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GENERALIZED HORSESHOES AND INDECOMPOSABILITY FOR ONE-DIMENSIONAL CONTINUA*

УЗАГАЛЬНЕНІ ПІДКОВИ ТА НЕРОЗКЛАДНІСТЬ ОДНОВИМІРНИХ КОНТИНУУМІВ

We consider dynamical systems given by a sequence of continuous maps of graphs. We obtain the results which generalize the known results concerning the existence of indecomposable subcontinua in terms of the corresponding maps of one-dimensional continua.

Розглядаються динамічні системи, що задаються послідовністю неперервних відображень графів. Одержано результати, які узагальнюють відомі результати стосовно існування нерозкладних підконтинуумів через відповідні відображення одновимірних континуумів.

1. Introduction. In the recent yers, there is a growing interest in investigating the connection between the dynamics of a continuous map of a graph and the topological structure of the inverse limit space using the map as a sole bonding map, as some attractor of a dynamical system can be shown to be the inverse limit space of a continuous map of a graph [1], see, e.g., [2 – 6]. In [3] Barge and Martin proved that if a continuous map of a closed interval has positive topological entropy then the inverse limit space using f as a sole bonding map contains an indecomposable subcontinuum. This result has been generalized to a continuous map of a graph by Barge and Diamond [2] and to a continuous map of a hereditarily decomposable chainable continuum by the author [5].

A finite graph or a graph is understood as a connected compact one-dimensional branched manifold. As we know, generally a one-dimensional continuum is the inverse limit space using a sequence of continuous maps of graphs as bonding maps [7]. Hence, it is interesting if we could describe the topological structure of the inverse limit space using a sequence of continuous maps as bonding maps from the information on the dynamics of the sequence and vice versa. The purpose of this paper is to try to understand some of the connections.

Let G_i be a graph and $f_i: G_{i+1} \to G_i$ be continuous for each $i \in \mathbb{N}$. We introduce the notion of the generalized horseshoe for $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ and show that if $f_{1,\infty}$ has a generalized horseshoe then the inverse limit space using $f_{1,\infty}$ as bonding maps contains an indecomposable subcontinuum. We also show that if $\{G_i : i \in \mathbb{N}\}$ is finite then the converse of the above statement is true. These results extend some known results concerning the existence of indecomposable subcontinua through the dynamics of the bonding maps for one-dimensional continua, for example, the above mentioned result of Barge and Diamond and the result in [6].

We apply the above result to show the existence of indecomposable subcontinua for certain continua. We also give a simple proof of the fact that if G is a graph and $f: G \to G$ is topologically mixing, then the inverse limit space using f as the sole bonding map is indecomposable, and then we use this fact to determine the inverse limit space of a transitive map of a graph.

The text is organized as follows: in the next section, we introduce necessary notion and preliminary results; in Section 3, we show that the existence of a generalized horseshoe implies the existence of an indecomposable subcontinuum in the inverse limit space and, in Section 4, we discuss the converse of the main result proved in Section 3. Finally, in Section 5, we give some application of the results obtained in the previous sections and give a simple proof of some known result and determine the

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inverse limit space of a transitive map of a graph.

2. Preliminary. Let X_i be a compact metric space and $f_i\colon X_{i+1}\to X_i$ be a continuous map for each $i\in\mathbb{N}$. Denote $\{f_i\}_{i=1}^{\infty}$ by $f_{1,\infty}$, and let

$$X_\infty = \lim\{X_i, f_i\}_{i=1}^{\infty} = \{(x_1, x_2, \dots): f_i(x_{i+1}) = x_i, \text{ for each } i\ge 1\}.$$

The space X_∞ is called the *inverse limit space* of $f_{1,\infty}$ and $f_{1,\infty}$ are called *bonding maps*. Note that X_∞ is a compact metric space as the subspace of the product space $\prod_{i=1}^{\infty} X_i$. If $X_i = X$ and $f_i = f$ for each $i\ge 1$, the space $\lim\{X, f\}$ is called *the inverse limit space* of f and f induces a homeomorphism $\hat{f}$ on $\lim\{X,f\}$ in the following way: $\hat{f}(x_1, x_2, \dots) = (f(x_1), x_1, x_2, \dots)$ for each $(x_1, x_2, \dots)\in \lim\{X,f\}$.

For each pair $i, j\in\mathbb{N}$ with $i+1\le j$, define $f_{i,j} = f_i\colon X_{i+1}\to X_i$ if $i+1=j$ and $f_{i,j} = f_i\circ f_{i+1}\circ\dots\circ f_{j-1}\colon X_j\to X_i$ if $i+1<j$. Note that $f_{i-1}\circ f_{i,j} = f_{i-1,j}$ for each $i\ge 2$ and $j\ge i+1$.

For each $i\in\mathbb{N}$, define $\pi_i\colon X_\infty\to X_i$ by $\pi_i(x) = x_i$ for each $(x_1, x_2, \dots)\in X_\infty$. π_i is an open continuous map and satisfies the relation $f_i\circ\pi_{i+1} = \pi_i$ for each $i\ge 1$.

A *continuum* is understoodas a non-empty connected compact metric space. A continuum is said to be *decomposable* (resp. *indecomposable*) if it can (resp. cannot) be written as the union of its two proper subcontinua. We say that a continuum is *hereditarily decomposable* (resp. *hereditarily indecomposable*) if each its nondegenerate subcontinuum is decomposable (resp. indecomposable). Let I be a closed interval. A continuum M is *chainable* if M is the inverse limit of $\{f_i\}_{i=1}^{\infty}$, where $f_i\colon I\to I$ is continuous for each $i\ge 1$. For the basic properties of chainable continua and one-dimensional continua, we refer [7, 8]. Let M be a hereditarily decomposable chainable continuum. Then there is a continuous map g from M onto $[0, 1]$ such that, for each $t\in[0, 1]$, $g^{-1}(t)$ is a maximal nowhere dense subcontinuum of M. The map g is called a *Kuratowski function* for M. $g^{-1}(0)$ and $g^{-1}(1)$ are called *end layers* of M.

Let G be a graph. We assume that each subset of G which is homeomorphic to S^1 contains at least 2 vertices. Thus, each edge of G is homeomorphic to $[0, 1]$. Let $f\colon G\to G$ be a continuous map of a graph G. If there are two closed non-degenerate intervals J_1 and J_2 contained in some edge of G with at most one common point such that $f(J_1)\cap f(J_2)\supset J_1\cup J_2$, then we say that f has a *horseshoe*. Not let G_i be a graph and let $f_i\colon G_{i+1}\to G_i$ be continuous for each $i\in\mathbb{N}$. We say that $f_{1,\infty}$ has a *generalized horseshoe* if there is a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers, edge $E_{n_i}\subset G_{n_i}$ and nondegenerate closed intervals $I(n_i, 1)$, $I(n_i, 2)$ of E_{n_i} with at most one common point such that

$$I(n_i, 1)\cup I(n_i, 2)\subset f_{n_i, n_{i+1}}(I(n_i, 1))\cap f_{n_i, n_{i+1}}(I(n_i, 2))$$

for each $i\ge 1$. If the above $I(n_i, 1)$ and $I(n_i, 2)$ are disjoint for each $i\ge 1$, then we say that $f_{1,\infty}$ has a *strongly generalized horseshoe*. It is easy to see that the following statement is true:

Remark 1. *Let G_i be a graph and let $f_i\colon G_{i+1}\to G_i$ be continuous for each $i\in\mathbb{N}$. Then $f_{1,\infty}$ has a generalized horseshoe if and only if it has a strongly generalized horseshoe.*

3. Generalized horseshoe → indecomposable subcontinuum. In this section, we show that if $f_{1,\infty}$ has a generalized horseshoe, then the inverse limit space using $\{f_i\}$ as bonding maps contains an indecomposable subcontinuum, where G_i is a graph and $f_i\colon G_{i+1}\to G_i$ is continuous for each $i\in\mathbb{N}$.

The following statements are basic facts in continuum theory [8]. As we shall use them several times, we represent them as lemmas here.

Lemma 1. *Let X_i be a compact metric space and let $f_i\colon X_{i+1}\to X_i$ be continuous for each $i\in\mathbb{N}$. Then* $\lim\{X_i, f_i\}$ *is homeomorphic to* $\lim\{X_{n_i}, f_{n_i,n_{i+1}}\}$ *for each strictly increasing sequence $\{n_i\}$ of positive integers.*

Lemma 2. *Let $X=\lim\{X_i, f_i\}$ with X_i being compact metric space and let $f_i\colon X_{i+1}\to X_i$ being cintinuous for each $i\in\mathbb{N}$. If A, B are two closed subsets of X then $A\cap B=\lim\{C_i, f_i|C_{i+1}\}$, where $C_i=\pi_i(A)\cap\pi_i(B)$ for each $i\in\mathbb{N}$.*

Lemma 4 is one of the key steps to the proof of the principal results of the section, namely, Theorem 1. To show it, we need the following statement:

Lemma 3. *Let I_i be a closed interval, let $f_i\colon I_{i+1}\to I_i$ be continuous for each $i\in\mathbb{N}$, and let $X=\lim\{I_i, f_i\}$ be a hereditarily decomposable chainable continuum. Assume that A is an end layer of X. Then there exists an $i_0\in\mathbb{N}$ such that $\pi_i(A)$ contains one and only ony end point of I_i for each $i\ge i_0$. Furthermore, if A is a subcontinuum of X and $j_0\in\mathbb{N}$ such that $\pi_i(A)$ contains an end point of I_i for $i\ge j_0$, then A intersects the union of the end layers of X.*

Proof. Let $g\colon X\to[0,1]$ be the Kuratowski function for X. Assume that $A = g^{-1}(0)$.

Let $s_n\in(0,1]$ with $s_n>s_{n+1}$ for each $i\in\mathbb{N}$ and $\lim\limits_{n\to\infty} s_n=0$.

Denote $A_n=g^{-1}([0,s_n])$ and $B_n=g^{-1}([s_n,1])$. Then $X=A_n\cup B_n$, $i\in\mathbb{N}$. Hence, $\pi_i(A_n)\cup\pi_i(B_n)=I_i$, and, consequently, $\pi_i(A_n)$ contains an end point of I_i for each $i,n\in\mathbb{N}$.

Let $I_i=[a_i,b_i]$, $i\in\mathbb{N}$. For fixed $i\in\mathbb{N}$, assume that $t_n\in\pi_i(A_n)$ with $t_n\in\{a_i, b_i\}$, $n\in\mathbb{N}$. Then there exists strictly increasing sequence n_j of $\mathbb{N}$ such that $t_{n_j}\in\pi_i(A_{n_j})$ and $t_{n_j}=t$ for each $j\in\mathbb{N}$.

For each $k\in\mathbb{N}$, as $A_k\supset A_{n_{j_0}}$ for some $j_0\in\mathbb{N}$, we have $t\in\pi_i(A_{n_{j_0}})\subset\pi_i(A_k)$. Hence, $t\in\pi_i(A_k)$, $k\in\mathbb{N}$. That is, $\pi_i(A)$ contains one end point of I_i because $A = \bigcap_{k=1}^{\infty} A_k$.

Since A is a proper subcontinuum of X, we know that there is $i_0\in\mathbb{N}$ such that $\pi_i(A)$ contains one and only one end point of I_i for $i\ge i_0$.

We now assume that A is a subcontinuum of X such that $\pi_i(A)$ contains an end point of I_i for $i\ge j_0$ and we show that A intersects the union of the end layers of X.

Otherwise, there are $0<c<d<1$ such that $A\subset B=g^{-1}([c,d])$. We claim that there is $n_0\in\mathbb{N}$ such that $\pi_n(B)$ does not contain end point of I_n for $n\ge n_0$. In fact, if $\pi_n(B)$ contains end point of I_n for infinitely many of $n\in\mathbb{N}$, then $\pi_n(B)\cap(\pi_n(g^{-1}(0))\cup\pi_n(g^{-1}(1)))\ne\varnothing$ for infinitely many $n\in\mathbb{N}$ (by the first part of the lemma), and, consequently, $B \cap (g^{-1}(0) \cup g^{-1}(1)) \neq \varnothing$, contradicting the fact that $B = g^{-1}([c, d])$. Hence, for some $n_0 \in \mathbb{N}$, $\pi_n(B)$ does not contain end point of I_n for $n \geq n_0$. This contradicts the assumption on A for $A \subset B$.

Lemma 4. *Let I_i be a closed interval, let $f_i : I_{i+1} \to I_i$ be continuous for each $i \in \mathbb{N}$, and let $X = \lim\{I_i, f_i\}$ be a hereditarily decomposable chainable continuum. If A and B are two subcontinua of X such that $\pi_i(A) \cap \pi_i(B) = \varnothing$ and $\pi_i(A) \cup \pi_i(B)$ contains the end points of I_i for each $i \in \mathbb{N}$, then there are two nonempty proper subcontinua C, D of X and some $i_0 \in \mathbb{N}$ with $X = C \cup \cup D$ such that $\pi_i(A) \subset \pi_i(C)$, $\pi_i(B) \subset \pi_i(D)$, and $\pi_i(C) \cap \pi_i(D) \subset I_i \setminus (\pi_i(A) \cup \cup \pi_i(B))$ for each $i \geq i_0$.*

Proof. Let $g : X \to [0, 1]$ be a Kuratowski function for X. By Lemma 3 and the assumption, we may assume that $A \cap g^{-1}(0) \neq \varnothing$ and $B \cap g^{-1}(1) = \varnothing$. Since $\pi_i(A) \cap \pi_i(B) = \varnothing$, $i \in \mathbb{N}$, we have $A \cap B = \varnothing$. Hence, it is impossible that there are $0 \leq c_1 < c_2 \leq 1$ with $g(A) = [0, c_2]$ and $g(B) = [c_1, 1]$. If there is $0 < c < 1$ such that $g(A) = [0, c]$ and $g(B) = [c, 1]$ then $A \cap B = (\bigcap_{b<c} g^{-1}([0, b])) \cap \cap (\bigcap_{b>c} g^{-1}([b, 1])) \neq \varnothing$, since $g^{-1}(c)$ is a subcontinuum of X and $(\bigcap_{b<c} g^{-1}([0, b])) \cup (\bigcap_{b>c} g^{-1}([b, 1])) = g^{-1}(c)$. Hence, $g(A) = [0, c]$ and $g(B) = [d, 1]$ for some $0 \leq c < d \leq 1$. Let e satisfy $c < e < d$ and let $C = g^{-1}([0, e])$, $D = g^{-1}([e, 1])$. Then C, D are subcontinua which we need.

Finally, we state a simple technical lemma which can be checked easily.

Lemma 5. *Let G be a graph and let E be an edge of G. Assume that I_1, I_2 are two non-degenerate closed intervals of E with at most one common point and A_1, A_2 are two proper subcontinua of G satisfying that $A_1 \cup A_2 = G$ and A_i does not contain I_j for $i, j = 1, 2$. Then there is $i_0 \in \{1, 2\}$ with $A_{i_0} \subset E$ and $A_1 \cap A_2$ has two connected components contained in I_1 and I_2, respectively.*

With the above preparation, we are ready to show the principal result of the section. Note that if X is a topological space and $A \subset X$, then we use $\mathrm{cl}(A)$ to denote the closure of A. We remark that some special case of Theorem 1 can be proved by using the notion "*indecomposable inverse sequence*" introduced in [8, p. 20].

Theorem 1. *Let G_i be a graph and let $f_i : G_{i+1} \to G_i$ be continuous for each $i \in \mathbb{N}$. Assume that $M = \lim\{G_i, f_i\}$ has a generalized horseshoe. Then M contains an indecomposable subcontinuum.*

Proof. Without loss of generality (by Lemma 1), we may assume that there are two closed intervals $I(i, 1)$, $I(i, 2)$ contained in some edge E_i of G_i with at most one common point satisfying the relations

$$I(i, 1) \cup I(i, 2) \subset f_i(I(i+1, 1)) \cap f_i(I(i+1, 2))$$

for each $i \geq 1$.

Let

$$A_i = \bigcup_{j=i+1}^{\infty} f_{i,j}(I(j, 1) \cup I(j, 2))$$

for each $i \in \mathbb{N}$. It is checked that

1) $f_{i,j}(I(j, 1) \cup I(j, 2)) \subset f_{i,j+1}(I(j+1, 1) \cup I(j+1, 2))$;

2) A_i is a connected subset of G_i;

3) $f_i(A_{i+1}) = A_i$, consequently $f_i(\mathrm{cl}(A_{i+1})) = \mathrm{cl}(A_i)$.

Let $M_1 = \lim\{\mathrm{cl}(A_i), f_i \,|\, \mathrm{cl}(A_{i+1})\}$. We show that M_1 contains an indecomposable subcontinuum.

Assume the contrary, that is, M_1 is hereditarily decomposable. Let $M_1 = A \cup B$ with A, B being nonempty proper subcontinua of M_1. If there are a strictly increasing sequence $\{n_i\}$ of positive integers and a sequence $\{j(i)\}$ of $\{1, 2\}$ such that either $\pi_{n_i}(A) \supset I(n_i, j(i))$ or $\pi_{n_i}(B) \supset I(n_i, j(i))$, then either $A = M_1$ or $B = M_1$, contradicting the assumption on A and B. Hence, we only have the following situation: there is $i_0 \in \mathbb{N}$ and some edge E_i of G_i such that $\pi_i(A)$ or $\pi_i(B) \subset E_i$ and $\pi_i(A) \cap \pi_i(B)$ has two connected components which are contained in $I(i, 1)$ and $I(i, 2)$, respectively, for each $i \geq i_0$ by Lemma 5.

By Lemma 1 and Lemma 5, we may assume that A is a chainable continuum with $\pi_i(A) \subset E_i$ for each $i \geq i_0$. It is easy to see (Lemma 2) that $A \cap B$ is the union of two disjoint subcontinua A_1, B_1 of A such that $\pi_i(A_1) \cup \pi_i(B_1)$ contains end points of $\pi_i(A)$. According to Lemma 4, there are some $i_1 \geq i_0$ and two nonempty proper subcontinua C, D of A with $A = C \cup D$ such that $\pi_i(A_1) \subset \pi_i(C)$, $\pi_i(B_1) \subset \pi_i(D)$, and $\pi_i(C) \cap \pi_i(D) \subset \pi_i(A) \setminus (\pi_i(A_1) \cup \pi_i(B_1))$ for each $i \geq i_1$.

Without loss of generality, we assume that $\pi_i(B \cup C) \supset I(i, 1)$ for infinitely many i. Let $A' = B \cup C$ and $B' = D$. On the one hand, A', B' are two proper subcontinua of M_1 with $A' \cup B' = B \cup C \cup D = B \cup A = M_1$ as $A' \cap B' = (B \cap D) \cup (C \cap D) \neq A'$ or B'. On the other hand, as $\pi_i(A') \supset I(i, 1)$ for infinitely many i, we have that $A' = = M_1$ which is a contradiction. Hence, M_1 contains an indecomposable subcontinuum, that is, X contains an indecomposable subcontinuum.

4. Indecomposable subcontinuum → generalized horseshoe. Let G_i be a graph and let $f_i\colon G_{i+1} \to G_i$ be continuous for each $i \in \mathbb{N}$. In this section, we show that if $M = \lim\{G_i, f_i\}$ contains an indecomposable subcontinuum and $\{G_i : i \in \mathbb{N}\}$ is finite, then $\{f_i\}$ has a strongly generalized horseshoe. Since $M = \lim\{\pi_i(M), f|\pi_{i+1}(M)\}$ (Lemma 2), we assume that $f_i\colon G_{i+1} \to G_i$ is surjective throughout this section. To start with the following lemma, we need some notion. Let X be a continuum and let $p, q \in X$. We say that X is *irreducible between p and q* if there is no proper subcontinuum of X containing both p and q.

Lemma 6. *Let G_i be a graph and let $f_i\colon G_{i+1} \to G_i$ be continuous for each $i \in \mathbb{N}$. Assume that $M = \lim\{G_i, f_i\}$ is irreducible between $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots)$ and J_i is a subcontinua of G_i such that $\{a_i, b_i\} \subset J_i$, $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, $\lim\limits_{j\to\infty} f_{i,j}(J_j) = G_i$.*

Proof. Assume that there is $i_0 \in \mathbb{N}$ such that $\lim f_{i_0,j}(J_j) = G_{i_0}$ does not hold. That is, there is a strictly increasing subsequence $\{n_i\}$ of positive integers such that

$$\lim f_{i_0,n_j}(J_{n_j}) = G'_{i_0} \neq G_{i_0}.$$

It is easy to see that $\{a_{i_0}, b_{i_0}\} \subset G'_{i_0}$. Choose a subsequence m_j of n_j such that

$$\lim f_{i_0+1,m_j}(J_{m_j}) = G'_{i_0+1},$$

then $G'_{i_0} = \lim f_{i_0, m_j}(J_{m_j}) = f_{i_0}(\lim f_{i+1, m_j}(J_{m_j})) = f_{i_0}(G'_{i_0+1})$, $\{a_{i_0+1}, b_{i_0+1}\} \subset G'_{i_0+1}$.

Continuing in this way, we get a sequence of subcontinua $\{G'_i \subset G_i : i \geq i_0\}$ with $f_j(G'_{j+1}) = G'_j$ and $j \geq i_0$.

Let $G'_{i_0-1} = f_{i_0-1}(G'_{i_0}), \ldots, G'_1 = f_1(G'_2)$. Then $\lim \{G'_j, f|G'_{j+1}\}$ is a proper subcontinuum of M with $\{a_j, b_j\} \subset G'_j$ for each $j \in \mathbb{N}$ which is a contradiction.

We use d_H to denote the Hausdorff metric induced by the metric d of some compact metric space.

Lemma 7. *Let G_i be a graph with metric d_i, let $f_i : G_{i+1} \to G_i$ be continuous for each $i \in \mathbb{N}$, and let $M = \lim \{G_i, f_i\}$ be indecomposable. If there are strictly increasing sequence $\{n_i\}$ of positive integers, edge $E_{n_i} \subset G_{n_i}$, and 12 points $x^1, \ldots, x^{12} \in M$ with $\pi_{n_i}(x^j) \subset E_{n_i}$ for $i \in \mathbb{N}$, $1 \leq j \leq 12$, and if M is irreducible between each pair from the 12 points, then there is a strongly generalized horseshoe.*

Proof. Without loss of generality, we may assume that $x^1_{n_i} < \ldots < x^{12}_{n_i}$ for some fixed orientation of E_{n_i}, $i \in \mathbb{N}$ (otherwise, we can take a subsequence of $\{n_i\}$ and relabel x^i, $1 \leq i \leq 12$).

Let $I^j_{n_i} = [x^{2j-1}_{n_i}, x^{2j}_{n_i}]$ for $1 \leq j \leq 6$ and $i \in \mathbb{N}$. Let $A_{n_i} = \{ I^{2j-1}_{n_i} \cup I^{2j}_{n_i} : 1 \leq j \leq 3\}$ and $j_1 = 1$.

By Lemma 6, for fixed $1 \leq j \leq 6$, $\lim\limits_{i \to \infty} f_{n_{j_1}, n_i}(I^j_{n_i}) = G_{n_{j_1}}$. Hence, there exists $j_2 \in \mathbb{N}$ such that

$$(d_{n_{j_1}})_H(f_{n_{j_1}, n_{j_2}}(I^j_{n_{j_2}}), G_{n_{j_1}}) < \min\{d_{n_{j_1}}(x^{2i}_{n_{j_1}}, x^{2i+1}_{n_{j_1}}) : i = 1, \ldots, 5\}$$

for each $1 \leq j \leq 6$. This implies that, for each $1 \leq j \leq 3$, $f_{n_{j_1}, n_{j_2}}(I^{2j-1}_{n_{j_2}}) \cap f_{n_{j_1}, n_{j_2}}(I^{2j}_{n_{j_2}})$ contains at least one element of $A_{n_{j_1}}$.

We now assume that we have obtained j_q for $1 \leq q \leq l$ with the property that, for $1 \leq j \leq 6$ and $1 \leq k \leq l-1$, $f_{n_{j_k}, n_{j_{k+1}}}(I^{2j-1}_{n_{j_{k+1}}}) \cap f_{n_{j_k}, n_{j_{k+1}}}(I^{2j}_{n_{j_{k+1}}})$ contains at least one element of $A_{n_{j_k}}$. Replacing j_1 by j_l in the above process, we may get j_{l+1} with $f_{n_{j_l}, n_{j_{l+1}}}(I^{2j-1}_{n_{j_{l+1}}}) \cap f_{n_{j_l}, n_{j_{l+1}}}(I^{2j}_{n_{j_{l+1}}})$ containing at least one element of $A_{n_{j_l}}$ for each $1 \leq j \leq 3$. This implies that there is a sequence $\{k_j\}$ of $\{1, 2, 3\}$ such that

$$f_{n_{j_l}, n_{j_{l+1}}}(I^{2k_{l+1}-1}_{n_{j_{l+1}}}) \cap f_{n_{j_l}, n_{j_{l+1}}}(I^{2k_{l+1}}_{n_{j_{l+1}}}) \supset I^{2k_l-1}_{n_{j_l}} \cup I^{2k_l}_{n_{j_l}}$$

for each $l \in \mathbb{N}$. Hence, there is a strongly generalized horseshoe for $\{f_i\}$.

Theorem 2. *Let G_i be a graph and let $f_i : G_{i+1} \to G_i$ be continuous for each $i \in \mathbb{N}$. Assume that $\{G_i : i \in \mathbb{N}\}$ is a finite set. Then there is strongly generalized horseshoe for $\{f_i\}$ if and only if $M = \lim \{G_i, f_i\}$ contains an indecomposable subcontinuum.*

Proof. The only if part of the theorem is a weak version of Theorem 1. Thus, we only need to show the if part of the theorem.

By the definition of strongly generalized horseshoe, we may assume that M is an indecomposable continuum. By Lemma 1, we can further assume that $M = \lim \{G_i, f_i\}$ with $G_i = G$ for each $i \in \mathbb{N}$.

Let e denote the number of edges of G. From the indecomposability of M [8, p. 204], we may take $12e$ points $\{y^1, y^2, \ldots, y^{12e}\}$ of M such that M is irreducible between each pair from the $12e$ points. Moreover, we may assume that $\pi_1(x^i) \neq \neq \pi_1(x^j)$, $1 \leq i < j \leq 12e$. Then, for each $i \in \mathbb{N}$, there is an edge E_i of G such that E_i contains at least 12 points from $\{\pi_i(y^j)\colon 1 \leq j \leq 12e\}$. Thus, there exist a strictly increasing sequence $\{n_i\}$ of positive integers and $\{x^1, x^2, \ldots, x^{12}\} \subset \{y^1, y^2, \ldots \ldots, y^{12e}\}$ such that $\pi_{n_i}(x^j) \subset E_{n_i}$ for each $i \in \mathbb{N}$ and $1 \leq j \leq 12$. Then Lemma 7 may be applied here and the theorem is proved.

5. An application. In this section, we give an application of Theorem 1. We need some notation. Let G be a graph and $C(G, G)$ denote the collection of all continuous maps of G. For $f, g \in C(G, G)$, let $d(f, g) = \sup\{|f(x) - g(x)|\colon x \in \in G\}$. d is a metric on $C(G, G)$. Recall that if $f \in C(G, G)$ and $s \geq 2$, then an *s-horseshoe for f* is an interval J contained in some edge of G and a partition $\mathcal{D}$ of J into s subintervals such that the closure of each element of $\mathcal{D}$ f— covers J (the image of the element contains J). A collection $J_1, J_2, \ldots, J_s$ of pairwise disjoint subintervals of some edge E which do not contain any endpoint of E is said to be a *strong s-horseshoe for f* if each of these intervals f— covers $\bigcup_{i=1}^{s} J_i$. We have the following theorem:

Theorem 3. *Let $f\colon G \to G$ be a continuous map of a graph G. If f has positive topological entropy, then there is a neighbourhood U of f in $C(G, G)$ such that $\lim\{G, f_i\}$ contains an indecomposable subcontinuum for any chosen $f_i \in U$, $i \geq 1$.*

To prove the theorem, we need a lemma from [9] (see also [10]).

Lemma 8. *Let $f \in C(G, G)$ have an s-horseshoe, $s \geq 4$. Then f has a strong $(s-2)$-horseshoe such that any map sufficiently close to f has the same strong $(s-2)$-horseshoe.*

***Proof of Theorem* 3.** By [10], there is an $n \in \mathbb{N}$ such that f^n has an s-horseshoe for $s \geq 4$. Hence, by Lemma 8, there is a neighbourhood U' of f^n such that any map of U' has a same strong $(s-2)$-horseshoe. Now let U be a neighbourhood of f such that, for any $g_1, g_2, \ldots, g_n \in U$, we have $g_1 \circ g_2 \circ \ldots \circ g_n \in \in U'$.

Let $\{f_i\}_{k=1}^{\infty}$ be a sequence chosen from U. Thus, for the sequence $\{n_k\}$ with $n_k = kn + 1$, $k \in \{0\} \cup \mathbb{N}$, $\{f_{n_k, n_{k+1}}\}_{k=1}^{\infty}$ has a strongly generalized horseshoe. This implies that $\lim\{G, f_i\}$ contains an indecomposable subcontinuum.

To end the paper, we determine the inverse limit space of a transitive map of a graph. First, we give a simple proof of the following Proposition [11]. We start with some notion. Let f be a continuous map of a topological space X. f is *topologically transitive* if, for each pair of non-empty open subsets U and V of X, there is $n = = n(U, V)$ such that $f^n(U) \cap V \neq \varnothing$ and f is *topologically mixing* if, for each pair of non-empty open subsets U and V of X, there is $n = n(U, V)$ such that $f^i(U) \cap V \neq \neq \varnothing$ for $i \geq n$. We note that if X is a metric space with metric d and $A \subset X$ then $\operatorname{diam}(A) = \sup\{d(x, y)\colon x, y \in A\}$.

Proposition 1. *Let f be a topologically mixing map of a graph G. Then the inverse limit space of $X = \lim\{G, f\}$ is indecomposable.*

Proof. Let $X = A \cup B$ with A, B being subcontinua of X. Let $A_n = \pi_n(A)$ and

$B_n = \pi_n(B)$, $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$\max\{\operatorname{diam}(A_n), \operatorname{diam}(B_n)\} \geq (1/2)\operatorname{diam}(G).$$

Without loss of generality, we assume that $\operatorname{diam}(A_{n_i}) \geq (1/2)\operatorname{diam}(G)$ for some strictly increasing sequence of positive integers and $A_{n_i} \to A \subset G$ in the Hausdorff metric. It is easy to see that $\operatorname{diam}(A) \geq (1/2)\operatorname{diam}(G)$. Hence, there exist $i_0 \in \mathbb{N}$ and a non-empty open subset U of G contained in A_{n_i} for $i \geq i_0$.

Let $x \in G$ and let V_x be a neighborhood of x. Then there exists $M \in \mathbb{N}$ such that $f^n(U) \cap V_x \neq \varnothing$ for $n \geq M$. Thus, $A_{n_i} \cap V_x \neq \varnothing$ for $i \geq i_0$ as $f^{n_j - n_i}(A_{n_j}) = A_{n_i}$ and $U \subset A_{n_j}$ for $i < j$. Since A_{n_i} is closed and connected, we conclude that $A_{n_i} = G$ for $i \geq i_0$. The fact that f is surjective implies that $A_n = G$ for each $n \in \mathbb{N}$. Hence, $A = X$. That is, X is indecomposable.

The following lemma was proved in [12]:

Lemma 9. *Let G be a graph and let $f\colon G \to G$ be continuous and transitive. Then*

a) *if $\operatorname{Per}(f) = \varnothing$ then G is homeomorphic to the unit circle and f is an irrational rotation of G;*

b) *if $\operatorname{Per}(f) \neq \varnothing$ then there are $n \in \mathbb{N}$ and subgraphs $G_1, G_2, \ldots, G_n$ of G such that*

1) *$G_i \cap G_j$ is a finite subset of G for $1 \leq i < j \leq n$;*

2) *$f(G_i) = G_{i+1(\mathrm{mod}\, n)}$ and $f^n | G_i$ is topologically mixing $1 \leq i \leq n$.*

We now are ready to prove

Theorem 4. *Let G be a graph and let $f\colon G \to G$ be continuous and transitive. Then*

a) *if $\operatorname{Per}(f) = \varnothing$ then $\lim\{G, f\}$ is homeomorphic to the unit circle;*

b) *if $\operatorname{Per}(f) \neq \varnothing$ then there is $n \in \mathbb{N}$ such that $\lim\{G, f\} = \bigcup_{i=1}^{n} K_i$, where each K_i is an indecomposable subcontinuum which is invariant under $\hat{f}^n$ and $K_i \cap K_j$ is a finite subset of $\lim\{G, f\}$ if $i \neq j$.*

Proof. a) follows from a) of Lemma 9.

Now we prove b). By b) of Lemma 9, there are $n \in \mathbb{N}$ and subgraphs $G_1, G_2, \ldots, G_n$ of G such that 1) $G_i \cap G_j$ is a finite subset of G for $1 \leq i < j \leq n$; 2) $f(G_i) = G_{i+1(\mathrm{mod}\, n)}$ and $f^n | G_i$ is topologically mixing $1 \leq i \leq n$.

For each $1 \leq i \leq n$, let $K_i = \lim\{A_i^j, f | A_i^{j+1}\}$ with $A_i^j = G_{i-j+1(\mathrm{mod}\, n)}$, $j \in \mathbb{N}$. It is checked that $\lim\{G, f\} = \bigcup_{i=1}^{n} K_i$ and each K_i is invariant under $\hat{f}^n$. The indecomposability of K_i follows from Proposition 1.

Let $A = \sup\{G_i \cap G_j \colon 1 \leq i < j \leq n\}$. Then A is a finite invariant subset of G. This implies that each point of A is an eventually periodic point of f, that is, for each point $x \in A$, there is $n_0 \in \mathbb{N}$ such that $f^{n_0}(x)$ is a periodic point of f. Let $1 \leq i < j \leq n$ and $(x_1, x_2, \ldots) \in K_i \cap K_j$. Then, for each $l \in \mathbb{N}$, $x_l \in (K_i \cap K_j) \subset A$. As $f(x_{l+1}) = x_l$, $l \in \mathbb{N}$, we get that $x_l \in A$ is a periodic point of f for each $i \in \mathbb{N}$. Thus, $K_i \cap K_j$ is a finite subset of $\lim\{G, f\}$ for each $i \neq j$.

1. *Williams R.* One-dimensional nonwandering sets // Topology. – 1967. – **6**. – P. 473–483.
2. *Barge M., Diamond B.* The dynamics of continuous maps on finite graphs through inverse limits // Trans. Amer. Math. Soc. – 1994. – **344**. – P. 773–790.
3. *Barge M., Martin J.* Chaos, periodicity, and snakelike continua // Proc. Amer. Math. Soc. – 1985. – **289**. – P. 355–365.
4. *Ye X.* The dynamics of homeomorphisms of hereditarily decomposable chainable continua // Topology Appl. – 1995. – **64**. – P. 85–93.
5. *Ye X.* Topological entropy of the induced maps of the inverse limits with bonding maps // Ibid. – 1995. – **67**. – P. 113–118.
6. *Ye X.* Dynamics of a sequence of continuous maps and indecomposability (Chinese) // Chin. Ann. Math. (to appear).
7. *Engelking.* Dimension theory. – New York: North-Holland, 1978. – 423 p.
8. *Nadler (Jr.) Sam B.* Continuum theory // Pure and Appl. Math. – 1992. – **158**.
9. *Kolyada S., Snoha L.* Topological entropy of nonautonomous dynamical systems // Random and Comput. Dynam. – 1996. – **4**. – P. 205–233.
10. *Llibre J., Misiurewicz M.* Horseshoe, entropy and periods for graph maps // Topology. – 1993. – **32**. – P. 649–664.
11. *Kato H.* Chaos of continuum-wise expansive homeomorphisms and dynamical properties of sensitive maps of graphs // Pacif. J. Math. (to appear).
12. *Blokh A. M.* On transitive transformations of one-dimensional branched manifolds // Differential-difference equations and problems in mathematical physics. – Kiev: Inst. Math. Ukr. Acad. Sci., 1984. – P. 3–9.
13. *Block L., Coopel W. A.* Dynamics is one dimension // Lec. Notes Math. – 1992. – **1513**. – 316 p.

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