\,ds\,dt + 2\int_E \frac{1}{t}\left(\int_\sigma^\infty \sin(st)\,ds\right)dt \le$$

$$\le 2\int_E |\psi_1'(\sigma)|\frac{1-\cos(\sigma t)}{t^2}\,dt + \int_\sigma^\infty (-\psi_1'(s))\int_0^\infty \frac{\sin(st)}{t}\,dt\,ds.$$

Noting that

$$\sigma|\psi_1'(\sigma)| \le (1+M_1)\psi_1(\sigma), \qquad \int_0^\infty \frac{\sin(st)}{t}\,dt = \int_0^\infty \frac{\sin t}{t}\,dt \in R,$$

we get

$$\int_0^\infty \frac{1}{t}\left|\int_0^\infty \psi_1'(s)\sin(st)\,ds\right| dt \le C\psi_1(\sigma).$$

Then

$$\int_{|t|<\pi/2\sigma}\left|\int_\sigma^\infty \psi_1(s)\cos(st)\,ds\right| dt = O(\psi_1(\sigma)).$$

Therefore,

$$\int_{|t|<\pi/2\sigma} |\hat r_\sigma(t)|\,dt \le C|\psi(\sigma)| + \frac{2}{\pi}\int_\sigma^\infty \frac{\psi_2(s)}{s}\,ds.$$

By (11′) and (12′), we have

$$\int_{\pi/2\sigma<|t|} |\hat r_\sigma(t)|\,dt \le \int_{\pi/2\sigma<|t|<\pi/2} \frac{|\psi(\sigma)|}{\pi t^2}\left|2\sin\frac{t}{2}\sin\left(\left(\sigma-\frac{1}{2}\right)t-\theta\right)\right| dt +$$

$$+ |\psi(\sigma)| + \int_{\pi/2\sigma<|t|}\left|\frac{1}{\pi t}\int_\sigma^\infty (-\psi_1'(s)\sin(st)+\psi_2'(s)\cos(st))\,ds\right| dt.$$

It is easy to see that

$$\int_{\pi/2\sigma<|t|<\pi/2} \frac{1}{\pi t^2}\left|2\sin\frac{t}{2}\sin\left(\left(\sigma-\frac{1}{2}\right)t-\theta\right)\right| dt \le \frac{4}{\pi^2}(\log\sigma + C).$$

By virtue of the monotonicity of ψ_1', $j = 1, 2$, and periodicity of trigonometric functions, we have

$$\left|\int_{\sigma}^{\infty} \psi_1'(s)\sin(st)\,ds\right| \le \int_{\sigma}^{\sigma+2\pi/|t|} |\psi_1'(\sigma)|\,ds = \frac{2\pi}{|t|}|\psi_1'(\sigma)|,$$

$$\left|\int_{\sigma}^{\infty} \psi_2'(s)\cos(st)\,ds\right| \le \int_{\sigma}^{\sigma+2\pi/|t|} |\psi_2'(\sigma)|\,ds = \frac{2\pi}{|t|}|\psi_2'(\sigma)|.$$

Then

$$\int_{|t|>\pi/2\sigma} \frac{1}{|t|}\left|\int_{\sigma}^{\infty}(-\psi_1'(s)\sin(st)+\psi_2'(s)\cos(st))\,ds\right| dt \le$$

$$\le C(|\psi_1'(\sigma)|+|\psi_2'(\sigma)|)\,\sigma \le C|\psi(\sigma)|.$$

Combining the above estimates, we complete the proof of Theorem 4.

1. *Stepanets A. I.* Approximation in spaces of locally integrable functions // Ukr. Math. J. – 1994. – **46**, № 5. – P. 638–670.
2. *Степанец А. И.* Классификация периодических функций и скорость сходимости их рядов Фурье // Изв. АН СССР. Сер. мат. – 1986. – **50**, № 1. – С. 101–136.
3. *Степанец А. И.* К неравенству Лебега на классах (ψ, β)-дифференцируемых функций // Укр. мат. журн. – 1989. – **41**, № 5. – С. 449–510.
4. *Степанец А. И.* Уклонения сумм Фурье на классах целых функций // Там же. – № 6. – С. 783–789.
5. *Stepanets A. I.* Classification and approximation of periodic functions. – London: Kluwer, 1995. – 360 p. (translation from Russian).
6. *Степанец А. И.* Скорость сходимости рядов Фурье на классах $\overline{\psi}$-интегралов // Укр. мат. журн. – 1997. – **49**, № 8. – С. 1069–1114.
7. *Степанец А. И.* Приближение $\overline{\psi}$-интегралов периодических функций суммами Фурье (небольшая гладкость) // Там же. – 1998. – **50**, № 2. – С. 274–291.
8. *Ахиезер Н. И.* Леции по теории аппроксимации. – М.: Наука, 1965. – 537 с.

Received 28.09.98