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D. Ya. Petrina, K. D. Petrina (Inst. Math. Nat. Acad. Sci. Ukraine, Kiev)

STOCHASTIC DYNAMICS AND BOLTZMANN HIERARCHY. III

СТОХАСТИЧНА ДИНАМІКА І ІЄРАРХІЯ БОЛЬЦМАНА. III

Stochastic dynamics corresponding to the Boltzmann hierarchy is constructed. The Liouville–Itô equations are obtained, from which we derive the Boltzmann hierarchy regarded as an abstract evolution equation. We construct the semigroup of evolution operators and prove the existence of solutions of the Boltzmann hierarchy in the space of sequences of integrable and bounded functions. On the basis of these results, we prove the existence of global solutions of the Boltzmann equation and the existence of the Boltzmann–Grad limit for an arbitrary time interval.

Побудована стохастична динаміка, яка відповідає ієрархії Больцмана. Отримані рівняння Ліувілля–Іто, а з них виведена ієрархія Больцмана, яка розглядається як абстрактне еволюційне рівняння. Побудована півгрупа еволюційних операторів і доведено існування розв'язків ієрархії Больцмана в просторі послідовностей інтегровних та обмежених функцій. На цій основі доведено існування глобальних розв'язків рівняння Больцмана та існування границі Больцмана–Греда на довільному інтервалі часу.

APPENDIX I. In this appendix, we give an exact mathematical meaning to the infinitesimal operator of the semigroups $S_N(-t)$ and to the derivation of the stochastic hierarchy.

1. Semigroup $S_2(t)$. Consider the following functional:

$$(S_2(t)\, f_2 - S_2^0(t)\, f_2, \varphi_2) =$$

$$= M \int [S_2(t)\, f_2(x_1, x_2) - S_2^0(t)\, f_2(x_1, x_2)]\, \varphi_2(x_1, x_2)\, dx_1\, dx_2, \qquad t \geq 0, \quad \text{(I.1)}$$

where $\varphi_2(x_1, x_2)$ is a test function from the Schwartz space S, $f_2(x_1, x_2)$ is continuous, $S_2^0(t)$ is the operator of the free evolution, M denotes the operation of averaging with respect to the random vector η. Later we show that functional (I.1) exists and is continuous.

Denote by D_{-t} the set in the phase space that is the collection of the points of the trajectories

$$X(-\tau, q_1, p_1, q_1, p_2), \quad 0 \leq \tau \leq t, \quad \eta \in S_+^2 \cup S_-^2 = S^2, \qquad \text{(I.2)}$$

$$q_1 \in R^3, \quad p_1 \in R^3, \quad p_2 \in R^3.$$

The set D_{-t} consists of the following points:

$$(x_1, x_2) = (q_1 - p_1^* \tau, p_1^*, q_1 - p_2^* \tau, p_2^*), \qquad \eta \in S_+^2, \quad 0 \leq \tau \leq t,$$

$$p_1^* = p_1 - \eta\, \eta \cdot (p_1 - p_2), \quad p_2^* = p_2 + \eta\, \eta \cdot (p_1 - p_2), \qquad \text{(I.3)}$$

$$(x_1, x_2) = (q_1 - p_1 \tau, p_1, q_1 - p_2 \tau, p_2), \qquad \eta \in S_-^2, \quad 0 \leq \tau \leq t.$$

Note that *the phase points of the set D_{-t} have the following characteristic property*: *the vectors of difference of positions are parallel to the vectors of difference of momenta*

$$q_1 - p_1^* \tau - q_1 + p_2^* \tau = (p_2^* - p_1^*)\, \tau, \qquad q_1 - p_1 \tau - q_1 + p_2 \tau = (p_2 - p_1)\, \tau.$$

It is obvious that the collection of points (I.3) is of full Lebesgue measure with respect to τ, η, q_1, p_1, p_2.

The function $S_2(t)\,f_2(x_1,x_2) - S_2^0(t)\,f_2(x_1,x_2)$ *is different from zero in the set* D_{-t}. Really, all points of D_{-t} "interact" at certain time on the interval $[0,t]$ during forward evolution. For $t=\tau-0$, the points $(q_1-p_1^*\tau, p_1^*, q_1-p_2^*\tau, p_2^*)$, $\eta\in S_+^2$, are shifted along the trajectory to the points (q_1, p_1^*, q_1, p_2^*); for $t=\tau+0$, they turn into the point (q_1, p_1, q_1, p_2); for time t, this point turns into the point

$$(q_1+p_1(t-\tau), p_1, q_1+p_2(t-\tau), p_2), \quad 0\le\tau\le t, \quad \eta\in S_+^2. \tag{I.4}$$

By analogy the points $(q_1-p_1\tau, p_1, q_1-p_2\tau, p_2)$, $\eta\in S_-^2$, turn into the points

$$(q_1+p_1^*(t-\tau), p_1^*, q_1+p_2^*(t-\tau), p_2^*), \quad 0\le\tau\le t, \quad \eta\in S_-^2. \tag{I.5}$$

Note that the phase points (I.4) and (I.5) have the same characteristic property as the points (I.3) of the set D_{-t}. Namely: the vectors of difference of positions are parallel to the vectors of difference of momenta. Thus, the hypersurface with this chatacteristic property is invariant with respect to the stochastic dynamics and function $S_2(t)\,f_2(x_1,x_2) - S_2^0(t)\,f_2(x_1,x_2)$ is different from zero on the set D_{-t} on this hypersurface.

Thus, we have

$$[S_2(t)\,f_2(x_1,x_2) - S_2^0(t)\,f_2(x_1,x_2)]\,\varphi(x_1,x_2) =$$
$$= [f_2(q_1+p_1(t-\tau), p_1, q_1+p_2(t-\tau), p_2) -$$
$$- f_2(q_1+p_1^*(t-\tau), p_1^*, q_1+p_2^*(t-\tau), p_2^*)]\,\varphi_2(q_1-p_1^*\tau, p_1^*, q_1-p_2^*\tau, p_2^*)$$

at the point $(x_1,x_2) = (q_1-p_1^*\tau, p_1^*, q_1-p_2^*\tau, p_2^*)\in D_{-t}$, $\eta\in S_+^2$;

$$[S_2(t)\,f_2(x_1,x_2) - S_2^0(t)\,f_2(x_1,x_2)]\,\varphi(x_1,x_2) =$$
$$= [f_2(q_1+p_1^*(t-\tau), p_1^*, q_1+p_2^*(t-\tau), p_2^*) -$$
$$- f_2(q_1+p_1(t-\tau), p_1, q_1+p_2(t-\tau), p_2)]\,\varphi_2(q_1-p_1\tau, p_1, q_1-p_2\tau, p_2)$$

at the point $(x_1,x_2) = (q_1-p_1\tau, p_1, q_1+p_2\tau, p_2)\in D_{-t}$, $\eta\in S_-^2$.

Functional (I.1) can be represented in the following form:

$$(S_2(t)\,f_2 - S_2^0(t)\,f_2, \varphi_2) =$$
$$= \frac{1}{4\pi}\frac{1}{2}\int_0^t d\tau \int_{S_+^2} d\eta \int dq_1\,dp_1\,dp_2\,|\eta\cdot(p_1-p_2)| \times$$
$$\times [f_2(q_1+p_1(t-\tau), p_1, q_1+p_2(t-\tau), p_2) -$$
$$- f_2(q_1+p_1^*(t-\tau), p_1^*, q_1+p_2^*(t-\tau), p_2^*)]\,\varphi_2(q_1-p_1^*\tau, p_1^*, q_1-p_2^*\tau, p_2^*) +$$
$$+ \frac{1}{4\pi}\frac{1}{2}\int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\,dp_1\,dp_2\,|\eta\cdot(p_1-p_2)| \times$$
$$\times [f_2(q_1+p_1^*(t-\tau), p_1^*, q_1+p_2^*(t-\tau), p_2^*) -$$
$$- f_2(q_1+p_1(t-\tau), p_1, q_1+p_2(t-\tau), p_2)]\,\varphi_2(q_1-p_1\tau, p_1, q_1-p_2\tau, p_2)$$

$$= \frac{1}{4\pi}\int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\, dp_1\, dp_2\, |\eta\cdot(p_1-p_2)| \times$$

$$\times \left[f_2(q_1 + p_1^*(t-\tau), p_1^*, q_1 + p_2^*(t-\tau), p_2^*) - \right.$$

$$\left. - f_2(q_1 + p_1(t-\tau), p_1, q_1 + p_2(t-\tau), p_2)\right] \varphi_2(q_1 - p_1\tau, p_1, q_1 - p_2\tau, p_2) =$$

$$= \frac{1}{4\pi}\int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\, dq_2\, dp_1\, dp_2\, \delta(q_1 + p_1\tau - q_2 - p_2\tau)\, |\eta\cdot(p_1-p_2)| \times$$

$$\times \left[f_2(q_1 + p_1\tau + p_1^*(t-\tau), p_1^*, q_2 + p_2\tau + p_2^*(t-\tau), p_2^*) - \right.$$

$$\left. - f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2)\right] \varphi_2(q_1, p_1, q_2, p_2). \tag{I.6}$$

The interpretation of the last formula is the following: the integral is taken over all initial phase points that interact during the time interval $[0, t]$ and, thus, belong to the hypersurface $q_1 + p_1\tau = q_2 + p_2\tau$, $0 \le \tau \le t$. This integral is averaged with respect to the random vectors η. We also used the fact that the Jacobian of the transformation $(p_1^*, p_2^*) \to (p_1, p_2)$ is equal to one and $\eta\cdot(p_1^* - p_2^*) = -\eta\cdot(p_1 - p_2)$.

Here, we take into account that in the set D_{-t} particles are shifted in direction η by the distance $|\eta\cdot(p_1 - p_2)|\tau$ and use in D_{-t} variables τ, η, q_1, p_1, p_2. (About the factor $\frac{1}{2}$, see Remark 1 in the end of this section.)

The derivative of functional (I.1), (I.6) at $t = 0$ is equal to

$$\frac{d}{dt}(S_2(t) f_2 - S_2^0(t) f_2, \varphi_2)\Big|_{t=0} = \frac{1}{4\pi}\frac{1}{2}\int_{S_+^2} d\eta \int dq_1\, dp_1\, dp_2\, |\eta\cdot(p_1-p_2)| \times$$

$$\times \left[f_2(q_1, p_1, q_1, p_2) - f_2(q_1, p_1^*, q_1, p_2^*)\right] \varphi_2(q_1, p_1^*, q_1, p_2^*) +$$

$$+ \frac{1}{4\pi}\frac{1}{2}\int_{S_-^2} d\eta \int dq_1\, dp_1\, dp_2\, |\eta\cdot(p_1-p_2)| \times$$

$$\times \left[f_2(q_1, p_1^*, q_1, p_2^*) - f_2(q_1, p_1, q_1, p_2)\right] \varphi_2(q_1, p_1, q_1, p_2). \tag{I.7}$$

Formula (I.7) yields

$$\frac{d}{dt}(S_2(t) f_2 - S_2^0(t) f_2, \varphi_2)\Big|_{t=0} = \frac{1}{4\pi}\int_{S_-^2} d\eta \int dq_1\, dp_1\, dp_2\, |\eta\cdot(p_1-p_2)| \times$$

$$\times \left[f_2(q_1, p_1^*, q_1, p_2^*) - f_2(q_1, p_1, q_1, p_2)\right] \varphi_2(q_1, p_1, q_1, p_2) =$$

$$= \int_{S_-^2} d\eta\, \chi(\eta) \int dq_1\, dp_1\, dp_2\, |\eta\cdot(p_1-p_2)| \times$$

$$\times \left[f_2(q_1, p_1^*, q_1, p_2) - f_2(q_1, p_1, q_1, p_2)\right] \varphi_2(q_1, p_1, q_1, p_2), \qquad \chi(\eta) = \frac{1}{4\pi}. \tag{I.8}$$

Here, we have again used the fact that the Jacobian of the transformation $(p_1^*, p_2^*) \to (p_1, p_2)$ is equal to one and $\eta\cdot(p_1^* - p_2^*) = -\eta\cdot(p_1 - p_2)$.

Formula (I.8) can be represented in the following final form:

$$\frac{d}{dt}(S_2(t)\,f_2 - S_2^0(t)\,f_2, \varphi_2)\Big|_{t=0} =$$

$$= \int_{S^2} d\eta\,\chi(\eta) \int dq_1\,dq_2\,dp_1\,dp_2\,\delta(q_1 - q_2)\,|\eta\cdot(p_1 - p_2)|\,\theta(-\eta\cdot(p_1 - p_2)) \times$$

$$\times [f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1, p_1, q_2, p_2)]\,\varphi_2(q_1, p_1, q_2, p_2). \tag{I.9}$$

In (I.9) we consider η as the random vector on S^2 with density $\frac{1}{4\pi}$. From (I.9), we have for a fixed random vector η

$$\frac{d}{dt}(S_2(t)\,f_2(x_1, x_2) - S_2^0(t)\,f_2(x_1, x_2))\Big|_{t=0} =$$

$$= \delta(q_1 - q_2)\,|\eta\cdot(p_1 - p_2)|\,\theta(-\eta\cdot(p_1 - p_2)) \times$$

$$\times [f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1, p_1, q_2, p_2)],$$

$$\tag{I.10}$$

$$\frac{d}{dt} S_2(t)\,f_2(x_1, x_2)\Big|_{t=0} = \left(p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}\right) f_2(x_1, x_2) +$$

$$+ \delta(q_1 - q_2)\,|\eta\cdot(p_1 - p_2)|\,\theta(-\eta\cdot(p_1 - p_2)) \times$$

$$\times [f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1, p_1, q_2, p_2)].$$

One should add the boundary condition in the Poisson bracket. The latter formula (I.10) coincides with (2.8).

Now we show that the stochastic dynamics is obtained from the dynamics of hard spheres by a specific averaging procedure over the sphere with radius a and the limit transitions as $a \to 0$ (details of proof will be published in a separate paper).

In order to explain the result of this averaging procedure, we consider the hypersurfaces (q_1, p_1, q_1, p_2) and $(q_1, p_1, q_1 - a\eta, p_2)$ with arbitrary q_1, p_1, p_2 and $\eta \in S^2$. Let us shift them backward at time $-\tau$ along the stochastic and Hamiltonian trajectories, respectively. We obtain

$$\begin{aligned} &(q_1 - p_1^*\tau, p_1^*, q_1 - p_2^*\tau, p_2^*), \quad \eta \in S_+^2, \\ &(q_1 - p_1\tau, p_1, q_1 - p_2\tau, p_2), \quad \eta \in S_-^2, \end{aligned} \tag{I.3$'$}$$

for the stochastic dynamics and

$$\begin{aligned} &(q_1 - p_1^*\tau, p_1^*, q_1 - a\eta - p_2^*\tau, p_2^*), \quad \eta \in S_+^2, \\ &(q_1 - p_1\tau, p_1, q_1 - a\eta - p_2\tau, p_2), \quad \eta \in S_-^2, \end{aligned} \tag{I.11}$$

for the Hamiltonian dynamics.

For the Hamiltonian dynamics every point $(q_1, p_1, q_1 - a\eta, p_2)$ for fixed τ is associated with the single point (I.11) while for the stochastic dynamics every point (q_1, p_1, q_1, p_2) is associated with the two-dimensional hypersurface (I.2).

Denote by D_{-t}^a the collection of points (I.11) with $\eta \in S_+^2 \cup S_-^2$, $0 \le \tau \le t$ and consider an analog of functional (1.I), (I.6) for hard spheres. We have

$$(S_2^a(t)f_2 - S_2^0(t)f_2, \varphi_2) =$$

$$= \int_{D_{-t}^a} [S_2^a(t)f_2(x_1,x_2) - S_2^0(t)f_2(x_1,x_2)]\,\varphi_2(x_1,x_2)\,dx_1\,dx_2 =$$

$$= \frac{a^2}{2}\int_0^t d\tau \int_{S_+^2} d\eta \int dq_1\,dp_1\,dp_2\,|\eta\cdot(p_1-p_2)| \times$$

$$\times [f_2(q_1+p_1(t-\tau), p_1, q_1-a\eta+p_2(t-\tau), p_2) -$$

$$- f_2(q_1+p_1^*(t-\tau), p_1^*, q_1-a\eta+p_2^*(t-\tau), p_2^*)] \times$$

$$\times \varphi_2(q_1-p_1^*\tau, p_1^*, q_1-a\eta-p_2^*\tau, p_2^*) +$$

$$+ \frac{a^2}{2}\int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\,dp_1\,dp_2\,|\eta\cdot(p_1-p_2)| \times$$

$$\times [f_2(q_1+p_1^*(t-\tau), p_1^*, q_1-a\eta+p_2^*(t-\tau), p_2^*) -$$

$$- f_2(q_1+p_1(t-\tau), p_1, q_1-a\eta+p_2(t-\tau), p_2)] \times$$

$$\times \varphi_2(q_1-p_1\tau, p_1, q_1-a\eta-p_2\tau, p_2). \tag{I.12}$$

We have used the variables τ, η, q_1, p_1, p_2 in the domain D_{-t}^a. It follows from (I.12) that the measure of D_{-t}^a is proportional to a^2. (Concerning the factor $\frac{1}{2}$, see Remark 1 at the end of this section.)

It is obvious that

$$\lim_{a\to 0}\frac{1}{4\pi a^2}(S_2^a(t)f_2 - S_2^0(t)f_2, \varphi_2) = (S_2(t)f_2 - S_2^0(t)f_2, \varphi_2)$$

for continuous $f_2 \in L_1$ and test functions φ_2.

This means that functional (I.1), (I.6) is the average of functional (I.12) over the sphere $q_2 = q_1 - a\eta$ as $a \to 0$.

The function $S_2^a(t)f_2(x_1,x_2) - S_2^0(t)f_2(x_1,x_2)$ is different from zero in the domain D_{-t}^a; the function $S_2(t)f_2(x_1,x_2) - S_2^0(t)f_2(x_1,x_2)$ is different from zero in the set D_{-t} which is the collection of the trajectories $X(-\tau,x)$, $q_1 = q_2$, $0 \le \tau \le t$.

Analogously, we can consider the semigroups $S_2(-t)$ its infinitesimal operator, and the set D_t as the collection of the points of the trajectories $X(\tau, q_1, p_1, q_1, p_2)$, $0 \le \tau \le t$.

Now consider the infinitesimal operator of $S_2^a(t)$.

It follows from (I.12) that

$$\frac{d}{dt}(S_2^a(t)f_2 - S_2^0(t)f_2, \varphi_2)\Big|_{t=0} =$$

$$= \frac{a^2}{2}\int_{S_+^2} d\eta \int dq_1\,dp_1\,dp_2\,|\eta\cdot(p_1-p_2)| \times$$

$$\times [f_2(q_1, p_1, q_1-a\eta, p_2) - f_2(q_1, p_1^*, q_1-a\eta, p_2^*)]\,\varphi_2(q_1, p_1^*, q_1-a\eta, p_2^*) +$$

$$+\ \frac{a^2}{2}\int\limits_{S_-^2} d\eta \int dq_1\, dp_1\, dp_2\, |\eta\cdot(p_1-p_2)| \times$$

$$\times\,[f_2(q_1,p_1^*,q_1-a\eta,p_2^*) - f_2(q_1,p_1,q_1-a\eta,p_2)]\,\varphi_2(q_1,p_1,q_1-a\eta,p_2) =$$

$$= a^2\int\limits_{S_-^2} d\eta \int dq_1\, dq_2\, dp_1\, dp_2\, |\eta\cdot(p_1-p_2)|\,\delta(q_1-q_2-a\eta)\times$$

$$\times\,[f_2(q_1,p_1^*,q_2,p_2^*) - f_2(q_1,p_1,q_2,p_2)]\,\varphi_2(q_1,p_1,q_2,p_2). \tag{I.13}$$

We have from (I.13)

$$\frac{d}{dt}(S_2^a(t)f_2 - S_2^0(t)f_2)(x_1,x_2)\Big|_{t=0} =$$

$$= a^2\int\limits_{S_-^2} d\eta\, |\eta\cdot(p_1-p_2)|\,\delta(q_1-q_2-a\eta)\times$$

$$\times\,[f_2(q_1,p_1^*,q_2,p_2^*) - f_2(q_1,p_1,q_2,p_2)] \tag{I.14}$$

and, finally,

$$\frac{d}{dt}S_2^a(t)f_2(x_1,x_2)\Big|_{t=0} = \sum_{i=1}^{2} p_i\frac{\partial}{\partial q_i}f_2(x_1,x_2) +$$

$$+\ a^2\int\limits_{S_-^2} d\eta\, |\eta\cdot(p_1-p_2)|\,\delta(q_1-q_2-a\eta)\times$$

$$\times\,[f_2(q_1,p_1^*,q_2,p_2^*) - f_2(q_1,p_1,q_2,p_2)]. \tag{I.15}$$

We have obtained a well-known formula (see [23–25]).

Comparing (I.9), (I.10) and (I.14) we get

$$\lim_{a\to 0}\frac{1}{4\pi a^2}\frac{d}{dt}(S_2^a(t)f_2 - S_2^0(t)f_2)(x_1,x_2)\Big|_{t=0} =$$

$$= M\frac{d}{dt}(S_2(t)f_2 - S_2^0(t)f_2)(x_1,x_2)\Big|_{t=0} \tag{I.16}$$

where M means the averaging operation with respect to η.

***Remark* 1.** The factor $\frac{1}{2}$ in (I.6) and (I.12) is connected with the following: the points $q_1=q_2,\ p_1^*,\ p_2^*,\ q_1,\ q_1-a\eta,\ p_1^*,\ p_2^*,\ \eta\in S_+^2,\ \eta\cdot(p_1-p_2)\geq 0$ are the states before the collision (for increasing time) as well as the points $q_1=q_2,\ p_1,\ p_2,\ q_1,\ q_1-a\eta,\ p_1,\ p_2,\ \eta\in S_-^2,\ \eta\cdot(p_1-p_2)\leq 0$. For the points $q_1=q_2,\ p_1^*,\ p_2^*,\ q_1,\ q_1-a\eta,\ p_1^*,\ p_2^*$ we have $\eta\cdot(p_1^*-p_2^*) = -\eta\cdot(p_1-p_2)\leq 0$. This means that the points $q_1=q_2,\ p_1^*,\ p_2^*,\ q_1,\ q_1-a\eta,\ p_1^*,\ p_2^*,\ \eta\cdot(p_1-p_2)\geq 0$ are associated with the random vector $\eta\in S_-^2,\ \eta\cdot(p_1^*-p_2^*) < 0$ with respect to the momenta $p_1^*,\ p_2^*$.

In functionals (I.6), (I.12) we have taken into account the two identic states before collision: $q_1=q_2,\ p_1,\ p_2,\ q_1,\ q_1-a\eta,\ p_1,\ p_2,\ \eta\in S_-^2,\ \eta\cdot(p_1-p_2)\leq 0$ and $q_1=q_2,\ p_1^*,\ p_2^*,\ q_1,\ q_1-a\eta,\ p_1^*,\ p_2^*,\ \eta\in S_-^2,\ \eta\cdot(p_1^*-p_2^*)\leq 0$.

Two terms in the right-hand side (I.6), (I.7) and (I.12), (I.13) coincide. It becomes obvious if one uses the momenta p_1^*, p_2^* as the new variables instead of p_1, p_2 in the first terms of the right-hand side (I.6), (I.7) and (I.12), (I.13).

This coincidence are connected with above mentioned two different representation of the identic states.

Therefore we used the factor $\frac{1}{2}$ in (I.6) and (I.12).

Thus, in this section, we have found the rigorous realization of the idea expressed in the important paper [24] of Cercignani, where he has written: "In order to introduce irreversibility, one has to consider an averaging process over the details of collisions; this can be obtained by a limiting process in which the space region, where a single collision takes place, vanishes and thus determinism is lost".

Namely, we have showed that the infinitesimal operator of the evolution group of the stochastic dynamics is the limit of the infinitesimal operator of the evolution group of the Hamiltonian dynamics of hard spheres, or, more precisely, the limit of the average of this operator over the spheres as its radius tends to zero. The obtained Liouville – Itô equation and the stochastic hierarchy are irreversible.

2. Connection between formulas (2.10) and (2.14), Liouville theorem. We start with the two-particle systems. Consider the functional $(S_2(t)f_2 - S_2^0(t)f_2, \varphi_2)$ and show that it is equal to $(f_2, S_2(-t)\varphi_2 - S_2^0(-t)\varphi_2)$.

According to (I.6) we have

$$(S_2(t)\, f_2, \varphi_2) = (S_2^0(t)\, f_2, \varphi_2) +$$

$$+ \frac{1}{4\pi}\int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta\cdot(p_1 - p_2)|\, \delta(q_1 + p_1\tau - q_2 - p_2\tau) \times$$

$$\times [\, f_2(q_1 + p_1\tau + p_1^*(t-\tau), p_1^*, q_2 + p_2\tau + p_2^*(t-\tau), p_2^*) -$$

$$- f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2)]\, \varphi_2(q_1, p_1, q_2, p_2) =$$

$$= (f_2, S_2^0(-t)\,\varphi_2) + \frac{1}{4\pi}\int_0^t d\tau \int_{S_+^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta\cdot(p_1 - p_2)| \times$$

$$\times\, \delta(q_1 - p_1\tau - q_2 + p_2\tau) f_2(q_1, p_1, q_2, p_2) \times$$

$$\times [\varphi_2(q_1 - p_1\tau - p_1^*(t-\tau), p_1^*, q_2 - p_2\tau - p_2^*(t-\tau), p_2^*) -$$

$$- \varphi_2(q_1 - p_1 t, p_1, q_2 - p_2 t, p_2)] =$$

$$= (f_2, S_2^0(-t)\,\varphi_2) + \frac{1}{4\pi}\int_0^t d\tau \int_{S_+^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta\cdot(p_1 - p_2)| \times$$

$$\times\, \delta(q_1 - q_2\,) f_2(q_1 + p_1\tau, p_1, q_2 + p_2\tau, p_2) \times$$

$$\times [\varphi_2(q_1 - p_1^*(t-\tau), p_1^*, q_2 - p_2^*(t-\tau), p_2^*) -$$

$$- \varphi_2(q_1 - p_1(t-\tau), p_1, q_2 - p_2(t-\tau), p_2)]. \qquad \text{(I.17)}$$

Formula (I.17) means that

$$(S_2(t)f_2 - S_2^0(t)f_2, \varphi_2) = (f_2, S_2(-t)\varphi_2 - S_2^0(-t)\varphi_2).$$

Finally we have

$$(S_2(t)\, f_2, \varphi_2) =$$
$$= M\int dx_1\, dx_2 [S_2(t) f_2(x_1, x_2)]\, \varphi_2(x_1, x_2) =$$
$$= M\int dx_1\, dx_2\, f_2(x_1, x_2)[S_2(-t)\varphi_2(x_1, x_2)] = (f_2, S_2(-t)\varphi_2). \quad \text{(I.18)}$$

Let put $\varphi_2 = 1$ in (I.18), then we obtain

$$M\int dx_1\, dx_2\, S_2(t) f_2(x_1, x_2) = M\int f_2(x_1, x_2)\, dx_1\, dx_2 = \int f_2(x_1, x_2)\, dx_1\, dx_2.$$

For $f_2 = 1$ one obtains

$$M\int dx_1\, dx_2\, S_2(-t)\varphi_2(x_1, x_2) = M\int \varphi_2(x_1, x_2)\, dx_1\, dx_2 =$$
$$= \int \varphi_2(x_1, x_2)\, dx_1\, dx_2.$$

This formulas can be considered as the Liouville theorem for the stochastic dynamics.

Now differentiate formulas (I.6), (I.17) with respect to time

$$\frac{d}{dt}(S_2(t) f_2, \varphi_2) = \left(\sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i} S_2^0(t)\, f_2, \varphi_2 \right) +$$
$$+ \frac{1}{4\pi} \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta\cdot(p_1 - p_2)|\, \delta(q_1 - q_2)\, \times$$
$$\times [f_2(q_1, p_1^*, q_1, p_2^*) - f_2(q_1, p_1, q_1, p_2)] \times$$
$$\times\, \varphi_2(q_1 - p_1 t, p_1, q_2 - p_2 t, p_2) +$$
$$+ \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta\cdot(p_1 - p_2)|\, \delta(q_1 - q_2)\, \times$$
$$\times \left[\sum_{i=1}^{2} p_i^* \frac{\partial}{\partial q_i} f_2(q_1 + p_1^*(t-\tau), p_1^*, q_2 + p_2^*(t-\tau), p_2^*) - \right.$$
$$\left. - \sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i} f_2(q_1 + p_1(t-\tau), p_1, q_2 + p_2(t-\tau), p_2) \right] \times$$
$$\times\, \varphi_2(q_1 - p_1 \tau, p_1, q_2 - p_2 \tau, p_2) =$$
$$= \left(\sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i} S_2^0(t)\, f_2, \varphi_2 \right) +$$
$$+ \frac{1}{4\pi} \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta\cdot(p_1 - p_2)|\, \delta(q_1 + p_1 t - q_2 - p_2 t)\, \times$$
$$\times [f_2(q_1 + p_1 t, p_1^*, q_2 + p_2 t, p_2^*) -$$
$$- f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2)]\, \varphi_2(q_1, p_1, q_2, p_2) +$$
$$+ \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta\cdot(p_1 - p_2)|\, \delta(q_1 + p_1 \tau - q_2 - p_2 \tau)\, \times$$

$$\times \left[\sum_{i=1}^{2} p_i^* \frac{\partial}{\partial q_i} f_2(q_1 + p_1\tau + p_1^*(t-\tau), p_1^*, q_2 + p_2\tau + p_2^*(t-\tau), p_2^*) - \right.$$

$$\left. - \sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i} f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2) \right] \varphi_2(q_1, p_1, q_2, p_2). \tag{I.19}$$

Formula (I.19) is the proof of the equality

$$\frac{d}{dt}(S_2(t) f_2, \varphi_2) = \left(\frac{d}{dt} S_2(t) f_2, \varphi_2 \right) \tag{I.20}$$

or

$$\frac{d}{dt} M \int dx_1 dx_2 [S_2(t) f_2(x_1, x_2)] \varphi_2(x_1, x_2) =$$

$$= M \int dx_1 dx_2 \left[\frac{d}{dt} S_2(t) f_2(x_1, x_2) \right] \varphi_2(x_1, x_2).$$

From (I.19) it follows that

$$\frac{d}{dt} S_2(t) f_2(x_1, x_2) = \sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i} S_2^0(t) f_2(x_1, x_2) \tag{I.21}$$

if during the time interval $[0, t)$ the particles with initial phase points (x_1, x_2) do not interact, i.e.

$$q_1 - q_2 \neq (p_1 - p_2)\tau \quad \text{for all} \quad \tau, \quad 0 \leq \tau \leq t.$$

If $q_1 - q_2 = (p_1 - p_2)\tau$ for some τ, $0 \leq \tau \leq t$, i.e. $(x_1, x_2) \in D_{-t}$, and the particles with these initial phase points interact, then from (I.19) we have

$$\frac{d}{dt} S_2(t) f_2(x_1, x_2) = [f_2(q_1 + p_1\tau, p_1^*, q_2 + p_2\tau, p_2^*) -$$

$$- f_2(q_1 + p_1\tau, p_1, q_2 + p_2\tau, p_2)] \delta(\tau - t) +$$

$$+ \sum_{i=1}^{2} p_i^* \frac{\partial}{\partial q_i} f_2(q_1 + p_1\tau + p_1^*(t-\tau), p_1^*, q_2 + p_2\tau + p_2^*(t-\tau), p_2^*) \tag{I.21$'$}$$

for fixed η.

Formulas (I.21) coincide with analogous formulas (2.5) – (2.10) obtained in Section 2.

Now we want to show that

$$\left(\frac{d}{dt} S_2(t) f_2, \varphi_2 \right) = \left(f_2, \frac{d}{dt} S_2(-t) \varphi_2 \right). \tag{I.22}$$

For this purpose we represent the last term in (I.19) as follows

$$\frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta \int dq_1 dp_1 dq_2 dp_2 \, |\eta \cdot (p_1 - p_2)| \, \delta(q_1 - q_2) \times$$

$$\times \left\{ -\frac{d}{d\tau} [f_2(q_1 + p_1^*(t-\tau), p_1^*, q_2 + p_2^*(t-\tau), p_2^*) - \right.$$

$$\left. - f_2(q_1 + p_1(t-\tau), p_1, q_2 + p_2(t-\tau), p_2)] \right\} \varphi_2(q_1 - p_1\tau, p_1, q_2 - p_2\tau, p_2) =$$

$$= \frac{1}{4\pi} \int_{S_-^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta \cdot (p_1 - p_2)| \, \delta(q_1 - q_2) \times$$

$$\times \Big\{ [f_2(q_1 + p_1^* t, p_1^*, q_1 + p_2^* t, p_2^*) -$$

$$- f_2(q_1 + p_1 t, p_1, q_1 + p_2 t, p_2)] \, \varphi_2(q_1, p_1, q, p_2) -$$

$$- [f_2(q_1, p_1^*, q_2, p_2^*) - f_2(q_1, p_1, q_2, p_2)] \, \varphi_2(q_1 - p_1 t, p_1, q_2 - p_2 t, p_2) +$$

$$+ \int_0^t d\tau [f_2(q_1 + p_1^*(t - \tau), p_1^*, q_2 + p_2^*(t - \tau), p_2^*) -$$

$$- f_2(q_1 + p_1(t - \tau), p_1, q_2 + p_2(t - \tau), p_2)] \times$$

$$\times \left(- p_1 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial q_2} \right) \varphi_2(q_1 - p_1 \tau, p_1, q_2 - p_2 \tau, p_2) \Big\}. \tag{I.23}$$

Inserting (I.23) into (I.19), we get

$$\frac{d}{dt} (S_2(t) f_2, \varphi_2) = \left(f_2, - \sum_{i=1}^{2} p_i \frac{\partial}{\partial q_i} S_2^0(-t) \, \varphi_2 \right) +$$

$$+ \frac{1}{4\pi} \int_{S_+^2} d\eta \int dq_1\, dp_1\, dq_2\, dp_2\, |\eta \cdot (p_1 - p_2)| \, \delta(q_1 - q_2) \times$$

$$\times \Big\{ f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2) [\varphi_2(q_1, p_1^*, q_2, p_2^*) - \varphi_2(q_1, p_1, q_2, p_2)] +$$

$$+ \int_0^t d\tau \, f_2(q_1 + p_1 \tau, p_1, q_2 + p_2 \tau, p_2) \times$$

$$\times \Big[\left(- p_1^* \frac{\partial}{\partial q_1} - p_2^* \frac{\partial}{\partial q_2} \right) \varphi_2(q_1 - p_1^*(t - \tau), p_1^*, q_2 - p_2^*(t - \tau), p_2^*) -$$

$$- \left(- p_1 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial q_2} \right) \varphi_2(q_1 - p_1(t - \tau), p_1, q_2 - p_2(t - \tau), p_2) \Big] \Big\} =$$

$$= \left(f_2, \frac{d}{dt} S_2(-t) \, \varphi_2 \right). \tag{I.24}$$

Equality (I.22) is proved.

From (I.24) we obtain the following formula

$$\left(\frac{d}{dt} S_2(t) \, f_2, \varphi_2 \right) =$$

$$= \int dq_1\, dp_1\, dq_2\, dp_2 \Big\{ f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2) +$$

$$+ \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta \, \delta(q_1 + p_1 \tau - q_2 - p_2 \tau) \, |\eta \cdot (p_1 - p_2)| \times$$

$$\times [f_2(q_1+p_1\tau+p_1^*(t-\tau),\, p_1^*,\, q_2+p_2\tau+p_2^*(t-\tau),\, p_2^*) -$$

$$- f_2(q_1+p_1t,\, p_1,\, q_2+p_2t,\, p_2)\Big\}\left(-p_1\frac{\partial}{\partial q_1} - p_2\frac{\partial}{\partial q_2}\right)\varphi_2(q_1,p_1,q_2,p_2) +$$

$$+ \int dq_1\, dp_1\, dq_2\, dp_2 \frac{1}{4\pi}\int_{S_-^2} d\eta\, |\eta\cdot(p_1-p_2)|\,\delta(q_1-q_2)\times$$

$$\times [f_2(q_1+p_1^*t,\, p_1^*,\, q_2+p_2^*t,\, p_2^*) -$$

$$- f_2(q_1+p_1t,\, p_1,\, q_2+p_2t,\, p_2)]\,\varphi_2(q_1,p_1,q_2,p_2). \tag{I.25}$$

Formula (I.25) can be interpretated in the sense of the generalized functions as follows

$$\frac{d}{dt}S_2(t)f_2(x_1,x_2) = \left(p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}\right)S_2(t)f_2(x_1,x_2)\times$$

$$\times\,\theta(-\eta\cdot(p_1-p_2))\,\delta(\tau)[f_2(q_1+p_1^*t,\, p_1^*,\, q_2+p_2^*t,\, p_2^*) -$$

$$- f_2(q_1+p_1t,\, p_1,\, q_2+p_2t,\, p_2)].$$

Formula (I.25) can be considered as justification of formulas (2.6), (2.9), (2.10).

Now consider the following test function

$$\varphi_2(x_1,x_2) = \varphi_1(x_1) + \varphi_2(x_2)$$

and consider again (I.25). Using symmetricity of $S_2(t)f_2(x_1,x_2)$ with respect to permutation $(f_2(x_1,x_2) = f_2(x_2,x_1),\ S_2(t)f_2(x_1,x_2) = S_2(t)f_2(x_2,x_1))$ we obtain

$$\frac{d}{dt}(S_2(t)f_2,\varphi_2) = \left(\frac{d}{dt}S_2(t)f_2,\varphi_2\right) =$$

$$= 2\int dq_1\, dp_1\, dq_2\, dp_2\left\{f_2(q_1+p_1t,\, p_1,\, q_2+p_2t,\, p_2) +\right.$$

$$+ \frac{1}{4\pi}\int_0^t d\tau\int_{S_-^2} d\eta\,\delta(q_1+p_1\tau-q_2-p_2\tau)\,|\eta\cdot(p_1-p_2)|\times$$

$$\times [f_2(q_1+p_1\tau+p_1^*(t-\tau),\, p_1^*,\, q_2+p_2\tau+p_2^*(t-\tau),\, p_2) -$$

$$- f_2(q_1+p_1t,\, p_1,\, q_2+p_2t,\, p_2)\Big\}\left(-p_1\frac{\partial}{\partial q_1}\varphi_1(q_1,p_1)\right) +$$

$$+ 2\int dq_1\, dp_1\, dq_2\, dp_2\frac{1}{4\pi}\int_{S_-^2} d\eta\,|\eta\cdot(p_1-p_2)|\,\delta(q_1-q_2)\times$$

$$\times [f_2(q_1+p_1^*t,\, p_1^*,\, q_2+p_2^*t,\, p_2^*) -$$

$$- f_2(q_1+p_1t,\, p_1,\, q_2+p_2t,\, p_2)]\,\varphi_1(q_1,p_1) =$$

$$= \int dq_1\, dp_1\, F_1(t,q_1,p_1)\left(-p_1\frac{\partial}{\partial q_1}\varphi_1(q_1,p_1)\right) +$$

$$+ \int dq_1\, dp_1 \frac{1}{4\pi} \int\limits_{S_-^2} d\eta \int dq_2\, dp_2\, |\eta \cdot (p_1 - p_2)|\, \delta(q_1 - q_2) \times$$

$$\times [F_2(t, q_1, p_1^*, q_2, p_2^*) - F_2(t, q_1, p_1, q_2, p_2)]\, \varphi_1(q_1, p_1). \qquad (I.26)$$

Here,

$$F_1(t, x_1) = F_1(t, p_1, q_1) = 2 \int dq_2\, dp_2 \Bigg\{ f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2) +$$

$$+ \frac{1}{4\pi} \int\limits_0^t d\tau \int\limits_{S_-^2} d\eta\, \delta(q_1 + p_1 \tau - q_2 - p_2 \tau)\; |\eta \cdot (p_1 - p_2)| \times$$

$$\times [\, f_2(q_1 + p_1 \tau + p_1^* (t - \tau), p_1^*, q_2 + p_2 \tau + p_2^* (t - \tau), p_2^*) -$$

$$- f_2(q_1 + p_1 t, p_1, q_2 + p_2 t, p_2)] \Bigg\}, \qquad (I.27)$$

$$F_2(t, x_1, x_2) = F_2(t, q_1, p_1, q_2, p_2) = 2 S_2(t) f_2(x_1, x_2),$$

$$(S_2(t) f_2, \varphi_2) = (F_1(t), \varphi_1) = \int dq_1\, dp_1\, F_1(t, q_1, p_1)\, \varphi_1(q_1, p_1).$$

The function $F_1(t, x_1)$ and $F_2(t, x_1, x_2)$ are the one-particle and two-particle distribution functions correspondingly.

Taking into account (I.26), relation (I.27) can be represented as equation connecting $F_1(t, x_1)$ and $F_2(t, x_1, x_2)$. Namely, we have, in the sense of generalized functions,

$$\frac{\partial}{\partial t} F_1(t, x_1) = p_1 \frac{\partial}{\partial q_1} F_1(t, x_1) +$$

$$+ \frac{1}{4\pi} \int\limits_{S_-^2} d\eta \int dq_2\, dp_2\, |\eta \cdot (p_1 - p_2)|\, \delta_2(q_1 - q_2) \times$$

$$\times [F_2(t, q_1, p_1^*, q_2, p_2^*) - F_2(t, q_1, p_1, q_2, p_2)],$$

$$\frac{\partial}{\partial t} F_2(t, x_1, x_2) = \left(p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2} \right) F_2(t, x_1, x_2) +$$

$$+ \delta(q_1 - q_2)\, \theta(-\eta \cdot (p_1 - p_2))\, |\eta \cdot (p_1 - p_2)| \times$$

$$\times [F_2(t, q_1, p_1^*, q_2, p_2^*) - F_2(t, q_1, p_1, q_2, p_2)]. \qquad (I.28)$$

Note that expression for $F_1(t, x_1)$ consists from two terms: the first one is the one-particle distribution function of the free two-particle system, the second one is contribution of the difference

$$S_2(t) f_2(x_1, x_2) - S_2^0(t) f_2(x_1, x_2)$$

which is different from zero on the hyperplane $q_1 - q_2 = \tau(p_1 - p_2)$, $0 \le \tau \le t$.

The contribution is equal to the integral over this hyperplane with the measure $|\eta \cdot (p_1 - p_2)|$ and the averaging procedure with respect to the random vector η is performed. The contribution was determined through the limit of average over sphere as diameter a tends to zero. The definition of the one-particle distribution function is based on the taking into account the contribution from the hyperplane $q_1 - q_2 = \tau(p_1 - p_2)$, $0 \le \le \tau \le t$. This circumstance is crucial in the definition of the one-particle distribution function, because in the classical statistical mechanics the sets of lower dimension than phase space are neglected.

Now consider N-particle systems. We have the following representation for an infinitesimal t, continuous $f_N(x_1, \ldots, x_N)$, and test function $\varphi_N(x_1, \ldots, x_N)$

$$(S_N(t) f_N, \varphi_N) = (S_N^0(t) f_N, \varphi_N) +$$
$$+ \sum_{i<j=1}^{N} \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta_{ij}\, dx_1 \ldots dx_N\, |\eta_{ij} \cdot (p_i - p_j)|\, \delta(q_i - q_j) \times$$
$$\times [f_N(q_1 + p_1(t-\tau), p_1, \ldots, q_i + p_i^*(t-\tau), p_i^*, \ldots,$$
$$q_j + p_j^*(t-\tau), p_j^*, \ldots, q_N + p_N(t-\tau), p_N) -$$
$$- f_N(q_1 + p_1(t-\tau), p_1, \ldots, q_i + p_i(t-\tau), p_i, \ldots,$$
$$q_j + p_j(t-\tau), p_j, \ldots, q_N + p_N(t-\tau), p_N)] \times$$
$$\times \varphi_N(q_1 - p_1\tau, p_1, \ldots, q_i - p_i\tau, p_i, \ldots, q_j - p_j\tau, p_j, \ldots, q_N - p_N\tau, p_N). \tag{I.29}$$

Unfortunately we have not yet the representation of the functional $(S_N(t) f_N, \varphi_N)$ for arbitrary t.

From representation (I.29) it follows that

$$(S_N(t) f_N, \varphi_N) = (f_N, S_N(-t) \varphi_N),$$
$$\frac{d}{dt}(S_N(t) f_N, \varphi_N)\Big|_{t=0} = \left(\frac{d}{dt} S_N(t) f_N, \varphi_N\right)\Big|_{t=0} =$$
$$= \frac{d}{dt}(f_N, S_N(-t) \varphi_N)\Big|_{t=0} = \left(f_N, \frac{d}{dt} S_N(-t) \varphi_N\right)\Big|_{t=0}. \tag{I.30}$$

These formulas are the analog of formulas (I.18), (I.20), (I.22) for two-particle systems.

Finally, we have

$$\frac{d}{dt}(S_N(t) f_N, \varphi_N)\Big|_{t=0} =$$
$$= \int dx_1 \ldots dx_N\, f_N(x_1, \ldots, x_N) \left(-\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} \varphi_N(x_1, \ldots, x_N)\right) +$$
$$+ \sum_{i<j=1}^{N} \int_{S_-^2} d\eta_{ij} \int dx_1 \ldots dx_N\, |\eta_{ij} \cdot (p_i - p_j)|\, \delta(q_i - q_j) \times$$
$$\times [f_N(q_1, p_1, \ldots, q_i, p_i^*, \ldots, q_j, p_j^*, \ldots, q_N, p_N) -$$
$$- f_N(q_1, p_1, \ldots, q_i, p_i, \ldots, q_j, p_j, \ldots, q_N, p_N)] \times$$
$$\times \varphi_N(q_1, p_1, \ldots, q_i, p_i, \ldots, q_j, p_j, \ldots, q_N, p_N). \tag{I.31}$$

If

$$\varphi_N(x_1, \ldots, x_N) = \sum_{i_1 < i_2 < \ldots < i_s} \varphi_s(x_{i_1}, \ldots, x_{i_s}),$$

then we obtain from (I.31)

$$(S_N(t) f_N, \varphi_N) = \frac{1}{s!} N(N-1)\ldots(N-s+1)(S_N(t) f_N, \varphi_s) =$$

$$= \frac{1}{s!} N(N-1)\ldots(N-s+1) \int dx_1 \ldots dx_s \times$$

$$\times \left\{ \int f_N(q_1 + p_1 t, p_1, \ldots, q_s + p_s t, p_s, q_{s+1}, p_{s+1}, \ldots, q_N, p_N) \, dx_{s+1} \ldots dx_N \right\} \times$$

$$\times \varphi_s(q_1, p_1, \ldots, q_s, p_s) +$$

$$+ \frac{1}{s!} N(N-1)\ldots(N-s+1) \int dx_1 \ldots dx_s \times$$

$$\times \left\{ \int \sum_{i<j=1}^{s} \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta_{ij} \, |\eta_{ij} \cdot (p_i - p_j)| \, \delta(q_i + p_i \tau - q_j - p_j \tau) \times \right.$$

$$\times [f_N(q_1 + p_1 t, p_1, \ldots, q_i + p_i \tau + p_i^*(t-\tau), p_i^*, \ldots,$$

$$q_j + p_j \tau + p_j^*(t-\tau), p_j^*, \ldots, q_s + p_s t, p_s, q_{s+1}, p_{s+1}, \ldots, q_N, p_N) -$$

$$- f_N(q_1 + p_1 t, p_1, \ldots, q_i + p_i t, p_i, \ldots,$$

$$\left. q_j + p_j t, p_j, \ldots, q_s + p_s t, p_s, q_{s+1}, p_{s+1}, \ldots, q_N, p_N)] \, dx_{s+1} \ldots dx_N \right\} \times$$

$$\times \varphi_s(q_1, p_1, \ldots, q_s, p_s) +$$

$$+ \frac{1}{s!} N(N-1)\ldots(N-s+1) \int dx_1 \ldots dx_s \times$$

$$+ \left\{ \sum_{i=1}^{s} \frac{1}{4\pi} \int_0^t d\tau \int_{S_-^2} d\eta_{is+1} \, |\eta_{is+1} \cdot (p_i - p_{s+1})| \, \delta(q_i + p_i \tau - q_{s+1} - p_{s+1} \tau) \times \right.$$

$$\times [f_N(q_1 + p_1 t, p_1, \ldots, q_i + p_i \tau + p_i^*(t-\tau), p_i^*, \ldots$$

$$\ldots, q_{s+1} + p_{s+1} \tau + p_{s+1}^*(t-\tau), p_{s+1}^*, q_{s+2}, p_{s+2}, \ldots, q_N, p_N) -$$

$$- f_N(q_1 + p_1 t, p_1, \ldots, q_i + p_i t, p_i, \ldots$$

$$\left. \ldots, q_{s+1} + p_{s+1} t, p_{s+1}, q_{s+2}, p_{s+2}, \ldots, q_N, p_N)] \, dx_{s+1} \ldots dx_N \right\} \times$$

$$\times \varphi_s(q_1, p_1, \ldots, q_s, p_s) =$$

$$= \frac{1}{s!} \int dx_1 \ldots dx_s \, F_s(t, x_1, \ldots, x_s) \, \varphi_s(x_1, \ldots, x_s). \tag{I.32}$$

The s-particle distribution function $F_s(t, x_1, \ldots, x_s)$ defined according to (I.32) does not depend on any random vectors. The s-particle distribution function which depends on the random vector of the s-particle subsystem has the following representation (we preserve for it the same denotation)

$$F_s(t, x_1, \ldots, x_s) = N(N-1)\ldots(N-s+1) \times$$

$$\times \frac{1}{s!}\int S_s(t, x_1, \ldots, x_s)\, f_N(x_1, \ldots, x_s, x_{s+1}, \ldots, x_N)\, dx_{s+1}\ldots dx_N \;+$$

$$+ \sum_{i=1}^{s} \int \left\{ \frac{1}{4\pi}\int_0^t d\tau \int_{S_-^2} d\eta_{is+1}\, |\eta_{is+1}\cdot(p_i - p_{s+1})|\, \delta(q_i + p_i\tau - q_{s+1} - p_{s+1}\tau) \times \right.$$

$$\times [\, f_N(q_1 + p_1 t, p_1, \ldots, q_i + p_i\tau + p_i^*(t-\tau), p_i^*, q_{i+1} + p_{i+1}t, p_{i+1}, \ldots$$

$$\ldots, q_{s+1} + p_{s+1}\tau + p_{s+1}^*(t-\tau), p_{s+1}^*, q_{s+2}, p_{s+2}, \ldots, q_N, p_N) \;-$$

$$-\; f_N(q_1 + p_1 t, p_1, \ldots, q_i + p_i t, p_i, \ldots$$

$$\left. q_{s+1} + p_{s+1}t, p_{s+1}, q_{s+2}, p_{s+2}, \ldots, q_N, p_N)] \right\} dx_{s+1}\ldots dx_N. \qquad \text{(I.33)}$$

Remark that the distribution functions $F_1(t, x_1), \ldots, F_s(t, x_1, \ldots, x_s), \ldots$ used in this Appendix differ from those defined in Section 3 by $(-t)$. For the sake of simplicity we preserve the same denotation as in Section 3.

APPENDIX II. In this appendix, we give a rigorous justification of derivation of relations (6.6) – (6.8).

In order to give mathematical meaning to derivative (6.7), (6.7′), it is necessary to prove that the integrand on the right-hand side of (6.7′) is well defined and the integral exists. It was shown in Section 2 that the integrand is a bounded function with compact support if $f_{s+1} \subset L_{s+1}^0$. Moreover, it is continuous with respect to time t and $(x_1, \ldots, x_s, x_{s+1})\,|_{q_{s+1}=q_i}$ on the intervals between collisions. Thus, the integral on the right-hand side of (6.7′) exists and relations (6.7), (6.7′) are proved.

In the general case, the expression

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \left\{ S(t_1)\mathcal{A}S(-t_1)\ldots S(t_n)\mathcal{A}S(-t_n) f \right\}_s (x_1, \ldots, x_s) \qquad \text{(II.1)}$$

is equal to the sum of the following integrals (for details, see [4, 5]):

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \int_{S_+^2} d\eta_1 \ldots \int_{S_+^2} d\eta_n \int dp_{s+1} \ldots \int dp_{s+n} \times$$

$$\times\, \eta_1\cdot(P_{i_1}(t_1) - p_{s+1}) \ldots \eta_n\cdot(P_{i_n}(t_n - t_{n-1}) - p_{s+n}) \times$$

$$\times\, f_{s+n}\Big(X^{(s+n)}(-t_n, X^{(s+n-1)}(t_n - t_{n-1}, X^{(s+n-2)}(t_{n-1} - t_{n-2}, \ldots$$

$$\ldots, X^{(s+1)}(t_2 - t_1, X^{(s)}(t_1, x^{(s)}); Q_{i_1}(t_1, x^{(s)}), p_{s+1}), \ldots); \ldots$$

$$\ldots; Q_{i_{n-1}}(t_{n-1} - t_{n-2}, \ldots), p_{s+n-1}); Q_{i_n}(t_n - t_{n-1}, \ldots), p_{s+n})\Big). \qquad \text{(II.2)}$$

Here, $X^{(s+k)} = (X_1, \ldots, X_{s+k})$, $i_1, \ldots, i_n \subset (1, \ldots, s)$, and some numbers $i_1, \ldots$ $\ldots, i_n$ may coincide.

In order to prove that integral (II.2) exists, we show that the integrand is a bounded function with compact support and that it is continuous with respect to $t_1, \ldots, t_n$, $\eta_1, \ldots, \eta_n$, $q_1, \ldots, q_s, p_1, \ldots, p_s, p_{s+1}$ on the intervals between pair collisions. We suppose that $f_{s+1} \subset L^0_{s+1}$.

It was shown in Section 2 that the integrand is a bounded function with compact support for $f_{s+1} \subset L^0_{s+1}$.

The trajectories

$$\begin{gathered}
X^{(s+n)}(-t_n, X^{(s+n-1)}(t_n - t_{n-1}, X^{(s+n-2)}(t_{n-1} - t_{n-2}, \ldots, X^{(s+1)}(t_2 - t_1, X^{(s)}(t_1, x^{(s)}); \\
Q_{i_1}(t_1, x^{(s)}), p_{s+1}), \ldots); \ldots; Q_{i_{n-1}}(t_{n-1} - t_{n-2}, \ldots), p_{s+n-1}); Q_{i_n}(t_n - t_{n-1}, \ldots), p_{s+n}), \\
\vdots \\
X^{(s+1)}(t_2 - t_1, X^{(s)}(t_1, x^{(s)}); Q_{i_1}(t_1, x^{(s)}), p_{s+1}), \\
X^{(s)}(t_1, x^{(s)})
\end{gathered} \tag{II.3}$$

are continuous with respect to time and the initial phase points lying outside the hyperplanes that define pair collisions in systems of $s+n$, $s+n-1, \ldots,$ s particles

$$\begin{aligned}
&Q_i(t_n) - Q_j(t_n) = 0, && (i,j) \subset (1, \ldots, s+n); \\
&Q_i(t_n - t_{n-1}) - Q_j(t_n - t_{n-1}) = 0, && (i,j) \subset (1, \ldots, s+n-1); \\
&\qquad \vdots \\
&Q_i(t_1) - Q_j(t_1) = 0, && (i,j) \subset (1, \ldots, s).
\end{aligned} \tag{II.4}$$

The corresponding initial phase points are indicated in (II.2) and in (II.3); but in (II.4), they are omitted for the sake of simplicity.

We show that the dimensionality of all these hyperplanes is lower than dimensionality of the subspace of the initial phase points $(q_1, \ldots, q_s, p_1, \ldots, p_s, p_{s+1}, \ldots, p_{s+n})$ and the random vectors η.

For this purpose, we consider hyperplanes (II.4) for fixed $q_1, \ldots, q_s$ in the spaces of the corresponding initial momenta

$$\begin{gathered}
P_i(t_n - t_{n-1}), \quad i = 1, \ldots, s+n, \\
P_i(t_{n-1} - t_{n-2}), \quad i = 1, \ldots, s+n-1, \ldots, \ p_1, \ldots, p_s.
\end{gathered} \tag{II.5}$$

It is obvious that the dimensionality of hyperplanes (II.4) is lower than the dimensionality of the space of the corresponding initial momenta (II.5) for their fixed initial positions

$$\begin{gathered}
Q_i(t_n - t_{n-1}), \quad i = 1, \ldots, s+n, \\
Q_i(t_{n-1} - t_{n-2}), \quad i = 1, \ldots, s+n-1, \ldots, q_1, \ldots, q_s.
\end{gathered}$$

Indeed, the hyperplanes (II.4) can be represented in the form

$$\begin{aligned}
&Q_i(t_n - t_{n-1}) + P_i(t_n - t_{n-1})\, t_n - \\
&\qquad - Q_j(t_n - t_{n-1}) - P_j(t_n - t_{n-1})\, t_n = 0, \quad (i,j) \subset (1, \ldots, s+n); \\
&Q_i(t_{n-1} - t_{n-2}) + P_i(t_{n-1} - t_{n-2})(t_n - t_{n-1}) - \\
&\quad - Q_j(t_{n-1} - t_{n-2}) - P_j(t_{n-1} - t_{n-2})(t_n - t_{n-1}) = 0, \quad (i,j) \subset (1, \ldots, s+n-1); \\
&\qquad \vdots \\
&q_i + p_i t_1 - q_i - p_i t_1 = 0, \qquad (i,j) \subset (1, \ldots, s),
\end{aligned} \tag{II.6}$$

if we have the first pair collisions in each system (the general case can be considered analogously).

Hyperplanes (II.6) considered in the space of momenta for fixed positions define the unique sequence of differences of vectors

$$P_i(t_n - t_{n-1}) - P_j(t_n - t_{n-1}),\ P_i(t_{n-1} - t_{n-2}) - P_j(t_{n-1} - t_{n-2}), \ldots, p_i - p_j. \tag{II.7}$$

Their components perpendicular to the differences

$$Q_i(t_n - t_{n-1}) - Q_j(t_n - t_{n-1}),\quad Q_i(t_{n-1} - t_{n-2}) - Q_j(t_{n-1} - t_{n-2}), \ldots, q_i - q_j \tag{II.8}$$

are equal zero, i.e., vectors (II.7) are parallel to the corresponding vectors (II.8).

For given times of collisions $\tau_n, \tau_{n-1}, \ldots, \tau_1$, the components parallel to vectors (II.8) are defined as follows:

$$\frac{Q_i(t_n - t_{n-1}) - Q_j(t_n - t_{n-1})}{\tau_n},\quad \frac{Q_i(t_{n-1} - t_{n-2}) - Q_j(t_{n-1} - t_{n-2})}{\tau_{n-1}},\ \ldots,\ \frac{q_i - q_j}{\tau_1}.$$

The momenta $P_i(t_n - t_{n-1})$, $i = 1, \ldots, s+n$, $P_i(t_{n-1} - t_{n-2})$, $i = 1, \ldots, s+n-1, \ldots$, are obtained from the momenta

$$\begin{aligned} &P_i(t_{n-1} - t_{n-2}), \quad i = 1, \ldots, s+n, \\ &P_i(t_{n-2} - t_{n-3}), \quad i = 1, \ldots, s+n-1, \ldots, \end{aligned} \tag{II.9}$$

respectively, by a linear transformation with Jacobian equal to unity. Therefore, hyperplanes of lower dimensionality with respect to the momenta $P_i(t_n - t_{n-1})$, $i = 1, \ldots, s+n$, $P_i(t_{n-1} - t_{n-2})$, $i = 1, \ldots, s+n-1, \ldots$, are also hyperplanes of lower dimensionality with respect to the momenta $P_i(t_{n-1} - t_{n-2})$, $i = 1, \ldots, s+n$, $P_i(t_{n-2} - t_{n-3})$, $i = 1, \ldots, s+n-1, \ldots$.

If momenta (II.7) are expressed by linear transformation (2.1) in terms of momenta fixed at previous collisions, then hyperplanes (II.6) define a set of lower dimensionality with respect to the random vectors η.

Repeating these considerations, we establish that the dimensionality of hyperplanes (II.4) is lower than the dimensionality of the momenta $p_1, \ldots, p_{s+n}$ and the random vectors η for fixed $q_1, \ldots, q_s$.

Outside hyperplanes (II.4), trajectories (II.3) are continuous functions of $t_1, \ldots, t_n$, $q_1, \ldots, q_s$, $p_1, \ldots, p_s, p_{s+1}, \ldots, p_{s+n}$ and the random vectors η that correspond to pair collisions. Hyperplanes (II.4) define times of collisions for fixed initial phase points and fixed random vectors η.

Note that, for fixed positions $q_1, \ldots, q_s$, the times of collisions are defined and finite only on the set of lower dimensionality in the space of momenta $p_1, \ldots, p_{s+n}$ and the random vectors η. Outside this set, the integrand is a continuous function with respect to the momenta $p_1, \ldots, p_{s+n}$, random vectors η, and variables $t_1, \ldots, t_n$.

If some particles have the same initial positions (II.8) and different momenta, then they do not collide with each other until they collide with other particles.

Therefore, the obtained result means that *the integrand in* (II.2) *is continuous with respect to the variables* $t_1, \ldots, t_n$, $p_1, \ldots, p_s, p_{s+1}, \ldots, p_{s+n}$, η *on the intervals between collisions for fixed* $q_1, \ldots, q_s$.

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