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STOCHASTIC DYNAMICS AND BOLTZMANN HIERARCHY. II

СТОХАСТИЧНА ДИНАМІКА І ІЄРАРХІЯ БОЛЬЦМАНА. II

Stochastic dynamics corresponding to the Boltzmann hierarchy is constructed. The Liouville–Itô equations are obtained, from which we derive the Boltzmann hierarchy regarded as an abstract evolution equation. We construct the semigroup of evolution operators and prove the existence of solutions of the Boltzmann hierarchy in the space of sequences of integrable and bounded functions. On the basis of these results, we prove the existence of global solutions of the Boltzmann equation and the existence of the Boltzmann–Grad limit for an arbitrary time interval.

Побудована стохастична динаміка, яка відповідає ієрархії Больцмана. Отримані рівняння Ліувілля–Іто, а з них виведена ієрархія Больцмана, яка розглядається як абстрактне еволюційне рівняння. Побудована півгрупа еволюційних операторів і доведено існування розв'язків ієрархії Больцмана в просторі послідовностей інтегровних та обмежених функцій. На цій основі доведено існування глобальних розв'язків рівняння Больцмана та існування границі Больцмана–Греда на довільному інтервалі часу.

4. Representation of the sequence $F(t)$ through $F(0)$. Formula (3.11) expresses the sequence $F(t)$ via the sequence

$$D(t) = (D_1(t, x_1), D_2(t, x_1, x_2), \ldots, D_N(t, x_1, \ldots, x_N), \ldots) =$$

$$= (S_1(-t)D_1(0, x_1), S_2(-t)D_2(0, x_1, x_2), \ldots, S_N(-t)D_N(0, x_1, \ldots, x_N), \ldots). \tag{4.1}$$

Let us express the sequence $F(t)$ via $F(0)$. For this purpose, we introduce the space L which is a direct sum of the spaces L_s,

$$L = \sum_{s=0}^{\infty} \oplus L_s. \tag{4.2}$$

The elements of L are the sequences

$$f = (f_0, f_1(x_1), \ldots, f_s(x_1, \ldots, x_s), \ldots) \tag{4.3}$$

of symmetric functions $f_s \in L_s$, and f_0 is a complex number.

The norm in L is the sum of the norms of f_s in L_s,

$$\|f\| = \sum_{s=0}^{\infty} \|f_s\|_s, \quad \|f_0\|_0 = |f_0|. \tag{4.4}$$

Define an operator $\int dx$ in L as follows:

$$\left(\int dx\, f\right)_s (x_1, \ldots, x_s) = \int f_{s+1}(x_1, \ldots, x_s, x_{s+1})\, dx_{s+1}. \tag{4.5}$$

The operator $\int dx$ is bounded in L and $\left\|\int dx\right\| = 1$. The exponents $e^{\int dx}$, $e^{-\int dx}$ exist and $\left\|e^{\pm\int dx}\right\| < e$.

Let us define the second operator $S(-t)$ in L as the direct sum of the operators $S_s(-t)$, $t > 0$,

$$S(-t) = \sum_{s=0}^{\infty} S_s(-t), \quad S_0(-t) = I. \tag{4.6}$$

The operator $S(-t)$ is formally defined as follows

$$S(-t)f = (S_0(-t)f_0, S_1(-t)f_1(x_1), \ldots, S_s(-t)f_s(x_1, \ldots, x_s), \ldots). \tag{4.7}$$

The operator $S(-t)$ forms a formal semigroup together with the operators $S_s(-t)$.

Now represent the sequence $F(t)$ via $S(-t)$, $D(0)$ and the operator $\int dx$. By definition (4.1), we have

$$D(t) = S(-t)D(0). \tag{4.8}$$

It follows from definition (3.11) that

$$F_s(0, x_1, \ldots, x_s) =$$

$$= \frac{1}{\Xi} M \sum_{N=0}^{\infty} \frac{1}{N!} \int D_{s+N}(0, x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+N})\, dx_{s+1} \ldots dx_{s+N} =$$

$$= \frac{1}{\Xi} M \left(e^{\int dx} D(0)\right)_s (x_1, \ldots, x_s), \tag{4.9}$$

$$F(0) = \frac{1}{\Xi} M\, e^{\int dx} D(0), \qquad \Xi = M \left(e^{\int dx} D(0)\right)_0.$$

The sequence $D(0)$ does not depend on the random variables η and, therefore, the operation M may be omitted in (4.9). In what follows, we will meet a situation where the initial time t_0 is different from zero, sequences $D(t_0)$ and $F(t_0)$ depend on the random vectors η, and the operation M cannot be omitted in (4.9) with t_0 instead of $t_0 = 0$. We represent formula (3.11) in the following way:

$$F_s(t, x_1, \ldots, x_s) = \frac{1}{\Xi} \left(M e^{\int dx} S(t)\, D(0)\right)_s (x_1, \ldots, x_s).$$

For the sequence $F(t)$, we get

$$F(t) = \frac{1}{\Xi}\, M e^{\int dx} S(-t)\, D(0). \tag{4.10}$$

The operator $M e^{\int dx}$ has an inverse one, which is equal to $M e^{-\int dx}$. The second formula in (4.9) yields

$$D(0) = \Xi\, M e^{-\int dx} F(0). \tag{4.11}$$

Substituting (4.11) into (4.10), we obtain a desired expression of $F(t)$ via $F(0)$

$$F(t) = M e^{\int dx} S(-t)\, e^{-\int dx} F(0) \tag{4.12}$$

or, componentwise,

$$F_s(t, x_1, \ldots, x_s) =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,k!} M \int S_{s+n-k}(-t, x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+n-k}) \times$$

$$\times\ F_{s+n}(0, x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+n-k}, \ldots, x_{s+n})\, dx_{s+1} \ldots dx_{s+n}. \qquad (4.13)$$

We perform average over the random parameters $\eta_{l_s+1}, \ldots, \eta_{l_{nk}}$ that correspond to collisions of the particles with the numbers $1, \ldots, s, s+1, \ldots, s+n-k$ except the collisions between the particles with the numbers $1, \ldots, s$ and the random vectors of the last subsystem are arbitrary fixed. The operator $S_{s+n-k}(-t, x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+n-k})$ corresponds to the system consisting of $s+n-k$ particles with the initial phase points $x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+n-k}$. The problem of convergence of series (4.13) in the space L and other spaces will be discussed in a separate paper.

5. Group of operators of evolution and derivation of infinitesimal operator. Define an operator of evolution as follows:

$$U(t) = M e^{\int dx} S(-t)\, e^{-\int dx}, \quad t \geq 0. \qquad (5.1)$$

Note that operation of averaging is carried out with respect to the random vectors η which correspond to collisions of the particles, over whose phase points we integrate, while phase points of all other particles and the other random vectors η are fixed.

In terms of the operator of evolution $U(t)$, the state $F(t)$ at time t is expressed via the state $F(0)$ at the initial time $t = 0$ (see formula (4.12))

$$F(t) = U(t) F(0). \qquad (5.2)$$

Let us show that the operators $U(t)$, $0 \leq t < \infty$, have the semigroup property

$$U(t_1 + t_2) = U(t_1)\, U(t_2) = U(t_2)\, U(t_1) \qquad (5.3)$$

for arbitrary $t_1 \geq 0$, $t_2 \geq 0$.

Indeed,

$$U(t_1)\, U(t_2) = M e^{\int dx} S(-t_1)\, e^{-\int dx} M e^{\int dx} S(-t_2)\, e^{-\int dx} =$$

$$= M e^{\int dx} S(-t_1)\, e^{-\int dx} e^{\int dx} S(-t_2)\, e^{-\int dx} =$$

$$= M e^{\int dx} S(-t_1 - t_2)\, e^{-\int dx} = U(t_1 + t_2). \qquad (5.4)$$

Here, we have used the semigroup property of $S(-t)$ and the commutativity of the operations $M \int dx$.

Thus, the operators $U(t)$ are defined in L for $t \geq 0$ and form a semigroup. The problem of the existence of the semigroup $U(t)$ will be discussed in a separate paper. In the given paper we restrict ourselves only to formal properties of $U(t)$. Let us determine the infinitesimal operator of the semigroup $U(t)$. By definition,

$$Bf = \lim_{t \to 0} \frac{1}{t} \left(U(t) f - f \right) = \lim_{t \to 0} \frac{1}{t} M \Big\{ S(-t)\, f - f + \int dx\, S(-t)\, f -$$

$$- S(t) \int dx\, f + \sum_{n=2}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)!} \left(\int dx \right)^{n-k} S(-t) \left(\int dx \right)^k f \Big\} =$$

$$= \lim_{t \to 0} \frac{1}{t} M \Big\{ [S(-t)\, f - f] + \int dx\, [S(-t)\, f - f] - [S(-t) - I] \int dx\, f +$$

$$+ \sum_{n=2}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \left(\int dx\right)^{n-k} [S(-t) - I] \left(\int dx\right)^k f \Big\} =$$

$$= \mathcal{H} f + M \int dx \mathcal{H} f - \mathcal{H} \int dx f +$$

$$+ M \sum_{n=2}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \left(\int dx\right)^{n-k} \mathcal{H} \left(\int dx\right)^k f, \tag{5.5}$$

where $\mathcal{H}$ is the infinitesimal operator of the semigroup $S(-t)$, which is equal to the direct sum of the infinitesimal operators of the semigroups $S_s(-t)$.

Consider the projection of (5.5) on the s-particle subspace

$$(Bf)_s(x_1, \ldots, x_s) = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} f_s(x_1, \ldots, x_s) +$$

$$+ \sum_{i<j=1}^{s} \delta(q_i - q_j)\, \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times [f_s(x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_s) - f_s(x_1, \ldots, x_s)] +$$

$$+ \sum_{i=1}^{s} \int dp_{s+1} d\eta_{is+1}\, \eta_{is+1} \cdot (p_i - p_{s+1})\, \theta(\eta_{is+1} \cdot (p_i - p_{s+1})) \times$$

$$\times [f_{s+1}(x_1, \ldots, x_i^*, \ldots, x_s, x_{s+1}^*) - f_{s+1}(x_1, \ldots, x_s, x_{s+1})]\big|_{q_{s+1} = q_i} +$$

$$+ M \sum_{n=2}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \int dx_{s+1} \ldots dx_{s+n} \mathcal{H}_{s+n-k}(x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+n-k}) \times$$

$$\times f_{s+n}(x_1, \ldots, x_{s+n}), \tag{5.6}$$

where $\mathcal{H}_{s+n-k}(x_1, \ldots, x_{s+n-k})$ is the infinitesimal operator of $S_{s+n-k}(-t, x_1, \ldots$ $\ldots, x_{s+n-k})$ (for the sake of simplicity, we denote $\mathcal{H}^-_{s+n-k}$ by $\mathcal{H}_{s+n-k}$).

Let us prove that the last term on the right-hand side of (5.6) is equal to zero. For this purpose, we represent the operator $\mathcal{H}_{s+n-k}(x_1, \ldots, x_{s+1}, x_{s+2}, \ldots, x_{s+n-k})$ as a sum of three operators

$$\mathcal{H}_{s+n-k}(x_1, \ldots, x_{s+1}, x_{s+2}, \ldots, x_{s+n-k}) =$$

$$= \mathcal{H}_s(x_1, \ldots, x_s) + \mathcal{H}_{n-k}(x_{s+1}, \ldots, x_{s+n-k}) +$$

$$+ \mathcal{H}_{s,n-k}(x_1, \ldots, x_s; x_{s+1}, \ldots, x_{s+n-k}), \tag{5.7}$$

where $\mathcal{H}_s(x_1, \ldots, x_s)$ is the infinitesimal operator of $S_s(-t, x_1, \ldots, x_s)$, $\mathcal{H}_{n-k}(x_{s+1}, \ldots$ $\ldots, x_{s+n-k})$ is the infinitesimal operator of $S_{s+n-k}(-t, x_{s+1}, \ldots, x_{s+n-k})$, and

$$\mathcal{H}_{s,n-k}(x_1, \ldots, x_s; x_{s+1}, \ldots, x_{s+n-k}) f_{s+n}(x_1, \ldots, x_{s+n}) =$$

$$= \sum_{i=1}^{s} \sum_{j=s+1}^{n-k} \delta(q_i - q_j)\, \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times [f_{s+n}(x_1, \ldots, x_i^*, \ldots, x_s, x_{s+1}, \ldots, x_j^*, \ldots, x_{s+n-k}, \ldots, x_{s+n}) -$$

$$- f_{s+n}(x_1, \ldots, x_i, \ldots, x_s, x_{s+1}, \ldots, x_j, \ldots, x_{s+n-k}, \ldots, x_{s+n})]. \tag{5.8}$$

$$U(t) = \prod_{i=1}^{n} U(t_i - t_{i-1})$$

where $t_i = i\frac{t}{n}$, $t_0 = 0$, and $\frac{t}{n}$ is considered as an infinitesimal time. As it follows from Appendix I, 1.2, the operators $U(t_i - t_{i-1})$ can be defined explicitly together with operator $S_N(t_i - t_{i-1})$. We define the sequence of the distribution functions by expression

$$F(t) = \prod_{i=1}^{n} U(t_i - t_{i-1})\, F(0).$$

The derivative of the sequence $F(t)$ is equal to

$$\frac{dF(t)}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(F(t+\Delta t) - F(t)) =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}(U(\Delta t) - I)\, F(t) = (\mathcal{H} + \mathcal{A})\, F(t). \tag{5.12}$$

We have obtained the formal derivation of the stochastic hierarchy.

In order to define the hierarchy for the distribtion function averaged with respect to the all random vectors, including the random vectors of s-particle system, one should also integrate with respect to η_{ij}, $i < j \leq s$, in (5.9).

Thus, the infinitesimal operator B of the semigroup $U(t)$ coincides with the operator on the right-hand side of the stochastic hierarchy (3.10). Henceforth, we *identify the stochastic hierarchy* (3.10) *with the evolution equation*

$$dF(t) = B\,F(t), \quad F(t)|_{t=0} = F(0), \tag{5.13}$$

where B is the infinitesimal operator of the semigroup $U(t)$. Then *the stochastic hierarchy* (5.13) *has a formal solution in the space L*, which can be represented in the form

$$F(t) = U(t)\,F(0) = M e^{\int dx} S(-t)\, e^{-\int dx} F(0). \tag{5.14}$$

6. Representation of solutions by iteration series. We have obtained the formal solutions of the stochastic hierarchy (3.10) in the space L which are represented by formula (5.14). We also need solutions represented by iteration series. For this purpose, we use the stochastic hierarchy represented in an abstract form as the evolution equation (5.13) with the infinitesimal operator B (5.9)

$$\frac{dF(t)}{dt} = (\mathcal{H} + \mathcal{A})\, F(t), \quad F(t)|_{t=0} = F(0). \tag{6.1}$$

The operators $\mathcal{H}$ and $\mathcal{A}$ are defined by formulas (5.10), (5.11).

Let us represent the solution of the stochastic hierarchy in the form

$$F(t) = S(-t)\,\tilde{F}(t), \quad \tilde{F}(0) = F(0). \tag{6.2}$$

Then we obtain the following equations for $\tilde{F}(t)$:

$$\frac{d\tilde{F}(t)}{dt} = S(t)\mathcal{A}S(-t)\,\tilde{F}(t), \quad \tilde{F}(t)|_{t=0} = \tilde{F}(0)$$

or in the integral form

$$\tilde{F}(t) = F(0) + \int_0^t S(\tau)\,\mathcal{A}\,S(-\tau)\,\tilde{F}(\tau)\,d\tau. \tag{6.3}$$

By analogy with the Boltzmann equation, the solution of the integral equation (6.3) is called a *mild solution of the stochastic hierarchy.*

For $F(t)$, we obtain the following integral equation:

$$F(t) = S(-t)F(0) + \int_0^t S(-t+\tau)\,\mathcal{A}\,F(\tau)\,d\tau. \tag{6.3$'$}$$

Iterating these equations, we obtain

$$\tilde{F}(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, S(t_1)\mathcal{A}S(-t_1)S(t_2)\mathcal{A}S(-t_2)\ldots$$

$$\ldots S(t_n)\mathcal{A}S(-t_n)F(0) \tag{6.4}$$

or

$$F(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, S(-t+t_1)\mathcal{A}S(-t_1+t_2)\mathcal{A}S(-t_2+t_3)\ \ldots$$

$$\ldots\ S(-t_{n-1}+t_n)\mathcal{A}S(-t_n)F(0). \tag{6.5}$$

It is obvious that series (6.4) is the unique formal solution of the integral equation (6.3) and series (6.5) is the unique formal solution of hierarchy (6.3$'$).

In the space L, the solution of the stochastic hierarchy (6.1) is represented by formula (5.14). It follows from the uniqueness of solutions in the space L that representations (6.5) and (5.14) coincide and (6.5) is also a solution in L and, moreover, the integration with respect to $t_1, \ldots, t_n$ in each term of (6.5) can be performed exactly

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, S(-t+t_1)\mathcal{A}S(-t_1+t_2)\mathcal{A}S(-t_2+t_3)\ \ldots$$

$$\ldots\ S(-t_{n-1}+t_n)\mathcal{A}S(-t_n)F(0) =$$

$$= M \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)!} \left(\int dx\right)^{n-k} S(-t) \left(\int dx\right)^{k} F(0). \tag{6.6}$$

Furthermore, the last formula and the coincidence of representations (6.5) and (5.14) can be proved directly by using the following obvious identity:

$$\frac{d}{dt_j}\left[S(-t_{j-1}+t_j)\,M\int dx\,S(-t_j+t_{j+1})\right] = S(-t_{j-1}+t_j)\mathcal{A}S(-t_j+t_{j+1}) \tag{6.7}$$

or the projection of this identity onto the s-particle subspace

$$\frac{d}{dt_j}\left[S(-t_{j-1}+t_j)\,M\int dx\,S(-t_j+t_{j+1})\right] f_{s+1}(x_1,\ldots,x_s,x_{s+1}) =$$

$$= S_s(-t_{j-1}+t_j)\sum_{i=1}^{s}\int dp_{s+1}\,d\eta_{is+1}\,\eta_{is+1}\cdot(p_i-p_{s+1})\,\theta(\eta_{is+1}\cdot(p_i-p_{s+1}))\times$$

$$\times\, S_{s+1}(-t_j + t_{j+1})\,[\,f_{s+1}(x_1, \ldots, x_i^*, \ldots, x_s, x_{s+1}^*) - \\ - f_{s+1}(x_1, \ldots, x_i, \ldots, x_s, x_{s+1})]\,|_{q_{s+1} = q_i}, \tag{6.7$'$}$$

where the operation of averaging M is carried out with respect to the random variables which correspond to the collisions of the $s+1$-th particle with the other s particles.

Moreover, the following identity holds:

$$\frac{d}{dt_j}\Big[S(-t_{j-1}+t_j)\,M\int dx\,S(-t_j+t_{j+1})\Big]\,S(-t_{j-1}+t_j)\,M\int dx\,S(-t_j+t_{j+1}) =$$

$$= S(-t_{j-1}+t_j)\,M\int dx\,S(-t_j+t_{j+1})\,\frac{d}{dt_j}\Big[S(-t_{j-1}+t_j)\,M\int dx\,S(-t_j+t_{j+1})\Big] \,. \tag{6.8}$$

Identities (6.7) and (6.8) yield identity (6.6).

For details, we refer the reader to the book [8], where analogous calculations were performed for the case of hard spheres. The proof of identity (6.8) is also analogous to the derivation of the infinitesimal operator B in Section 5.

Thus, the representation by the evolution operator (4.13) and the representation by the iteration series (6.5) coincide term by term. This implies that the iteration series is convergent in the space L for all initial data $F(0) \subset L$ for which each term of the iteration series (6.5) is meaningful and series (4.13) converge.

It follows from Appendix II (see also Lemma 1 in our paper [5]) that, for the initial data $F(0) \subset L^0$ of differentiable functions with compact supports the integrand in each term of series (6.5) for $F_i(t, (x)_s)$ is a continuous function, with respect to the variables $q_1, \ldots, q_s, p_1, \ldots, p_s, p_{s+1}, \ldots, p_{s+n}, \eta_{l_2}, \ldots, \eta_{l_{s+n}}$ and $t_1, t_2, \ldots, t_n$, on the intervals between pair collisions, concentrated on compactum. Thus, for the initial data $F(0) \subset L^0$ the integration in each term of series (6.5) can be carried out exactly and, as a result, each term of series (6.5) coincides with the corresponding term of series (4.13).

It is easy to show that the seies (6.5) for $F_s(t, x_1, \ldots, x_s)$, $s = 1, 2, \ldots$, are uniformly convergent with respect to $(x_1, \ldots, x_s)$ on compacta globally in time for the set $X_{\xi, \beta}$ of sequences from the space L which consist from functions exponentially decreasing with respect to the squared momenta and coordinates [21, 22] with norm

$$\|f\| = \sup_{n \ge 0} \xi^{-n} \sup_{x_1, \ldots, x_n} |f_n(x_1, \ldots, x_n)| \exp\left\{\beta \sum_{i=1}^{n} (p_i^2 + q_i^2)\right\}. \tag{6.9}$$

7. Convergence of iteration series in the space E_ξ. Note that solutions in the space L describe states of finite systems with finite average number of particles. Indeed, we have for the average number $\langle N \rangle$

$$\langle N \rangle \sim \int F_1(t, x_1)\, dx_1 < \infty$$

because $F_1(t, x_1) \subset L^1$ if $F(0) \subset L$.

In order to describe the evolution of states of systems with infinite number of particles, it is necessary to use initial data from a functional space other than L. It is natural to regard the initial data $F(0)$ as certain perturbations of the equilibrium states of systems of particles interacting via a regular stable pair potential. It is well known

[8] that the equilibrium states of such systems (sequences of equilibrium distribution functions) belong to the space E_ξ which consists of the sequences

$$f = (f_1(x_1), f_2(x_1, x_2), \dots, f_s(x_1, \dots, x_s), \dots) \tag{7.1}$$

of symmetric functions with the norm

$$\|f\| \le \sup_{s \ge 1} \sup_{(x)_s} \xi^{-s} e^{\beta \sum_{i=1}^{s} p_i^2} |f_s(x_1, \dots, x_s)| < \infty. \tag{7.2}$$

The parameter $\xi > 0$ depends on the density and potential of the system and β is inverse temperature.

In this section, we suppose that initial states of the stochastic hierarchy (5.13) belong to the space E_ξ.

Let us show that, for $F(0) \subset E_\xi$, solutions of the stochastic hierarchy (5.13) exist and can be represented by the iteration series (6.5). This follows directly from our results obtained for the systems of hard spheres [5, 8]. According to these results, series (6.5) for $F_s(t, x_1, \dots, x_s)$ are uniformly convergent with respect to $(x_1, \dots, x_s)$ on arbitrary compacta and $|t| < t_0$, where t_0 is some finite number determined by the parameters of systems (density, temperature, etc.) for arbitrary initial states $F(0) \subset E_\xi$.

The constructed solutions describe nonequilibrium states of infinite systems and can be obtained from nonequilibrium states of finite systems by thermodynamic limit procedure.

The proof of the existence of the thermodynamic limit for systems with the stochastic dynamics is the same as for systems of hard spheres [5]. The obtained solutions are weak solutions of the stochastic hierarchy. The proof of this statement is completely analogous to the proof of the corresponding statement for systems of hard spheres [5].

8. Connection between the stochastic hierarchy and the Boltzmann hierarchy. In the previous sections, the solutions of the stochastic hierarchy (3.10) were constructed in the space L and E_ξ. In this section, we clarify the close connection between the stochastic hierarchy and the Boltzmann hierarchy. To do this, we write both hierarchies. The stochastic hierarchy has the form

$$\begin{aligned}
\frac{\partial F_s(t, x_1, \dots, x_s)}{\partial t} &= -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \dots, x_s) + \\
&+ \sum_{i<j=1}^{s} \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j))\, \delta(q_i - q_j) \times \\
&\times [F_s(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_s) - F_s(t, x_1, \dots, x_i, \dots, x_j, \dots, x_s)] + \\
&+ \sum_{i=1}^{s} \int d\eta_{is+1}\, dp_{s+1}\, \eta_{is+1} \cdot (p_i - p_{s+1})\, \theta(\eta_{is+1} \cdot (p_i - p_{s+1})) \times \\
&\times [F_{s+1}(t, x_1, \dots, q_i, p_i^*, \dots, x_s, q_i, p_{s+1}^*) - \\
&- F_{s+1}(t, x_1, \dots, q_i, p_i, \dots, x_s, q_i, p_{s+1})],
\end{aligned} \tag{8.1}$$

$$s = 1, 2, \dots,$$

with the boundary condition in Poisson bracket (or simply the boundary condition).

The Boltzmann hierarchy has the form

$$\frac{\partial F_s^0(t, x_1, \dots, x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s^0(t, x_1, \dots, x_s) +$$

$$+ \sum_{i=1}^{s} \int d\eta_{is+1}\, dp_{s+1}\, \eta_{is+1} \cdot (p_i - p_{s+1})\, \theta(\eta_{is+1} \cdot (p_i - p_{s+1}))$$

$$\times [F_{s+1}^0(t, x_1, \ldots, q_i, p_i^*, \ldots, x_s, q_i, p_{s+1}^*) -$$

$$- F_{s+1}^0(t, x_1, \ldots, q_i, p_i, \ldots, x_s, q_i, p_{s+1})], \tag{8.2}$$

$$s = 1, 2, \ldots .$$

Usually, the Boltzmann hierarchy (8.2) is considered without the boundary conditions, but we will also consider it with the boundary conditions (in Poisson bracket). Note that the boundary conditions are not imposed on one-particle distribution function for both hierarchy.

We suppose that the stochastic hierarchy is the true Boltzmann hierarchy in the entire phase space and solutions of the stochastic hierarchy are the Boltzmann–Grad limit of solutions of the BBGKY hierarchy for hard spheres in the entire phase space.

It is well known that the Boltzmann hierarchy without the boundary condition preserves the chaos property. In other words, if the initial s-particle distribution functions $F_s^0(0, x_1, \ldots, x_s)$ have the chaos property and are products of the one-particle distribution functions

$$F_s^0(0, x_1, \ldots, x_s) = F_s^0(0, x_1) \ldots F_s^0(0, x_s), \tag{8.3}$$

then the time dependent distribution functions $F_s(t, x_1, \ldots, x_s)$ also have the chaos property

$$F_s^0(t, x_1, \ldots, x_s) = F_s^0(t, x_1) \ldots F_s^0(t, x_s), \tag{8.4}$$

where the one-particle distribution function is the solution of the nonlinear Boltzmann equation

$$\frac{\partial F_1^0(t, p_1, q_1)}{\partial t} = - p_1 \frac{\partial}{\partial q_1} F_1^0(t, p_1, q_1) +$$

$$+ \int dp_2\, d\eta\, \theta(\eta \cdot (p_1 - p_2))\, \eta \cdot (p_1 - p_2) \times$$

$$\times [F_1^0(t, p_1^*, q_1)\, F_1^0(t, p_2^*, q_1) - F_1^0(t, p_1, q_1)\, F_1^0(t, p_2, q_1)]. \tag{8.5}$$

Now consider the solutions of the stochastic hierarchy (8.1) with initial data (8.3) and $F(0) \in X_{\xi,\beta}$.

The sequence

$$F(t) = (F_1(t, x_1), \ldots, F_s(t, x_1, \ldots, x_s), \ldots)$$

is a solution of the stochastic hierarchy (8.1) global in time. In the next section we consider in detail the connection between solutions of the stochastic and the Boltzmann hierarchy (for the latter without the boundary condition) represented by the iteration series in the entire phase space and show that the one-particle distribution functions of both hierarchy $F_1(t, x_1)$ and $F_1^0(t, x_1)$ coincide in the entire phase space, globally in time, and for arbitrary initial data $F(0) \subset X_{\xi,\beta}$.

The many-particle distribution functions of both hierarchy are different in certain set of zero Lebesgue measure (see the next section and Appendix I.1). These considerations also will mean that there exist solutions of the Boltzmann equation for the initial data $F_1(0, x_1) \subset X_{\xi,\beta}$ in the same sense as solutions of the stochastic hierarchy.

Analogous considerations are also true for solutions of the stochastic hierarchy (8.1) with the initial data (8.3) which belong to the space E_ξ. Indeed, in this case, solutions of the stochastic hierarchy (8.1) exist and are represented by the iteration series (6.5) for $|t| < t_0$.

The problem of construction of solutions of the stochastic hierarchy (8.1) by solutions of the Boltzmann equation (8.5) in the whole phase space (including the hyperplanes $q_i = q_j$, $(i,j) \subset (1, \ldots, s)$) is open. It resembles the problem of construction of eigenfunctions of the Schrödinger equations with the δ-functions as a potential [17, 18].

9. Coincidence of solutions of the Boltzmann hierarchy and the stochastic hierarchy represented by iteration series. It is well known that solutions of the Boltzmann hierarchy (8.2) without the boundary condition can be represented by the iteration series

$$F^0(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \, S^0(-t+t_1) \mathcal{A} S^0(-t_1+t_2) \mathcal{A} \ldots$$
$$\ldots S^0(-t_{n-1}+t_n) \mathcal{A} S^0(-t_n) F(0), \tag{9.1}$$

where $S^0(-t)$ is the direct sum of the s-particle operators of evolution of free particles

$$S^0(-t) = \sum_{s=0}^{\infty} \oplus S_s^0(-t), \tag{9.2}$$

and the operator $\mathcal{A}$ is defined by (5.11).

Solutions of the stochastic hierarchy (8.1) can be represented by the iteration series

$$F(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \, S(-t+t_1) \mathcal{A} S(-t_1+t_2) \mathcal{A} \ldots$$
$$\ldots S(-t_{n-1}+t_n) \mathcal{A} S(-t_n) F(0), \tag{9.3}$$

where $S(-t)$ is the direct sum of the s-particle operators of evolution of particles with the stochastic dynamics

$$S(-t) = \sum_{s=0}^{\infty} \oplus S_s(-t). \tag{9.4}$$

The iteration series (9.1) of the Boltzmann hierarchy is convergent for the initial data $F(0)$ from the space E_ξ in the sense that for every $F_s^0(t, x_1, \ldots, x_s)$ series (9.1) is uniformly convergent with respect to $(x_1, \ldots, x_s)$ on compacta for $|t| < t_0$. Here, t_0 is a certain number depending on the thermodynamic parameters of the system [5]. The iteration series (9.3) of the stochastic hierarchy is convergent in the same sense.

It follows from the results obtained in the proof of the existence of the Boltzmann–Grad limit that the following statement is true:

For arbitrary $\varepsilon > 0$ *the inequality*

$$|F_s^0(t, x_1, \ldots, x_s) - F_s(t, x_1, \ldots, x_s)| < \varepsilon \tag{9.5}$$

holds on compacts outside any neighborhood of the hyperplanes

$$q_i = q_j, \quad (i, j) \subset (1, \ldots, s)$$

and outside any neighborhood of the hyperplanes such that vectors $p_i - p_j$ *are parallel to* $q_i - q_j \neq 0$ *and for* $|t| < t_0$.

In other words, the functions $F_s^0(t, x_1, \ldots, x_s)$ and $F_s(t, x_1, \ldots, x_s)$ *coincide outside the collection of these hyperplanes.*

According to the definition of the stochastic dynamics this collection forms the set D_t of zero Lebesgue measure in the phase space (see Appendix I, Section 1).

The one-particle distribution functions $F_1^0(t, x_1)$ *and* $F_1(t, x_1)$ *coincide on entire phase space.*

In order to prove this proposition we recall the main ideas of the proof of the existence of the Boltzmann–Grad limit [4, 5].

In that proof, we used the iteration series of the BBGKY hierarchy for systems of hard spheres with diameter a

$$F^a(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \, S^a(-t+t_1)\mathcal{A}_a S^a(-t_1+t_2)\mathcal{A}_a \ldots$$

$$\ldots S^a(-t_{n-1}+t_n)\mathcal{A}_a S^a(-t_n)F(0), \tag{9.6}$$

where $S^a(-t)$ is defined by (9.4) but with the operators $S_s^a(-t)$ of evolution of s hard spheres. The operator $\mathcal{A}_a$ is defined by the following formula:

$$(\mathcal{A}_a x)_s(x_1, \ldots, x_s) = \sum_{i=1}^{s} \int dp_{s+1}\, d\eta\, \eta\cdot(p_i - p_{s+1})\,\theta(\eta\cdot(p_i - p_{s+1})) \times$$

$$\times [f_{s+1}(x_1, \ldots, q_i, p_i^*, \ldots, x_s, q_i - a\eta, p_{s+1}^*) -$$

$$- f_{s+1}(x_1, \ldots, q_i, p_i, \ldots, x_s, q_i + a\eta, p_{s+1})]. \tag{9.7}$$

Series (9.6) is convergent as well as series (9.1) and (9.3) [5].

In [4, 5], it was shown that the iteration series for the BBGKY hierarchy (9.6) converges to the iteration series of the Boltzmann hierarchy (9.1) in the following sense:

Consider the functions $F_s^0(t, x_1, \ldots, x_s)$ for arbitrary $s \geq 1$. Let the variables $(q_1, \ldots, q_s)$ belong to an arbitrary compactum K_s such that $|q_i - q_j| > a + a_0(a)$ for all pairs $(i, j) \subset (1, \ldots, s)$. The function $a_0(a)$ tends to zero as $a \to 0$ so that

$$\lim_{a \to 0} \frac{a}{a_0(a)} = 0.$$

The variables $(p_1, \ldots, p_s)$ belong to a compactum outside the cones with respect to all differences $p_i - p_j$, $(i, j) \subset (1, \ldots, s)$, whose volumes are proportional to $\left(\frac{a}{a_0(a)}\right)^2$ and whose axes are parallel to the vectors $q_i - q_j$.

As $a \to 0$, the function $F_s^a(t, x_1, \ldots, x_s)$ tends to the function $F_s^0(t, x_1, \ldots, x_s)$ on the above-described compacta uniformly with respect to $(q_1, \ldots, q_s)$ and weakly with respect to $(p_1, \ldots, p_s)$ on the finite interval $|t| < t_0$. Here, the number t_0 is determined by the thermodynamic parameters of the initial state $F(0)$. (In [4, 5] the weak convergence with respect to momenta $p_1, \ldots, p_s$ was considered, but it is easy to prove that the uniform convergence also holds.)

To prove inequality (9.5) for the difference $F_s(t, x_1, \ldots, x_s) - F_s^0(t, x_1, \ldots, x_s)$, it suffices to use the fact that the operator $S(-t)$ (9.4) for the stochastic hierarchy is determined by the dynamics obtained as a certain limit of the dynamics of hard spheres as $a \to 0$, and therefore particles interact only for the initial positions and momenta such that the vectors $p_i - p_j$ are parallel to the vectors $q_i - q_j \neq 0$, i.e., the cones are reduced to the vectors $p_i - p_j$. One can repeat the proof of the convergence of $F_s^a(t, x_1, \ldots, x_s)$ to $F_s^0(t, x_1, \ldots, x_s)$ as $a \to 0$.

The collection of these vectors forms the set D_t of zero measure, and $F_s(t, x_1, \ldots, x_s) \neq F_s^0(t, x_1, \ldots, x_s)$ on the set D_t (see Section 1 and Appendix I.1). The set D_t is empty for one-particle systems and, therefore, $F_1(t, x_1)$ and $F_1^0(t, x_1)$ coincide on the entire phase space.

These results are restricted to the finite interval of time $0 < t < t_0$ because the iteration series for the both hierarchies are convergent on this finite interval for $F(0) \subset E_\xi$. But for the stochastic and Boltzmann hierarchies, the solutions in the space $X_{\xi,\beta}$ $(F(0) \in X_{\xi,\beta})$ exist for arbitrary time $t \geq 0$. The existence of the Boltzmann–Grad limit global in time and its coincidence in certain sense with the solutions of the stochastic hierarchy can be proved as follows:

We represent the solutions of the Boltzmann hierarchy and the solutions of the stochastic hierarchy by the series of iterations (9.1) and (9.3), respectively. These series of iterations for the both hierarchies are convergent in the space $X_{\xi,\beta}$ and we can repeat the proof given above.

Note that the sum corresponding to the stochastic hierarchy (9.3) identically coincides term by term with the analogous sum of the Boltzmann hierarchy (9.1) (outside all hyperplanes $q_i = q_j$ and outside all hyperplanes such that vectors $p_i - p_j$ are parallel to $q_i - q_j \neq 0$, i.e., outside of the domain D_t). The proof of this statement is the same as for a bounded interval of time.

This means that *solutions of the Boltzmann hierarchy* $F_s^0(t, x_1, \ldots, x_s)$, $s \geq 1$, *global in time exist for* $F(0) \subset X_{\xi,\beta}$ *and coincide outside the set* D_t *with the solutions of the stochastic hierarchy* $F_s(t, x_1, \ldots, x_s)$, $s \geq 1$, *and for* $F(0) \in E_\xi$ *the same holds on the finite time interval* $|t| < t_0$. *The one-particle distribution functions* $F_1^0(t, x_1)$ *and* $F_1(t, x_1)$ *coincide in the entire phase space (the set* D_t *is empty for* $s = 1$).

10. New representation of solutions of the Boltzmann equation. Consider in detail the one-particle distribution functions $F_1(t, x_1)$ and $F_1^0(t, x_1)$ that correspond to the stochastic hierarchy and the Boltzmann hierarchy, respectively. We restrict ourselves to the factorized initial data

$$F_s(0, x_1, \ldots, x_s) = F_1(0, x_1) \ldots F_1(0, x_s).$$

It was shown above that $F_1(t, x_1)$ coincides with $F_1^0(t, x)$, i.e. the iteration series (9.3) and (9.1) for them coincide term by term in the entire phase space. Series (9.3) for $F_1(t, x_1)$ is convergent for $F(0) \in X_{\xi,\beta}$. Series (9.1) for $F_1^0(t, x_1)$ is also convergent for $F(0) \in X_{\xi,\beta}$, and $F_1^0(t, x_1)$ as well as $F_1(t, x_1)$ exists globally in time for $F(0) \in X_{\xi,\beta}$.

As known, $F_1^0(t, x_1)$ represented by series (9.1) is the mild solution of the nonlinear Boltzmann equation (8.5) and, thus, *we have proved the existence of mild solutions global in time of the Boltzmann equation for initial data* $F(0) \in X_{\xi,\beta}$.

Analogously, it can be proved that mild solutions of the Boltzmann equation exist on the finite interval $|t| < t_0$ for initial data $F(0) \in E_\xi$.

Usually the mild solutions of the Boltzmann equation (or the one-particle distribution function of the Boltzmann hierarchy) are represented by the iteration series

$$F_1^0(t, x_1) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, S_1^0(-t+t_1, x_1)\, \mathcal{A}_1\, S_2^0(-t_1+t_2, x_1, x_2)\, \mathcal{A}_2 \ldots$$

$$\ldots \mathcal{A}_n\, S_n^0(-t_{n-1}+t_n, x_1, \ldots, x_n)\, \mathcal{A}_{1+n}\, S_{1+n}^0(-t_n, x_1, x_2, \ldots, x_n, x_{n+1}) \times$$

$$\times F_1^0(0, x_1)\, F_1^0(0, x_2) \ldots F_1^0(0, x_n)\, F_1^0(0, x_{1+n}), \tag{10.1}$$

where

$$(\mathcal{A}f)_i(x_1, \ldots, x_i) = (\mathcal{A}_i f_i)(x_1, \ldots, x_i),$$

the operator $\mathcal{A}$ is defined by (5.11), and $S_i^0(-t_{i-1}+t_i, x_1, \ldots, x_i)$ is the evolution operator of free particles.

As shown, series (10.1) for $F_1^0(t, x_1)$ coincides with the series for $F_1(t, x_1)$ of the stochastic hierarchy

$$F_1(t, x_1) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, S_1(-t+t_1, x_1)\, \mathcal{A}_1\, S_2(-t_1+t_2, x_1, x_2)\, \mathcal{A}_2 \ldots$$

$$\ldots \mathcal{A}_n\, S_n(-t_{n-1}+t_n, x_1, \ldots, x_n)\, \mathcal{A}_{1+n}\, S_{1+n}(-t_n, x_1, x_2, \ldots, x_n, x_{n+1}) \times$$

$$\times F_1(0, x_1)\, F_1(0, x_2) \ldots F_1(0, x_n)\, F_1(0, x_{1+n}), \tag{10.2}$$

$$F_1^0(0, x) = F_1(0, x),$$

where $S_i(-t_{i-1}+t_i, x_1, \ldots, x_i)$ is the evolution operator associated with the stochastic hierarchy.

Indeed, as mentioned above, the term by term coincidence of series (10.1) and (10.2) follows directly from the fact that, according to the stochastic dynamics, particles can interact if and only if the differences of their initial momenta are parallel to the differences of their initial positions. Then, repeating our proof of the existence of the Boltzmann – Grad limit word by word and taking into account that the cones in the momentum space degenerate to the vectors parallel to differences of positions, we obtain the proof of the coincidence of series (10.1) and (10.2).

Taking into account the importance of this assertion, we give the detailed proof.

Consider the integrand of the n-th term of series (10.2) for $F_1(t, x_1)$

$$S_1(-t+t_1, x_1)\, \mathcal{A}_1\, S_2(-t_1+t_2, x_1, x_2)\, \mathcal{A}_2\, S_3(-t_2+t_3, x_1, x_2, x_3) \ldots$$

$$\ldots \mathcal{A}_n\, S_n(-t_{n-1}+t_n, x_1, \ldots, x_n)\, \mathcal{A}_{1+n}\, S_{1+n}(-t_n, x_1, x_2, \ldots, x_n, x_{1+n}) \times$$

$$\times F_1(0, x_1)\, F_1(0, x_2) \ldots F_1(0, x_n)\, F_1(0, x_{1+n}) \tag{10.3}$$

and the action of the operators S_i and $\mathcal{A}_i$.

After the action of the operator $S_1^0(-t+t_1, x_1)$ all particles are shifted by the vector $p_1(-t+t_1)$ or $p_1^*(-t+t_1)$. After the action of the operator $S_2(-t_1+t_2, x_1, x_2)$ the first particle is shifted again by the vector $p_1(-t_1+t_2)$ or $p_1^*(-t_1+t_2)$ and the other particles are shifted by the vectors $p_2(-t_1+t_2)$ or $p_2^*(-t_1+t_2)$. The first and the second particles do not interact because they are in the state of collision according to the action of the operator $\mathcal{A}_2$ and $S_2(-t_1+t_2, x_1, x_2) = S_2^0(-t_1+t_2, x_1, x_2)$.

Consider now the action of the operator $S_2(-t_2+t_3, x_1, x_2, x_3)$. The third particle is in the state of collision with the second one and they do not interact, but the third particle may interact with the first one if the difference of their momenta $p_1 - p_3$ ($p_1^* - p_3^*$) is parallel to the difference of their positions after the action of the operators $S_1(-t+t_1, x_1)$ and $S_2(-t_1+t_2, x_1, x_2)$ expressed in terms of the momenta p_1, p_2 (p_1^*, p_2^*). For example, the position of the first particle may be $p_1(-t+t_2)$ and the position of the third one may be $p_1(-t+t_1) + p_2(-t_1+t_2)$. Thus, the first and third particles may interact only if one vector $p_3 - p_1$ is parallel to the vector $(p_2 - p_1)(-t_1+t_2)$. We can omit these vectors in the integrand because they form a hyperplane of lower dimension in the momentum space (p_1, p_2, p_3). This means that, outside this hyperplane, the operator $S_3(-t_2+t_3, x_1, x_2, x_3)$ coincides with the free evolution operator $S_3^0(-t_2+t_3, x_1, x_2, x_3)$.

By analogy, we can replace successively all operators $S_i(-t_{i-1}+t_i, x_1, \ldots, x_i)$ by the operators $S_i^0(-t_{i-1}+t_i, x_1, \ldots, x_i)$, omitting the hyperplanes of lower dimension in integrand (10.3) and using the fact that since $q_1 = q_2 = \ldots = q_n = q_{n+1}$, the differences of the positions of all particles after the action of the operators $S_1, \ldots, S_{i-1}$ do not depend on q_1 and on the momenta of the i-th particle, and depend only on the momenta of the particles with numbers $1, 2, \ldots, i-1$.

Note that, in the case of s-particle distribution functions $F_s(t, x_1, \ldots, x_s)$ and $F_s^0(t, x_1, \ldots, x_s)$, the proof of the term by term coincidence of the corresponding iteration series (9.3) and (9.1) outside the set D_t is absolutely the same as for $F_1(t, x_1)$ and $F_1^0(t, x_1)$. If one fixes initial points $x_1, \ldots, x_s$ outside the set D_t, then all stochastic evolution operators coincide with the free evolution operators outside the degenerate cones such that the differences of positions are parallel to the differences of momenta. These sets have Lebesgue measure zero with respect to $p_{s+1}, \ldots, p_{s+i}$ and the random vectors η.

For solutions of the Boltzmann equation (10.1), (10.2), we have the equivalent new representation

$$F_1^0(t, x_1) = F_1(t, x_1) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} M \int S_{1+n-k}(-t, x_1, \ldots, x_{1+n-k}) \times$$

$$\times F_1(0, x_1) F_1(0, x_2) \ldots F_1(0, x_n) F_1(0, x_{1+n}) \, dx_2 \ldots dx_n \, dx_{1+n}, \qquad (10.4)$$

which was obtained from (10.2) by integration with respect to $t_1, \ldots, t_n$, as shown in Section 6. (Note that in (10.4) we have the evolution operators of the stochastic dynamics which differ from the evolution operators of the free dynamics on the sets D_t described in Appendix I.1. The many-particle distribution functions of the stochastic and Boltzmann hierarchies also differ in the many-particle set D_t.)

1. *Bogoliubov N. N.* Problems of dynamical theory in statistical physics. – M.: Gostekhizdat, 1946 (also in Bogoliubov N. N., Selected Works. – Kiev: Naukova Dumka,1970. – Vol. 2. – P. 99–196.
2. *Cercignani C.* The Grad limit for a system of soft spheres // Communs Pure and Appl. Math. – 1983. – **36**. – P. 479–494.
3. *Lanford O. E.* Time evolution of large classical systems // Lect. Notes Phys. – 1975. – **38**.
4. *Gerasimenko V. I., Petrina D. Ya.* Existence of the Boltzmann–Grad limit for infinite system of hard sphere // Teoret. i Mat. Fiz. – 1990. – **83**. – P. 92–114.
5. *Petrina D. Ya., Gerasimenko V. I.* Mathematical problems of statistical mechanics of hard sphere particle system // Usp. Mat. Nauk. – 1990. – **45**. – P. 192–214.
6. *Petrina D. Ya., Gerasimenko V. I.* Evolution of states of infinite systems in classical statistical mechanics // Sov. Sci. Rev. Ser. C Math. Phys. – 1984. – **5**. – P. 1–60.
7. *Petrina D. Ya., Gerasimenko V. I., Malyshev P. V.* Thermodynamic limit for solutions of Bogoliubov's equations // Ibid. – 1987. – **7**. –P. 280–335.
8. *Petrina D. Ya., Gerasimenko V. I., Malyshev P. V.* Mathematical foundations of classical statistical mechanics. – continuous systems. – New York: Gordon and Breach, 1989.
9. *Tanaka H.* Probabilistic treatment of the Boltzmann equation of Maxwellian molecules // Z. Warscheinlichkeitstheor. und verw. Geb. – 1978. – **46**. – P. 67–105.
10. *Sznitman A. S.* Equations de type de Boltzmann, spatialement homogenes // Ibid. – 1984. – **66**. – P. 559–592.
11. *Arsen'ev A. A.* On the approximation of the solution of the Boltzmann equation by solution of Itô stochastic differential equations // Z. Vychisl. Mat. i Mat. Fiz. – 1987. – **27**. – P. 400–410.
12. *Arsen'ev A. A.* Approximation of the Boltzmann equation by stochastic equations // Ibid. – 1988. – **28**. – P. 560–567.
13. *Skorokhod A. V.* Stochastic equation for complex systems. – Moscow: Nauka, 1983.
14. *Lachowicz M., Pulvirenti M.* A stochastic particle system modeling the Euler equation // Arch. Ration. Mech. and Anal. – 1990. – **109**. – P. 81–93.
15. *Caprino S., De Masi A., Presutti E., Pulvirenti M.* A stochastic particle system modeling the Carleman equation // J. Statist. Phys. – 1989. – **55**. – P. 625–638.
16. *Lachowicz M.* A system of stochastic differential equations modeling the Euler and the Navier–Stokes hydrodynamic equations // Jap. J. Industr. and Appl. Math. – 1993. – **10**. – P. 109–131.
17. *Albeverio S., Gesztesy S., Høegh-Krohn R., Holden H.* Solvable models in quantum mechanics. – New York; Berlin: Springer, 1988.
18. *Koshmanenko V. D.* Singular quadratic forms in the theory of perturbations of self-adjoint operators. – Kiev: Naukova Dumka, 1993.
19. *Gelfand I. M., Graev M. I., Vilenkin N. Ya.* Generalized functions. – Moscow: Nauka, 1962.
20. *Krein S. G.* Linear differential equations in Banach space. – Moscow: Nauka, 1967.
21. *Cercignani C., Gerasimenko V. I., Petrina D. Ya.* Many-particle dynamics and kinetic equations. – Dordrecht: Kluwer, 1997.
22. *Cercignani C., Illner R., Pulvirenti M.* The mathematical theory of dilute gases. – New York: Springer-Verlag, 1997.
23. *Bogolyubov N. N., Bogolyubov N. N. (jr.)* Introduction to statistical mechanics. – Moscow: Nauka, 1984.
24. *Cercignani C.* On the master equation in the space in homogeneous case // Pichon G. (ed.) Theories cinetiques classiques et relativistes // Colloq. Int. C. N. R. S. – 1975. – P. 209–221.
25. *Dorfman J. R., Ernst M. H.* Hard-sphere binary-collision operators // J. Stat. Phys. – 1989. – **57**. – 1989. – P. 581–593.
26. *Di Perna R. J., Lions P. L.* Global solutions of Boltzmann's equation and the entropy inequality // Arch. Ration. Mech. and Anal. – 1991. – P. 47–55.
27. *Di Perna R. J., Lions P. L.* On the cauchy problem for Boltzmann equations: Global existence and weak stability // Ann. Math. – 1989. – **130**. – P. 321–366.

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