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SPECTRAL THEORY OF SOME MATRIX DIFFERENTIAL OPERATORS OF MIXED ORDER

СПЕКТРАЛЬНА ТЕОРІЯ МАТРИЧНИХ ДИФЕРЕНЦІАЛЬНИХ ОПЕРАТОРІВ МІШАНОГО ПОРЯДКУ

We develop spectral and scattering theory for a class of self-adjoint matrix operators of a mixed order.

Розвивається спектральна теорія та теорія розсіяння для одного класу самоспряжених матричних диференціальних операторів мішаного порядку.

1. Introduction. Our purpose in this work is to study the spectral properties and the scattering theory of self-adjoint operator matrices of the form

$$H = \begin{pmatrix} -\Delta + q & -\nabla b + v \\ \bar{b}\,\mathrm{div} + v^* & h \end{pmatrix} \tag{1}$$

acting in the product of Hilbert spaces $\mathcal{H} = L_2(\mathbb{R}^n; \mathbb{C}^n) \times L_2(\mathbb{R}^n)$, $n > 1$. Here, $L_2(\mathbb{R}^n; \mathbb{C}^n)$ is L_2-space of vector functions $u: \mathbb{R}^n \to \mathbb{C}^n$, q is an $n \times n$ Hermitian matrix function, b is a complex function, v is an $n \times 1$ matrix function, h is a real function. In the recent paper [1] (see also [2]), an abstract approach was proposed to study the essential spectrum of 2×2 block operator matrices acting in a Banach space. In particular, this approach was proved to be very useful for dealing with the spectral theory of some mixed order differential operators occurring in magnetohydrodynamics [1, 3, 4]. On the other hand, the structure of the continuous spectrum of such matrix differential operators has not been investigated. Here, we give a sufficiently complete analysis of the self-adjoint operators of form (1). In particular, we prove (under certain assumptions) the limiting absorption principle and the absence of singular continuous spectrum. Moreover, we describe the possible points of the eigenvalue accumulation. In Section 2, we study the essential spectrum of some general self-adjoint operator matrices of the form

$$L = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. \tag{2}$$

In particular, we prove (cf. [2]) that, under some natural assumptions, $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$, where

$$H_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & h_0 \end{pmatrix}, \tag{3}$$

$$h_0 = h - |b|^2. \tag{4}$$

In Section 3, we present some variant of Mourre's commutator method, which is suitable for dealing with the operators of form (1). The limiting absorption principle for the operator H is proved in Section 4. In Section 5, we show that, for the pair H, H_0, there exist complete wave operators. Note that the case $b(x) \equiv 0$ was investigated in [5]. The results of the paper were partially announced in [6].

2. The essential spectrum of some matrix operators. First introduce some notations. Let T be a linear closed operator in a Hilbert space with the domain $D(T)$. Denote by $\rho(T)$ the resolvent set of T, $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is the spectrum of T. The set $\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda$ is not a Fredholm operator$\}$ is called the essential spectrum of T. If T is a self-adjoint operator, then $\sigma_{\text{ess}}(T) = \{\lambda \in \sigma(T) \mid \lambda$ is not an isolated eigenvalue of finite multiplicity$\}$. Denote by $\mathcal{L}(\mathcal{H}, \mathcal{K})$ $(S_\infty(\mathcal{H}, \mathcal{K}))$ the space of bounded (respectively compact) linear operators from $\mathcal{H}$ into $\mathcal{K}$, $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$, $S_\infty(\mathcal{H}) = S_\infty(\mathcal{H}, \mathcal{H})$.

Consider symmetric operator matrices of form (2) acting in the product of Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. Suppose that A and C are self-adjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let B be a densely defined closable operator from $\mathcal{H}_2$ into $\mathcal{H}_1$ such that $D(B)$ is a core of C. We assume that

$$D(B^*) \supset D(|A|^{1/2}). \tag{5}$$

Under these conditions, L is essentially self-adjoint on $D(A) \times D(B)$ [7, 8]. We preserve the notation L for the closure of operator (2). Define

$$S_0(\lambda) := B^*(A - \lambda)^{-1} B, \quad \lambda \in \rho(A).$$

By (5), the closure $S(\lambda) := \overline{S_0(\lambda)}$ is a bounded operator in $\mathcal{H}_2$. On $D(C)$, define

$$T(\lambda) = C - S(\lambda), \quad \lambda \in \rho(A). \tag{6}$$

Note that $\lambda \in \rho(T(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ [9].

Define

$$F(\lambda) := B^*(A - \lambda)^{-1} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2), \quad \lambda \in \rho(A).$$

Then, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$(L - \lambda)^{-1} = \begin{pmatrix} (A-\lambda)^{-1} + \left(F(\bar{\lambda})\right)^* (T(\lambda) - \lambda)^{-1} F(\lambda) & -\left(F(\bar{\lambda})\right)^* (T(\lambda) - \lambda)^{-1} \\ -(T(\lambda) - \lambda)^{-1} F(\lambda) & (T(\lambda) - \lambda)^{-1} \end{pmatrix} \tag{7}$$

(see [7–9]). Denote

$$L(\mu) = \begin{pmatrix} A & 0 \\ 0 & T(\mu) \end{pmatrix}, \quad \mu \in \rho(A).$$

Theorem 1. *Suppose that, for some (and, hence, for all)* $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$G(\lambda) := (C - \lambda)^{-1} B^*(A - \lambda)^{-1} \in S_\infty(\mathcal{H}_1, \mathcal{H}_2). \tag{8}$$

Then, for all $\lambda \in (\mathbb{C} \setminus \mathbb{R}) \cap \rho(T(\mu))$, *the difference of resolvents* $D(\lambda) := (L - \lambda)^{-1} - (L(\mu) - \lambda)^{-1} \in S_\infty(\mathcal{H})$. *In particular,* $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}((L(\mu)) = \sigma_{\text{ess}}(A) \cup \sigma_{\text{ess}}(T(\mu))$.

***Remark* 1.** It is clear (see (6)) that, for t large enough, $it \in \rho(T(\mu))$.

***Remark* 2.** Under the assumptions of Theorem 1, $\sigma_{\text{ess}}(T(\mu))$ does not depend on $\mu \in \rho(A)$ (see the proof).

***Remark* 3.** The essential spectrum of 2×2 block operator matrices acting in the product of Banach spaces was studied by F. V. Atkinson, H. Langer, R. Menicken and A. A. Shkalikov [1, 2]. Their compactness conditions are more restrictive then (8). It was assumed that A has a compact resolvent or $B^*(A - \mu)^{-1} \in S_\infty(\mathcal{H}_1, \mathcal{H}_2)$. On the other hand, they did not suppose (5). To weaken (5) and extend Theorem 1 to the case of Banach spaces is beyond the scope of this paper.

Proof. By (7),

$$D(\lambda) = \begin{pmatrix} F(\bar{\lambda})^* (T(\lambda) - \lambda)^{-1} F(\lambda) & F(\bar{\lambda})^* (T(\lambda) - \lambda)^{-1} \\ -(T(\lambda) - \lambda)^{-1} F(\lambda) & (T(\lambda) - \lambda)^{-1} - (T(\mu) - \lambda)^{-1} \end{pmatrix}.$$

Consider

$$(T(\lambda)-\lambda)^{-1}F(\lambda) = (C-\lambda)^{-1}F(\lambda) + \left((T(\lambda)-\lambda)^{-1}-(C-\lambda)^{-1}\right)F(\lambda) =$$
$$= G(\lambda) + (T(\lambda)-\lambda)^{-1}S(\lambda)G(\lambda) \in S_\infty(\mathcal{H}_1, \mathcal{H}_2). \tag{9}$$

Analogously, $\left(F(\bar{\lambda})\right)^*(T(\lambda)-\lambda)^{-1} \in S_\infty(\mathcal{H}_1, \mathcal{H}_2)$. Furthermore, by (9)

$$(T(\lambda)-\lambda)^{-1}-(T(\mu)-\lambda)^{-1} = (T(\lambda)-\lambda)^{-1}(T(\mu)-T(\lambda))(T(\mu)-\lambda)^{-1} =$$
$$= (\lambda-\mu)(T(\lambda)-\lambda)^{-1}F(\lambda)(F(\bar{\mu}))^*(T(\mu)-\lambda)^{-1} \in S_\infty(\mathcal{H}_2, \mathcal{H}_2).$$

Hence, $D(\lambda) \in S_\infty(\mathcal{H})$.

Let us apply Theorem 1 to operator (1). Denote by Q_j the operator of multiplication by the variable x_j acting in $L_2(\mathbb{R}^n)$. Let $Q = (Q_1, \dots, Q_n)$ be the position operator, $a(Q)$ be an operator of multiplication by a $s \times t$ matrix function a acting from $L_2(\mathbb{R}^n; \mathbb{C}^t)$ to $L_2(\mathbb{R}^n; \mathbb{C}^s)$ $(s, t \in \mathbb{N})$. For simplicity, we assume throughout the paper that (matrix) functions q, b, v are bounded over $\mathbb{R}^n$. Moreover, we assume that b is a Lipschitz function and h is continuous on $\mathbb{R}^n$. Denote $\mathcal{H}_1 = L_2(\mathbb{R}^n; \mathbb{C}^n)$, $\mathcal{H}_2 = L_2(\mathbb{R}^n)$. Now A is a vector Schrödinger operator in $\mathcal{H}_1$ defined by

$$A = -\Delta + q(Q), \qquad D(A) = D(-\Delta).$$

$B = -\nabla b(Q) + v(Q)$ with $D(B) = C_0^\infty\left(\mathbb{R}^n\right)$ and $C = h(Q)$. Clearly, (5) is satisfied and H is essentially self-adjoint on $\left(C_0^\infty\left(\mathbb{R}^n\right)\right)^n \times C_0^\infty\left(\mathbb{R}^n\right)$. Below, we denote by the same symbol $|\cdot|$ the norm in $\mathbb{R}^n \left(|x| = \left(x_1^2 + \ldots + x_n^2\right)^{1/2},\ x = (x_1, \dots, x_n)\right)$ and the norm of $s \times t$ (complex) matrices. Here, s, t are arbitrary positive integers.

Theorem 2. *Suppose that*

$$|q(x)| + \frac{|v(x)| + |b(x)|}{|h(x)| + 1} \to 0 \quad \text{as} \quad |x| \to \infty. \tag{10}$$

Then

$$(H+i)^{-1} - (H_0+i)^{-1} \in S_\infty(\mathcal{H}). \tag{11}$$

In particular, $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [0, \infty) \cup \overline{h_0\left(\mathbb{R}^n\right)}$.

Proof. Define the self-adjoint operator $\tilde{H}$ in $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ by

$$\tilde{H} = \begin{pmatrix} -\Delta & B \\ B^* & C \end{pmatrix}. \tag{12}$$

We first prove that

$$(H+i)^{-1} - \left(\tilde{H}+i\right)^{-1} \in S_\infty(\mathcal{H}).$$

As $D(H) = D\left(\tilde{H}\right)$ (q is bounded), it suffices to prove that

$$(H+i)^{-1}\left(H - \tilde{H}\right)(H+i)^{-1} \in S_\infty(\mathcal{H}).$$

As $D(H) \subset D\left((-\Delta)^{1/2}\right) \times D(h(Q))$ (see [1]), this follows from the condition

$$(-\Delta+1)^{-1/2} q(Q) (-\Delta+1)^{-1/2} \in S_\infty(\mathcal{H}_1), \tag{13}$$

which is a consequence of (10). We now apply Theorem 1 to the operator $\tilde{H}$. By (10) and (8),

$$\left(\tilde{H}+i\right)^{-1}-\left(L(-1)+i\right)^{-1} \in S_\infty(\mathcal{H}),$$

where $L(-1) = \operatorname{diag}\{-\Delta, T(-1)\}$, $T(-1) = C - B^*(-\Delta+1)^{-1}B$ (see (6)). To prove (11), it suffices to show that

$$(T(-1)+i)^{-1}-(h_0(Q)+i)^{-1} \in S_\infty(\mathcal{H}_2). \tag{14}$$

We have

$$T(-1) = h(Q)-\left(\bar{b}(Q)\operatorname{div}+v^*(Q)\right)(-\Delta+1)^{-1}(-\nabla b(Q)+v(Q)) =$$

$$= h_0(Q) + b(Q)(-\Delta+1)^{-1}b(Q) - v^*(Q)(-\Delta+1)^{-1}v(Q) -$$

$$- \bar{b}(Q)\operatorname{div}(-\Delta+1)^{-1}v(Q) + v^*(Q)(-\Delta+1)^{-1}\nabla b(Q). \tag{15}$$

As $D(T(-1)) = D(h(Q)) = D\left(h_0(Q)\right)$, (14) is equivalent to the condition

$$(h(Q)+i)^{-1}(T(-1)-h_0(Q))(h(Q)+i)^{-1} \in S_\infty(\mathcal{H}_2),$$

which follows from (10) and (15).

***Remark* 4.** Clearly, condition (10) can be weakened. We need only (8), (13) and (14).

3. Conjugate operator method. Here, we give a version of the abstract Mourre's method [10 – 12], which is suitable for the study of the operators of form (1).

Let M be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})$. We say that $T \in C^1(M)$ if the bounded operator function

$$T(t) := e^{itM}\, T e^{-itM} \tag{16}$$

is differentiable in the strong operator topology. $T \in C^1(M)$ if and only if the commutator $[T, M] := TM - MT$ (defined in the form sense) extends to a bounded operator in H. Analogously, one can define a class $C^k(M)$, $k \in \mathbb{N}$. We say that a self-adjoint operator $H \in C^k(M)$ if, for some (and, hence, for all) $z \in \rho(H)$, the resolvent $(H-z)^{-1} \in C^k(M)$. Throughout this section, we assume that $H \in C^1(M)$. We say that M is conjugate to H on an open set J if there is a compact operator K and $\alpha > 0$ such that

$$E_H(J)[H, iM]E_H(J) \geq \alpha E_H(J) + K. \tag{17}$$

Here, E_H is the spectral measure of H. If we have estimate (17) with $K = 0$, we say that M is strictly conjugate to H. We also need the following definition. An operator $T \in \mathcal{L}(\mathcal{H})$ is of class $C^{+0}(M)$ if

$$\int_0^1 \| T(t) - T \| t^{-1}\, dt < \infty. \tag{18}$$

Here, $T(t)$ is defined by (16). Clearly, $C^{+0}(M) \subset C^1(M)$.

Theorem 3. *Assume that a self-adjoint operator* $H \in C^1(M)$ *and* M *is conjugate to* H *on an open set* $J \subset \mathbb{R}$. *Suppose also that, for any* $\varphi \in C_0^\infty(\mathbb{R})$ *and some* $z \in \rho(H)$,

$$\varphi(H)\left[(H-z)^{-1}, iM\right]\varphi(H) \in C^{+0}(M). \tag{19}$$

Then H *has in* J *at most a finite number of eigenvalues (counting multiplicities) and no singular continuous spectrum. Moreover, for any compact* $\Delta \subset J \setminus \sigma_\rho(H)$ *and* $\varepsilon > 0$,

$$\sup_{\lambda \in \Delta, \mu \neq 0} \left\| (|M|+1)^{-1/2-\varepsilon} (H - \lambda - i\mu)^{-1} (|M|+1)^{-1/2-\varepsilon} \right\| < \infty. \tag{20}$$

If, in addition, M *is strictly conjugate to* H *on* J, *then* H *has no eigenvalues in* J.

Corollary 1. *The assertion of Theorem* 3 *remains true if* (19) *is replaced by the assumption*

$$(H-z)^{-1}\left[(H-z)^{-1},\, iM\right](H-z)^{-1} \in C^{+0}(M), \qquad z \in \rho(H). \tag{21}$$

Proof. We have

$$\varphi(H)\left[(H-z)^{-1},\, iM\right]\varphi(H) = \psi(H) T \psi(H),$$

where $\psi(x) = \varphi(x)(x-z) \in C_0^\infty(\mathbb{R})$ and T is the right side of (21). As $H \in C^1(M)$, we have $\psi(H) \in C^1(M)$ [12]. Therefore, (21) implies (19).

***Remark* 5.** Conditions (19), (21) weaken the condition

$$\left[(H-z)^{-1},\, M\right] \in C^{+0}(M), \tag{22}$$

which was used in [13, 14]. Note that, in our applications to the operators of form (1), condition (22) is not satisfied. Moreover, we cannot apply the more general results of [14]. On the other hand, we can verify (21). Note at last that, under the assumptions of Theorem 3, the limiting absorption principle (see (20)) can be proved in some Besov spaces [12, 14].

The proof of Theorem 3 can be obtained following the line of [13, 15]. It will be given in the later paper of the author devoted to the abstract Mourre theory.

4. Limiting absorption principle. In what follows, we assume that h and b are of class C^2. Moreover, we suppose that h_0'' (see (4)) is bounded. We say that a real number λ is a threshold of h_0 if, for any open neighbourhood J of λ, $\inf\left\{ |h_0'(x)| \,\middle|\, h_0(x) \in J \right\} = 0$. The set of all thresholds of h_0 is denoted by $\tau(h_0)$. $\tau(h_0)$ is a closed subset of $\mathbb{R}$ such that $\tau(h_0) \supset c(h_0)$, where $c(h_0) = \left\{ h_0(x) \mid h_0'(x) = 0 \right\}$ is the set of critical values of h_0. Denote $P_j = -i\dfrac{\partial}{\partial x}$, $P = -i\nabla = (P_1, \ldots, P_n)$ is a momenum operator. As usual, $f(P) = \mathcal{F}^{-1} f(Q)\, \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform. Denote $\tau = \tau(h_0) \cup \{0\}$. Let J be a bounded open interval such that its closure $\bar{J} \subset \mathbb{R} \setminus \tau$. Let us construct an operator M conjugate to the unperturbed operator H_0 on J. Let $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta(x) = 1$ if $x \in J$. Suppose that $\operatorname{supp}\theta \subset \mathbb{R} \setminus \tau$. Define

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \tag{23}$$

where

$$M_1 = \frac{1}{2}\left(F_1(P)Q + QF_1(P)\right), \tag{24}$$

$$M_2 = \frac{1}{2}\left(F_2(Q)P + PF_2(Q)\right), \tag{25}$$

$$F_1(x) = \frac{1}{2}\theta\left(|x|^2\right)|x|^{-2}x, \qquad F_2(x) = -\theta(h_0(x))\left|h_0'(x)\right|^{-2} h_0'(x),$$

$$|x| = \left(x_1^2 + \ldots + x_n^2\right)^{1/2}, \qquad x \in \mathbb{R}^n.$$

Since F_i are Lipschitz functions, M is essentially self-adjoint on $\left(S\left(\mathbb{R}^n\right)\right)^n \times S\left(\mathbb{R}^n\right)$

[14]. Here, $S(\mathbb{R}^n)$ is the space of temperate test functions. It is easy to see that

$$[H_0, iM] = \theta(H_0), \qquad [[H_0, iM], iM] = \theta(H_0)\theta'(H_0), \tag{26}$$

so both commutators are bounded in $\mathcal{H}$ and $H_0 \in C^2(M)$. Moreover, (26) implies that M is conjugate to H_0 on J. Introduce the diagonal operator

$$\Lambda = \begin{pmatrix} \left(1+|Q|^2\right)^{1/2} & 0 \\ 0 & \left(1+|P|^2\right)^{1/2} \end{pmatrix} \tag{27}$$

acting in $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$.

Theorem 4. *Let H, H_0 be self-adjoint operators defined by* (1) *and* (3), *respectively. Assume that*

(i) for some $c > 0, \delta > 0$

$$|q(x)| + |b(x)| + |v(x)| \le c(1+|x|)^{-1-\delta}, \qquad x \in \mathbb{R}^n;$$

(ii) b, v, h are bounded functions of class C^2 with bounded first and second derivatives.

Then

(a) the only possible (finite) accumulation points for the eigenvalues of H are in τ. Any eigenvalue not in τ has finite multiplicity. H_0 has no eigenvalues in τ;

(b) the singular continuous spectrum of H lies in τ. In particular, if τ is countable, then H has no singular continuous spectrum;

(c) for any compact $\Delta \subset \mathbb{R} \setminus \left(\tau \cup \sigma_\rho(H)\right)$ and $\varepsilon > 0$,

$$\sup_{\lambda \in \Delta,\, \mu \ne 0} \left\| \Lambda^{-1/2-\varepsilon}(H - \lambda - i\mu)^{-1}\Lambda^{-1/2-\varepsilon} \right\| < \infty. \tag{28}$$

***Remark* 6.** It is possible to generalize Theorem 4 to some unbounded functions h [5]. Moreover, in this situation, we can weaken (i). It suffices to assume that, for some $c > 0$ and $\delta > 0$,

$$|q(x)| + \frac{|b(x)| + |v(x)|}{|h(x)| + 1} \le c(1+|x|)^{-1-\delta}.$$

Proof. Let J be arbitrary bounded interval such that $\bar{J} \subset \mathbb{R} \setminus \tau$ and let the operator M be defined by (23). The direct calculations (see (7)) show that the commutator $\left[(H-\lambda)^{-1}, iM\right] \in \mathcal{L}(\mathcal{H})$ $(\lambda \in \rho(H))$ and, hence, $H \in C^1(M)$. Furthermore, by Theorem 2

$$(H+i)^{-1} - (H_0+i)^{-1} \in S_\infty(\mathcal{H}).$$

Hence, M is conjugate to H on J [14]. Now verify condition (21). Since $H_0 \in C^2(M)$, we have (for sufficiently large λ)

$$(H+\lambda)^{-1}\left[(H_0+\lambda)^{-1}, iM\right](H+\lambda)^{-1} \in C^1(M) \subset C^{+0}(M).$$

Therefore, it suffices to show that

$$(H+\lambda)^{-1}\left[(H+\lambda)^{-1} - (H_0+\lambda)^{-1}, iM\right](H+\lambda)^{-1} \in C^{+0}(M). \tag{29}$$

To simplify calculations, we prove (29) in the case $q(x) \equiv 0$. The proof in the general case is of the same type (see also the proof of Theorem 2). Note that, in order to verify that a symmetric bounded operator S is of class $C^{+0}(M)$, it suffices to show [14] that, for some function $\xi \in C^\infty(\mathbb{R})$ with $\xi(t) = 0$ near zero and $\xi(t) = 1$ near infinity,

$$\int_1^\infty \|\xi(\Lambda/r)S\| r^{-1}\,dr < \infty. \tag{30}$$

We have (for sufficiently large λ)

$$(H+\lambda)^{-1}-(H_0+\lambda)^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{pmatrix}, \tag{31}$$

where (see (7) and (15))

$$R_{11} = (-\Delta+\lambda)^{-1}(-\nabla b(Q)+v(Q))T\left(\bar b(Q)\operatorname{div} + v^*(Q)\right)(-\Delta+\lambda)^{-1}, \tag{32}$$

$$R_{12} = -(-\Delta+\lambda)^{-1}(-\nabla b(Q)+v(Q))T, \tag{33}$$

$$R_{22} = \left(h_0(Q)+\lambda\right)^{-1}\left(v^*(Q)(-\Delta+\lambda)^{-1}v(Q) - \lambda\bar b(Q)(-\Delta+\lambda)^{-1}b(Q) + \right.$$
$$\left. + \bar b(Q)\operatorname{div}\left(-\Delta+\lambda\right)^{-1}v(Q) - v^*(Q)(-\Delta+\lambda)^{-1}\nabla b(Q)\right)T, \tag{34}$$

$$T = \left(h(Q)-\left(\bar b(Q)\operatorname{div}+v^*(Q)\right)(-\Delta+\lambda)^{-1}(-\nabla b(Q)+v(Q))+\lambda\right)^{-1}. \tag{35}$$

We write

$$(H+\lambda)^{-1}-(H_0+\lambda)^{-1} = R_1+R_2,$$

where

$$R_1 = \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & R_{12} \\ R_{12}^* & 0 \end{pmatrix}.$$

Simple calculations show that

$$\begin{pmatrix} \left(1+|Q|^2\right)^{\delta/2} & 0 \\ 0 & (-\Delta+1)^{-1/2} \end{pmatrix}[R_1,\, iM] \in \mathcal{L}(\mathcal{H}).$$

Here, $\delta>0$ is defined by (*i*). In particular, $\Lambda^\delta[R_1,\, iM]\in\mathcal{L}(\mathcal{H})$ and (see (30))

$$[R_1,\, iM] \in C^{+0}(M). \tag{36}$$

On the other hand, condition (30) is not sutisfied for the operator $S=[R_2,\, iM]$. Therefore, we need to consider the operator

$$U = (H+\lambda)^{-1}[R_2,\, iM](H+\lambda)^{-1} = \begin{pmatrix} U_{11} & U_{12} \\ U_{12}^* & U_{22} \end{pmatrix}.$$

We have

$$[R_2,\, iM] = \begin{pmatrix} 0 & T_{12} \\ T_{12}^* & 0 \end{pmatrix},$$

where $T_{12}=i(R_{12}M_2-M_1R_{12})$. Therefore,

$$U_{11} = R_{12}T_{12}^*\left((-\Delta+\lambda)^{-1}+R_{11}\right) + \left((-\Delta+\lambda)^{-1}+R_{11}\right)T_{12}R_{12}^*,$$

$$U_{12} = R_{12}T_{12}^*R_{12} + \left((-\Delta+\lambda)^{-1}+R_{11}\right)T_{12}\left((h_0(Q)+\lambda)^{-1}+R_{22}\right),$$

$$U_{22} = \left((h_0(Q)+\lambda)^{-1}+R_{22}\right)T_{12}^*R_{12} + R_{12}^*T_{12}\left((h_0(Q)+\lambda)^{-1}+R_{22}\right).$$

Using (*ii*), it is easy prove that $\Lambda^\delta U\in\mathcal{L}(\mathcal{H})$. Therefore (see also (36)), condition (29) is verified. Now Theorem 4 directly follows from Corollary 1. Note only that, as $D(M)\supset D(\Lambda)$, estimate (20) implies (28).

5. Wave operators. We see that the unperturbed operator H_0 acts in one channel as the (vector) Laplace operator and in the other channel as a multiplication operator.

Therefore, we can expect existence and completeness of wave operators for the pair H, H_0 if the perturbation is of short-range in Q-variable in the first channel and in P-variable in the other one. In general, R_{22} (see (31), (34) and (35)) has only Coulomb decay in P-variable, and we cannot prove the existence of wave operators. On the other hand, we shall see that if $\bar{b}(x)v(x) \in \mathbb{R}^n$ $\left(x \in \mathbb{R}^n\right)$, R_{22} has a decay like $|P|^{-2}$. We below prove that, in this situation, the wave operators for the pair H, H_0 exist and are complete.

Denote by $P_{ac}(H)$ the projector on the (spectrally) absolutely continuous subspace $\mathcal{H}_{ac}(H)$ of the operator H. Let $H_{ac} = HP_{ac}(H)$ be the absolutely continuous part of H and let $\sigma_{ac}(H) = \sigma(H_{ac})$ be the absolutely continuous spectrum of H.

Theorem 5. *Suppose that*

(i) the assumptions of theorem 4 *are true;*

(ii) $\bar{b}v : \mathbb{R}^n \to \mathbb{R}^n$;

(iii) for some $c > 0$ *and* $\delta > 0$,

$$\left|b'(x)\right| + \left|v'(x)\right| \le c(1+|x|)^{-1/2-\delta}, \qquad x \in \mathbb{R}^n;$$

(iv) τ *is a countable set.*

Then

(a) $\sigma_{ac}(H) = \sigma_{ac}(H_0)$;

(b) H *has no singular continuous spectrum;*

(c) the wave operators

$$W^{\pm} = \underset{t\to\pm\infty}{s-\lim}\, e^{itH} e^{-itH_0} P_{ac}(H_0)$$

exist and are complete: $\operatorname{Ran}(W^{\pm}) = \mathcal{H}_{ac}(H)$. *In particular, operators* H_{ac} *and* $H_{0,ac}$ *are unitary equivalent.*

Proof. Assertion (b) was proved in Theorem 4, (a) follows from (c). By Theorem 4 and Theorem 2, it suffices to show that, for some $\delta > 0$ and $\lambda > 0$ (sufficiently large),

$$\Lambda^{1/2+\delta}\left((H+\lambda)^{-1} - \left(H_0+\lambda\right)^{-1}\right)\Lambda^{1/2+\delta} \in \mathcal{L}(\mathcal{H}).$$

Let $\tilde{H}$ be the operator defined by (12). By (i) of Theorem 4, for sufficiently small $\delta > 0$,

$$\Lambda^{1/2+\delta}\left((H+\lambda)^{-1} - \left(\tilde{H}+\lambda\right)^{-1}\right)\Lambda^{1/2+\delta} \in \mathcal{L}(\mathcal{H}).$$

Therefore, it is sufficient to show that, for small $\delta > 0$,

$$\Lambda^{1/2+\delta}\left((\tilde{H}+\lambda)^{-1} - (H_0+\lambda)^{-1}\right)\Lambda^{1/2+\delta} \in \mathcal{L}(\mathcal{H}). \tag{37}$$

Relation (37) is equivalent to the following conditions (see (31)):

$$\left(1+|Q|^2\right)^{1/4+\delta/2} R_{11}\left(1+|Q|^2\right)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_1), \tag{38}$$

$$\left(1+|Q|^2\right)^{1/4+\delta/2} R_{12}(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1), \tag{39}$$

$$(-\Delta+1)^{1/4+\delta/2} R_{22}(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2). \tag{40}$$

Condition (38) easily follows from the assumptions of Theorem 4. Let us prove (39). Note that commutators $\left[\left(1+|Q|^2\right)^{1/4+\delta/2}, (-\Delta+1)^{-1}\right]$, $\left[\left(1+|Q|^2\right)^{1/4+\delta/2}, (-\Delta+1)^{-1}\nabla\right]$, $\left[(-\Delta+1)^{1/4+\delta/2}, T\right]$ (T defined by (35)) are bounded for $\delta \in (0, 1/2]$. Therefore, it suffices to show that, for some $\delta > 0$,

$$(-\Delta+1)^{-1}\nabla\left(1+|Q|^2\right)^{1/4+\delta/2} b(Q)(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2),$$

$$(-\Delta+1)^{-1}\left(1+|Q|^2\right)^{1/4+\delta/2} v(Q)(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2).$$

These conditions directly follow from (iii). It remains to prove (40). Since the functions b, v, h are smooth with bounded first and second derivatives, it suffices to verify that, for some $\delta > 0$,

$$(-\Delta+1)^{1/4+\delta/2}\Big(\bar{b}(Q)\operatorname{div}(-\Delta+\lambda)^{-1}v(Q) - - v^*(Q)(-\Delta+\lambda)^{-1}\nabla b(Q)\Big)(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2). \tag{41}$$

Clearly, for $\delta \in (0, 1/2]$,

$$(-\Delta+1)^{1/4+\delta/2}\left[\bar{b}(Q), \operatorname{div}(-\Delta+\lambda)^{-1}\right]v(Q)(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2)$$

and

$$(-\Delta+1)^{1/4+\delta/2}\left[v^*(Q), (-\Delta+\lambda)^{-1}\nabla\right]b(Q)(-\Delta+1)^{1/4+\delta/2} \in \mathcal{L}(\mathcal{H}_2).$$

Therefore, (41) follows from (ii).

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1. *Atkinson F. V., Langer H., Mennicken R., Shkalikov A. A.* The essential spectrum of some matrix operators // Math. Nachr. – 1994. – **167**. – P. 5 – 20.
2. *Shkalikov A. A.* On essential spectrum of matrix operators // Math. Zametki. – 1995. – **58**. – P. 945 – 949 (Russian).
3. *Faierman M., Mennicken R., Möller M.* A boundary eigenvalue problem for a system of partial differential operators occurring in magnetohydrodynamics // Math. Nachr. – 1995. – **173**. – P. 141 – 167.
4. *Langer H., Möller M.* The essential spectrum of a non-elliptic boundary value problem // Ibid. – 1996. – **178**. – P. 233 – 248.
5. *Konstantinov A. Yu.* Spectral and scattering theory of some matrix operators related to Schrödinger equations with potentials depending rationally on spectral parameter // Meth. Function. Anal. and Topology. – 1996. – **2**, № 3, 4. – P. 69 – 77.
6. *Konstantinov A. Yu.* Spectral analysis of one class of matrix differential operators // Function. Anal. and Appl. – 1997. – **31**, № 3. – P. 209 – 211.
7. *Adamyan V., Langer H., Mennicken R., Saurer J.* Spectral components of self-adjoint block operator matrices with unbounded entries // Math. Nachr. – 1996. – **178**. – P. 43 – 80.
8. *Mennicken R., Shkalikov A. A.* Spectral decomposition of symmetric operator matrices // Ibid. – **179**. – P. 259 – 273.
9. *Adamyan V., Langer H.* Spectral properties of a class of rational operator valued functions // J. Operator Theory. – 1995. – **33**. – P. 259 – 277.
10. *Mourre E.* Absence of singular continuous spectrum for certain self-adjoint operators // Communs Math. Phys. – 1981. – **78**. – P. 391 – 408.
11. *Perry P. A., Sigal I. M., Simon B.* Spectral analysis of N-body Schrödinger operators // Ann. Math. – 1981. – **114**. – P. 519 – 567.
12. *Amrein W., Boutet de Monvel A., Georgescu V.* C_0-groups, commutator methods and spectral theory of N-body Hamiltonians. – Basel: Birkhäuser Verlag, 1996. – 440 p.
13. *Boutet de Monvel A., Georgescu V., Mantoiu M.* Locally smooth operators and the limiting absorption principle for N-body Hamiltonians // Rew. Math. Phys. – 1993. – **5**, № 1. – P. 105 – 189.
14. *Boutet de Monvel-Berthier A., Georgescu V.* Spectral and scattering theory by the conjugate operator method // Algebra i Analiz. – 1992. – **4**, № 3. – P. 79 – 116.
15. *Yafaev D. R.* Remarks on spectral theory for multiparticle Schrödinger operators // Zap. Nauchn. Sem. LOMI. – 1984. – **133**. – P. 277 – 298.

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