UDC 517.9 + 531.19

D. Ya. Petrina, K. D. Petrina (Inst. Math. Nat. Acad. Sci. Ukraine, Kiev)

STOCHASTIC DYNAMICS AND BOLTZMANN HIERARCHY. I

СТОХАСТИЧНА ДИНАМІКА І ІЄРАРХІЯ БОЛЬЦМАНА. I

Stochastic dynamics corresponding to the Boltzmann hierarchy is constructed. The Liouvillr–Itô equations are obtained, from which we derive the Boltzmann hierarchy regarded as an abstract evolution equation. We construct the semigroup of evolution operators and prove the existence of solutions of the Boltzmann hierarchy in the space of sequences of integrable and bounded functions. On the basis of these results, we prove the existence of global solutions of the Boltzmann equation and the existence of the Boltzmann–Grad limit for an arbitrary time interval.

Побудована стохастична динаміка, яка відповідає ієрархії Больцмана. Отримані рівняння Ліувілля–Іто, а з них виведена ієрархія Больцмана, яка розглядається як абстрактне еволюційне рівняння. Побудована півгрупа еволюційних операторів і доведено існування розв'язків ієрархії Больцмана в просторі послідовностей інтегровних та обмежених функцій. На цій основі доведено існування глобальних розв'язків рівняння Больцмана та існування границі Больцмана–Греда на довільному інтервалі часу.

Introduction. About hundred years ago, Boltzmann deduced his famous equation. Since that time, the researchers discuss the problem of irreversibility, i.e., how to pass from reversible solutions of the Hamiltonian equations to irreversible solutions of the Boltzmann equation.

In doing this, it is customary to assume (explicitly or implicitly) that the Boltzmann equations can be derived directly from the Hamiltonian equations. At the same time, this assumption is not evident and, maybe, even not true. Indeed, the BBGKY hierarchy follows from the Hamiltonian equation (via the Liouville equation) and, in this sense, they are equivalent. In its turn, the Boltzmann equation can be obtained from the BBGKY hierarchy as a result of specific limit transitions (the Bogolyubov limit and the Boltzmann–Grad limit) and, therefore, should be equivalent to certain limiting Hamiltonian equations. To clarify in what sense one should understand limiting Hamiltonian equations, let us analyze the cited Bogolyubov and Boltzmann–Grad limit transitions used to derive Boltzmann equations from the BBGKY hierarchy.

The relationship between the Boltzmann equations and the BBGKY hierarchy was first established by Bogolyubov [1] who showed that the Boltzmann equations can be obtained from the BBGKY hierarchy in the first order of perturbation theory in density if one restricts himself to the solutions satisfying the principle of decay of correlations. It follows just from the procedure of derivation of the Boltzmann equations that one uses only the asymptotics of the solutions of Hamiltonian equations for two particles as the distance between them tends to infinity and this asymptotics is reflected in the Boltzmann equation by the scattering cross section. The computation of the collision integral is pure probabilistic and this means, in turn, that the limiting evolution of particles that corresponds to the Boltzmann equation is governed by a certain random process. Note that Bogolyubov's method has not yet found its mathematical justification.

In recent years, significant progress has been achieved in the mathematical justification of the derivation of the Boltzmann equations from the BBGKY hierarchy for an infinite system of hard spheres in the Boltzmann–Grad limit [2–5]. It was shown that, in the Boltzmann–Grad limit, solutions of the BBGKY hierarchy for a system of hard spheres approach, in a certain sense, solutions of the Boltzmann hierarchy. In its turn, the Boltzmann hierarchy possesses the property of chaos, i.e., it has solutions such that all distribution functions are equal to the product of one-particle functions and the one-particle distribution function satisfies a nonlinear Boltzmann equation.

The right-hand side of each equation in the Boltzmann hierarchy has two terms, one of which is simply the right-hand side of the Liouville equation for free particles and the second integral term contains the scattering cross section of hard spheres. Note that the solutions of this Boltzmann hierarchy can be regarded as a limit of solutions of the BBGKY hierarchy only outside hyperplanes where the coordinates of particles coincide and the momenta are outside certain cones and, hence, the real Boltzmann hierarchy, i.e., the limit of the BBGKY hierarchy in the entire phase space may also contain terms with δ-functions concentrated in these hyperplanes.

It is thus natural to ask the following question:

What kind of particle dynamics corresponds to the Boltzmann hierarchy? In other words, we want to establish an analog of the Hamiltonian equations such that the Boltzmann hierarchy is obtained from these equations via the Liouville equations by applying the standard procedure.

This would enable one to clarify the problem connected with the contradiction between the reversible nature of Hamiltonian equations and the irreversible nature of Boltzmann equations. But if the equations of motion that correspond to the Boltzmann hierarchy are also irreversible, no contradiction appears.

Another problem connected with particle dynamics that corresponds to the Boltzmann hierarchy is the problem of construction of its global solutions. Actually, global solutions of the BBGKY hierarchy were constructed in the space L of sequences of summable functions and in certain sets in the space E of sequences of functions bounded in coordinates and exponentially decreasing in momenta [6–8]. For this purpose, the group of evolution operators for the BBGKY hierarchy was constructed by using detailed information about Hamiltonian dynamics of systems of particles. The evolutionary semigroup for the Boltzmann hierarchy has not yet been constructed mainly because the corresponding particle dynamics was unknown. Solutions of the Boltzmann hierarchy were constructed by the iteration method. The operator on the right-hand side of this hierarchy (written as an abstract evolutionary equation) is unbounded and, therefore, the iterative series converges in a finite time interval and under essential restrictions imposed on the initial data.

This paper is aimed at the construction of stochastic dynamics that corresponds to the Boltzmann hierarchy for hard spheres. On the basis of this dynamics, we construct a semigroup of evolutionary operators and establish the existence of global (in time) solutions both for the Boltzmann hierarchy and Boltzmann equations. Having global solutions of the BBGKY and Boltzmann hierarchies, one can prove the existence of the Boltzmann–Grad limit for an arbitrary interval of time and in the entire phase space.

Note that random processes and stochastic equations were used earlier for the construction of smoothed Boltzmann equations [9–12]. However, the Boltzmann hierarchy was not studied in these works. Probably, the first hierarchy for the distribution functions that correspond to certain stochastic equations was constructed in the monograph [13]. Further progress in the approach founded in [13] is connected with the works [14–16] based on the use of the method of iterations in the mean-field model. The ideas used in this paper are similar to those developed in [13] but we take physically meaningful stochastic dynamics.

Let us now dwell upon the content of the present work.

We construct stochastic dynamics according to which particle move freely unless they collide. As a result of collisions, particles change their momenta jumpwise, as in the case of hard spheres, but reflection angles can be arbitrary (with the same probability). This assumption is responsible for the stochastic nature of particle dynamics. After collision, particles move freely up to the next collision. We say that this dynamics is stochastic. Stochastic dynamics can be regarded as a certain limit of the dynamics of hard spheres as their diameters tend to zero.

Let us now write the Liouville–Itô equations that corresponds to the stochastic dynamics described above. It has the form

$$\frac{\partial}{\partial t} D_N(t, x_1, \ldots, x_N) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} D_N(x_1, \ldots, x_N) +$$

$$+ \sum_{i<j=1}^{n} \theta(\eta_{ij}\cdot(p_i-p_j))\,\eta_{ij}\cdot(p_i-p_j)\,\delta(q_i-q_j)\big[D_N(t,x_1,\ldots,x_i^*,\ldots$$

$$\ldots, x_j^*,\ldots,x_N) - D_N(t,x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N)\big], \qquad t\ge 0, \tag{1}$$

where $D_N(t, x_1, \ldots, x_N)$ is the distribution function of a system of N particles in the phase space at time t, η_{ij} is a random vector that characterizes the postcollision momenta of particles, $|\eta_{ij}| = 1$, $x_i^* = (q_i, p_i^*)$, $x_j^* = (q_i, p_j^*)$,

$$p_i^* = p_i - \eta_{ij}\ (\eta_{ij}\cdot(p_i-p_j)), \quad \text{and} \quad p_j^* = p_i + \eta_{ij}\cdot(\eta_{ij}\ (p_i-p_j)).$$

The other notation is standard for statistical mechanics.

We introduce a sequence of s-particle (reduced) distribution functions, both in the canonical and grand canonical ensembles, by integrating the function $D_N(t, x_1, \ldots$ $\ldots, x_N)$ with respect to $N-s$ phase variables and averaging over the random vectors η_{ij} that correspond to the collisions of $N-s$ particles.

This new conception of distribution functions takes into account the set of lower dimension where particles interact and the contribution of this set is different from zero.

For the sequences of distribution functions, we have the following hierarchy of equations:

$$\frac{\partial F_s(t,x_1,\ldots,x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t,x_1,\ldots,x_s) +$$

$$+ \sum_{i<j=1}^{s} \theta(\eta_{ij}\cdot(p_i-p_j))\,\eta_{ij}\cdot(p_i-p_j)\,\delta(q_i-q_j)\times$$

$$\times\ [F_s(t,x_1,\ldots,x_i^*,\ldots,x_j^*,\ldots,x_s) - F_s(t,x_1,\ldots,x_i,\ldots,x_j,\ldots,x_s)] +$$

$$+ \sum_{i=1}^{s}\int \theta(\eta_{is+1}\cdot(p_i-p_{s+1}))\,\eta_{is+1}\cdot(p_i-p_{s+1})\times$$

$$\times\ [F_{s+1}(t,x_1,\ldots,q_i,p_i^*,\ldots,x_s,q_i,p_{s+1}^*) -$$

$$- F_{s+1}(t,x_1,\ldots,q_i,p_i,\ldots,x_s,q_i,p_{s+1})]\,d\eta_{is+1}\,dp_{s+1}, \tag{2}$$

$$t\ge 0, \quad s = 1, 2, \ldots .$$

with initial and certain boundary conditions.

The chain of equations (2) is called the stochastic Boltzmann hierarchy or simply the stochastic hierarchy. It differs from the well-known Boltzmann hierarchy

$$\frac{\partial F_s(t,x_1,\ldots,x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t,x_1,\ldots,x_s) +$$

$$+ \sum_{i=1}^{s}\int \theta(\eta_{is+1}\cdot(p_i-p_{s+1}))\,\eta_{is+1}\cdot(p_i-p_{s+1})\times$$

$$\times\ [F_{s+1}(t,x_1,\ldots,q_i,p_i^*,\ldots,x_s,q_i,p_{s+1}^*) -$$

$$- F_{s+1}(t,x_1,\ldots,q_i,p_i,\ldots,x_s,q_i,p_{s+1})]\,d\eta_{is+1}\,dp_{s+1}, \tag{3}$$

by the terms with $\delta(q_i - q_j)$ and coincides with (3) for $q_i \ne q_j$, $(i,j)\subset(1,\ldots,s)$.

The problem of existence of formal solutions of the stochastic Boltzmann hierarchy is positively solved by using the semigroup of evolutionary operators. Indeed, we have

$$F(t) = M e^{\int dx} S(-t) e^{-\int dx} F(0) = U(t)F(0), \qquad t>0, \tag{4}$$

where $F(t)$ and $F(0)$ are sequences of distribution functions at time $t>0$ and at $t=0$, M denotes the averaging over random vectors η, $S(-t)$ is a direct sum of evolutionary operators of s-particle subsystems, and $\int dx$ denotes the operator of integration.

The evolution operators $U(t)$ form a semigroup and its infinitesimal generator coincides with the operator that determines the right-hand side of the stochastic hierarchy (2). On this basis, we can identify the stochastic Boltzmann hierarchy (2) with an abstract evolutionary equation

$$\frac{dF(t)}{dt} = (\mathcal{H} + \mathcal{A})F(t), \quad F(t)|_{t=0} = F(0). \tag{5}$$

where $\mathcal{H}$ is determined by the first and second terms and $\mathcal{A}$ is given by the third term on the right-hand side of hierarchy (2). Therefore, relation (4) defines the formal solution of hierarchy (2).

The solutions given by formula (4) are made meaningful for initial data $F(0)$ from the space E_ξ of sequences of functions bounded in coordinatewise and exponentially decreasing in momenta and for initial data $F(0)$ from the space L of sequences of functions exponentially decreasing in squared coordinates and momenta. For general initial data $F(0) \in E_\xi$, this program is realized for a finite time interval, while for a certain above described subset of L, this can be proved for an arbitrarily large time interval.

It is proved that the Boltzmann–Grad limit for solutions of the BBGKY hierarchy for a system of hard spheres exists and coincides with solutions of the stochastic Boltzmann hierarchy in the following sense. The one-particle distribution function of the stochastic Boltzmann hierarchy (2) coincides with the one-particle distribution function of the Boltzmann hierarchy (3) that satisfies the Boltzmann equation for factorized initial data (conditions of chaos). In this sense, the Boltzmann equation, is rigorously deduced for an arbitrarily large interval of time.

Note that the many-particle distribution functions of both hierarchies differ from each other on certain set of zero measure in the phase space.

One can expect that these results remain true and can be established for the case of the Boltzmann equation for particles interacting via a short range potential.

The authors express their gratitude to Dr. P. V. Malyshev and Dr. D. V. Malyshev for the help in translation, editing, and preparing this paper for publishing.

In this introduction we have briefly described the contents of the entire work devoted to the stochastic dynamics and the Boltzmann hierarchy. The work will be published in several parts. In the first part we describe the stochastic dynamics, the Liouville – Ito equation, the stochastic Boltzmann hierarchy.

1. Dynamics of finitely many particles. Let us consider N identical particles of unit mass situated in a three-dimensional Euclidean space R^3. Their positions and momenta are denoted by q_i and p_i, and $x_i = (q_i, p_i)$ is the phase point of the i-th particle, $i = 1, \ldots, N$.

Denote by $Q_i(t)$, $P_i(t)$, $X_i(t) = (P_i(t), Q_i(t))$ the position, momentum, and phase point of the i-th particle, respectively, at time $0 \le t < \infty$ (this means that time changes from 0 to $+\infty$, or, more generally, that time increases),

$$Q_i(0) = q_i, \quad P_i(0) = p_i, \quad X_i(0) = x_i, \quad i = 1, \ldots, N.$$

The phase point of the entire system of N particles is denoted by $X(t) = (X_1(t), \ldots$ $\ldots, X_N(t))$, $X(0) = (x_1, \ldots, x_N) = x = (q, p)$, $q = (q_1, \ldots, q_N)$, $p = (p_1, \ldots$ $\ldots, p_N)$. To indicate the dependence of $X(t)$ on the initial value x we shall write $X(t) = X(t, x) = X(t, x_1, \ldots, x_N)$, $X_i(t) = X_i(t, x) = X_i(t, x_1, \ldots, x_N)$. $X(t, x)$ is the trajectory of our system which passes through the point x at $t = 0$.

We suppose that particles move as free ones until their positions coincide:

$$Q_i(t) = q_i + p_i t, \quad P_i(t) = p_i, \quad i = 1, \ldots, N.$$

If the positions of the i-th and j-th particles coincide at time t,

$$Q_i(t) = Q_j(t), \tag{1.1}$$

then they collide instantaneously and their momenta change jumpwise.

After the collision, for time $t + 0$, the momenta have the following form:

$$\begin{aligned} P_i^*(t+0) &= P_i(t) - \eta_{ij}\eta_{ij}\cdot(P_i(t) - P_j(t)), \\ P_j^*(t+0) &= P_j(t) + \eta_{ij}\eta_{ij}\cdot(P_i(t) - P_j(t)), \end{aligned} \tag{1.2}$$

where η_{ij} is an unit vector, $|\eta_{ij}| = 1$, $\eta_{ij}\cdot(P_i(t) - P_j(t))$ is the scalar product of the vector η_{ij} and $P_i(t) - P_j(t)$. We consider only vectors η_{ij} that satisfy the condition

$$\eta_{ij}\cdot(P_i(t) - P_j(t)) \le 0, \tag{1.3}$$

for positive time $t \ge 0$. Denote the unit semisphere (1.3) by S^2_-.

If the vectors η_{ij} satisfy the condition

$$\eta_{ij}\cdot(P_i(t) - P_j(t)) \ge 0, \tag{1.3$'$}$$

then, after the collision, the momenta $P_i(t)$, $P_j(t)$ do not change, i.e., they are the same as before the collision. Denote semisphere (1.3′) by S^2_+ and the sphere $|\eta_{ij}| = 1$ by S^2, $S^2 = S^2_- \cup S^2_+$.

We suppose that the vector η_{ij} is random with constant density of probability on the semisphere S^2. Denote the density of probability by $\chi(\eta_{ij})$. Then

$$\int_{S^2} \chi(\eta_{ij})\, d\eta_{ij} = 1, \quad \chi(\eta_{ij}) = \frac{1}{4\pi}. \tag{1.4}$$

It is obvious that two particles (i-th and j-th) can collide only if the vector $p_i - p_j$ is parallel to the vector $q_i - q_j$.

We suppose that under a simultaneous collision of more than two particles they move as free particles.

For negative time $-\infty < t \le 0$ (this means that time changes in direction from 0 to $-\infty$, or, more generally, that time decreases), all considerations are the same as for $t \ge 0$, but the random vector η_{ij} satisfies the condition

$$\eta_{ij}\cdot(P_i(t) - P_j(t)) \ge 0, \tag{1.5}$$

i.e., the vector η_{ij} belongs to the unit semisphere S^2_+ (1.5), and, after the collision, for time $t \le 0$, the momenta of the i-th and j-th particles have the following form:

$$P_i^*(t-0) = P_i(t) - \eta_{ij}\eta_{ij}\cdot(P_i(t) - P_j(t)),$$
$$P_j^*(t-0) = P_j(t) + \eta_{ij}\eta_{ij}\cdot(P_i(t) - P_j(t)), \tag{1.2'}$$

If the vectors η_{ij} satisfy the condition

$$\eta_{ij}\cdot(P_i(t) - P_j(t)) \leq 0, \quad \eta_{ij} \subset S_-^2, \tag{1.5'}$$

then, after the collision, the momenta $P_i(t)$, $P_j(t)$ do not change. The vector η_{ij} is again random with constant density $\chi(\eta_{ij}) = \dfrac{1}{4\pi}$.

We suppose that random vectors η_{ij} related to different collisions are independent.

Note that the trajectories $X(t,x)$ constructed above are continuous functions of time t and phase point x on the intervals of time between collisions. At time of collision, the trajectory $X(t,x)$ has a jump, and it is left continuous for $t>0$ and right continuous for $t<0$ (or for increasing and decreasing time, respectively).

We call the above defined evolution the *stochastic dynamics.*

Let us present a motivation of the definition of the concept of the stochastic dynamics. For this purpose, we recall that dynamics of hard spheres with diameter a is as follows:

They move as free particles if the distances between centers of two spheres are greater than a: $|Q_i(t) - Q_j(t)| > a$. If $|Q_i(t) - Q_j(t)| = a$, then they collide instantaneously and, after collision, their momenta $P_i^*(t)$, $P_j^*(t)$ are given by (1.2), where a vector η_{ij} is determined by the formula

$$\eta_{ij} = \frac{Q_i(t) - Q_j(t)}{|Q_i(t) - Q_j(t)|}. \tag{1.6}$$

For fixed p_i, p_j and $a \neq 0$, the collision of the i-th and j-th particles may take place for all η_{ij} defined according to (1.6) and belonging to semisphere (1.3) for $t>0$ or (1.5) for $t<0$ (or for increasing and decreasing time, respectively).

Let diameter a tend to zero, $a \to 0$, with fixed η_{ij}. In this limit, particles become pointwise ones and their dynamics coincides with the stochastic dynamics defined above with the same η_{ij} and with conditions (1.3) for $t>0$ or (1.5) for $t<0$. For the obtained point particles, momenta (1.2) after collision are determined for arbitrary η_{ij} from the corresponding semisphere S_-^2 or S_+^2 with equal probability. (Details of the proof will be published in a separate article.)

It is obvious that $X(t,x)$ considered as a function of t for fixed x and arbitrary fixed vectors η_{ij}, related to pair collisions has the semigroup property

$$X(t_1+t_2, x) = X(t_1, X(t_2,x)) = X(t_2, X(t_1,x)) \tag{1.7}$$

for arbitrary $t_1>0$, $t_2>0$ or $t_1<0$, $t_2<0$.

Note that $X(t,x)$ does not possess the group property because the condition of collision (1.3) for positive time $t>0$ does not coincide with the condition of collision (1.5) for negative time $t<0$ and the random vectors η_{ij} for different collisions are independent.

If $q_i = q_j$ for some pair $(i,j) \subset (1,\ldots,N)$ at initial time $t=0$, then the trajectory $X(t,x)$ is defined as follows:

$$X(t,x) = (q + p^*t, p^*), \quad p^* = (p_1, \ldots, p_i^*, \ldots, p_j^*, \ldots, p_N), \tag{1.8}$$

where p_i^*, p_j^* are defined as in (1.2) or (1.2′) with $\eta_{ij} \in S_-^2$ for $t > 0$ or $\eta_{ij} \in S_+^2$ for $t < 0$. If $\eta_{ij} \in S_+^2$ for $t > 0$ or $\eta_{ij} \in S_-^2$ for $t < 0$, then

$$X(t, x) = (q + pt, p), \tag{1.8′}$$

It is obvious that above mentioned properties of the trajectories are compatible with conditions (1.3), (1.3′) for increasing time because

$$(p_i^* - p_j^*) \cdot \eta = -\eta \cdot (p_i - p_j) < 0, \quad \eta \in S_+^2,$$

and with conditions (1.5), (1.5′) for decreasing time because

$$(p_i^* - p_j^*) \cdot \eta = -\eta \cdot (p_i - p_j) > 0, \quad \eta \in S_-^2.$$

Note that we consider point particles which may interact only if the vectors $p_i - p_j$ are parallel to the vectors $q_i - q_j$. The points $(q_i + p_i^* t, p_i^*, q_i + p_j^* t, p_j^*)$ satisfy these conditions.

This means that if we consider the hypersurface all points of which satisfy the condition that the vectors of difference of positions $q_1 - q_2$ *are parallel to the vectors of difference of momenta* $p_1 - p_2$, *then this hypersurface is invariant with respect to the stochastic dynamics.*

The union of the hypersurfaces (with respect to time)

$$Q_i(t, x) = Q_j(t, x), \quad (i, j) \subset (1, \dots, N), \quad t > 0 \ (t < 0), \tag{1.9}$$

which correspond to all pair collisions with fixed random vectors η_{ij}, has lower dimensionality than the phase space. The union of the corresponding points $X^*(t, x)$ after collisions, with respect to the random vectors $\eta_{ij} \in S_-^2$ and $t > 0$ or $\eta_{ij} \in S_+^2$ and $t < 0$, forms a set of the same dimensionality as the phase space (see Appendix I.1).

In what follows, we denote by η the collection of random vectors η_{ij} related to all pair collision in our system during the interval of time $[0, t]$, $t > 0$ or $[t, 0]$, $t < 0$.

One can say that stochastic trajectories are defined by initial momenta and positions and by random vectors η, but depend explicitly on η only after collisions.

2. Operator of evolution of finitely many particles. Consider a function $f_N(x_1, \dots, x_N) \equiv f(x)$ defined on the phase space of N particles. The operator of evolution of N particles $S_N(t)$ is formally defined by the following formula:

$$(S_N(t) f_N)(x_1, \dots, x_N) = (S_N(t) f_N)(t) =$$
$$= f_N(X(t, x)) = f_N(X_1(t, x), \dots, X_N(t, x)) \tag{2.1}$$

for arbitrary time $t > 0$ or $t < 0$.

The function $f(X(t, x))$ depends on time, initial phase point x, and random vectors η related to all pair collisions during the time interval $[0, t]$.

Consider the function $f_N(X(t, x))$ for arbitrary fixed realization of random vectors η. This means that we also consider the operator $S_N(t)$ for some realization of η and, in what follows, we consider the properties of the operator $S_N(t)$ associated with some realization of η.

Outside hypersurfaces (1.9), the trajectory $X(t, x)$ coincides with the trajectory of free particles, i.e., is a continuous function of time t and phase points x. Therefore, the function $f_N(X(t,x))$ is defined outside hypersurface (1.9), i.e., almost everywhere in the phase space (for fixed realization of η).

The semigroup property of the operator $S_N(t)$

$$S_N(t_1+t_2) = S_N(t_1)\,S_N(t_2) = S_N(t_2)\,S_N(t_1) \tag{2.2}$$

for arbitrary $t_1>0$, $t_2>0$ or $t_1<0$, $t_2<0$, follows from the semigroup property of $X(t,x)$.

Let us proceed to the determination of an infinitesimal operator of the semigroup $S_N(t)$ on the set of differentiable functions $f(x)$.

First, we consider the case $t\geq 0$ and differentiate the function $S_N(t)f_N(x)\equiv$ $\equiv S(t)f(x)$ at time $t=0$

$$\left.\frac{dS(t)f(x)}{dt}\right|_{t=0} = \lim_{\Delta t\to 0}\frac{f(X(\Delta t,x)) - f(x)}{\Delta t}. \tag{2.3}$$

If $q_i\neq q_j$ for arbitrary $(i,j)\in(1,\ldots,s)$, then

$$X(\Delta t,x) = (q+p\Delta t,p),$$

and

$$\left.\frac{dS(t)f(x)}{dt}\right|_{t=0} = \sum_{i=1}^{N} p_i\frac{\partial}{\partial q_i}f(x). \tag{2.4}$$

If $q_i=q_j$ for some pair $(i,j)\in(1,\ldots,s)$ and $\eta_{ij}\in S_-^2$, then

$$X(\Delta t,x) = (q+p^*\Delta t,p^*),$$

where

$$p^* = (p_1,\ldots,p_i^*,\ldots,p_j^*,\ldots,p_N),$$

and p_i^*, p_j^* are defined by (1.2).

In order to calculate (2.3), we choose the coordinate system so that the first component of the vector (q_i-q_j) is directed along the vector η_{ij}. In this coordinate system, we have

$$\begin{aligned} p_i^* &= \left(p_i^1-\left(p_i^1-p_j^1\right),p_1^2,p_i^3\right) = \left(p_j^1,p_i^2,p_i^3\right),\\ p_j^* &= \left(p_j^1-\left(p_j^1-p_i^1\right),p_j^2,p_j^3\right) = \left(p_i^1,p_j^2,p_j^3\right), \end{aligned} \tag{2.5}$$

and the function $f(X(t,x))$ has a jump at $t=0$ which is connected with jumps of $P_i^1(t)$ and $P_j^1(t)$ and is equal to $f(x^*)-f(x)$, $x^*=(q,p^*)$.

In this sense, the problem of calculation of derivative (2.3) is reduced to the one-dimensional case, where i-th and j-th particles change only the first components of their momenta. Taking this into account, we obtain from (2.3) for $q_i=q_j$

$$\begin{aligned} \left.\frac{dS(t)f(x)}{dt}\right|_{t=0} &= \lim_{\Delta t\to 0}\Big\{\big(f(q+p^*\Delta t,p^*) - f(q+p^*O(\Delta t),p^*) +\\ &\quad + f(q+p^*O(\Delta t),p^*) - f(q+pO(\Delta t),p)\big)\Big\}\Big/\Delta t =\\ &= \sum_{i=1}^{N} p_i\frac{\partial}{\partial q_i}f(x)\Big|_{p\to p_i^*} + \delta(t-\tau)\,[f(x^*)-f(x)]\,|_{t=0}, \end{aligned} \tag{2.6}$$

where τ is the time of collision,

$$\tau = \frac{q_i^1 - q_j^1}{p_i^1 - p_j^1}, \tag{2.7}$$

and $O(\Delta t)$ is an infinitesimal value of higher order.

By using the definition of the time of collision, one can represent the first term on the right-hand side of (2.6) as follows (for $q_i^2 = q_j^2$, $q_i^3 = q_j^3$):

$$\delta(t-\tau)[f(x^*) - f(x)] =$$

$$= \delta\left(q_i^1 - q_j^2\right) \left|p_i^1 - p_j^1\right| [f(x^*) - f(x)]|_{q_i^2 = q_j^2, q_i^3 = q_j^3}.$$

This expression considered as a generalized function in the three-dimensional space of the variables $q_i = q_j$ is concentrated on the first axis $q_i^1 - q_j^1$ (for $q_i^2 - q_j^2 = 0$, $q_i^3 - q_j^3 = 0$) and, therefore, is equal to

$$\delta\left(q_i^1 - q_j^1\right) \left|p_i^1 - p_j^1\right| [f(x^*) - f(x)]\, \delta\left(q_i^2 - q_j^2\right) \delta\left(q_i^3 - q_j^3\right) =$$

$$= \delta(q_i - q_j) \left|p_i^1 - p_j^1\right| [f(x^*) - f(x)].$$

(for analogous calculation, see [19, p. 48]).

Recalling that $p_i^1 - p_j^1 = \eta_{ij} \cdot (p_i - p_j)$, we obtain from (2.6) the following formula:

$$\left.\frac{dS(t)f(x)}{dt}\right|_{t=0} =$$

$$= \sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f(x)\Big|_{p \to p^*} + \delta(q_i - q_j)|\eta_{ij} \cdot (p_i - p_j)| [f(x^*) - f(x)]. \tag{2.8}$$

Formula (2.8) does not depend on the choice of a coordinate system because the scalar product $\eta_{ij} \cdot (p_i - p_j)$ is invariant under rotation.

If $\eta_{ij} \in S_+^2$, $\eta_{ij} \cdot (p_i - p_j) > 0$, then $X(t, x) = (q + pt, p)$, and the second term is absent in (2.8). In the general case where $\eta_{ij} \subset S^2$, we obtain

$$\left.\frac{dS(t)f(x)}{dt}\right|_{t=0} = \sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f(x) +$$

$$+ \delta(q_i - q_j)\theta(-\eta_{ij} \cdot (p_i - p_j)) |\eta_{ij} \cdot (p_i - p_j)| [f(x^*) - f(x)], \tag{2.8$'$}$$

where $\theta(a) = 1$, $a > 0$, $\theta(a) = 0$, $a < 0$.

In order to take into account all possible pair collisions of all particles, one should sum over all pairs $(i, j) \in (1, \ldots, N)$ in (2.8). After summation, we obtain the final formula for arbitrary $q = (q_1, \ldots, q_N)$

$$\left.\frac{dS(t)f(x)}{dt}\right|_{t=0} = \sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f(x) +$$

$$+ \sum_{i<j=1}^{N} \delta(q_i - q_j)\theta(-\eta_{ij} \cdot (p_i - p_j)) |\eta_{ij} \cdot (p_i - p_j)| \times$$

$$\times [f(x^*) - f(x)] = \mathcal{H}_N^+ f(x). \tag{2.9}$$

For $q_i = q_j$, one must replace $p = (p_1, \ldots, p_N)$ in the first term on the right-hand side by $p^* = (p_1, \ldots, p_i^*, \ldots, p_j^*, \ldots, p_N)$. We regard it as the boundary condition in the Poisson bracket of free particles.

Thus, the second term on right-hand side of (2.8) is taken into account in (2.9).

By using the semigroup property (2.2), one can prove the following formula for the derivative of $S(t)f(x)$ at arbitrary time $t > 0$

$$\frac{dS(t)f(x)}{dt} = \sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} S(t)f(x) +$$

$$+ \sum_{i<j=1}^{N} \delta(q_i - q_j)\theta(-\eta_{ij}\cdot(p_i - p_j))\,|\eta_{ij}\cdot(p_i - p_j)| \times$$

$$\times [S(t)f(x^*) - S(t)f(x)] = \mathcal{H}_N^+ S(t)f(x),$$

$$\frac{dS(t)f(x)}{dt} = \sum_{i=1}^{N} P_i(t)\frac{\partial}{\partial Q_i(t)} f(X(t)) + \sum_{i<j=1}^{N} \delta(Q_i(t) - Q_j(t)) \times$$

$$\times \theta(-\eta_{ij}\cdot(P_i(t) - P_j(t)))\,|\eta_{ij}\cdot(P_i(t) - P_j(t))|\,[f(X^*(t)) - f(X(t))]$$

$$= S(t)\,\mathcal{H}_N^+ f(x), \tag{2.10}$$

Indeed, formula (2.10) follows directly from the identity

$$S(t+\Delta t) - S(t) = (S(\Delta t) - I)\,S(t) = S(t)(S(\Delta t) - I). \tag{2.11}$$

We stress that in the expressions $S(t)f(x^*)$ and $S(t)f(x)$ in (2.10) the initial phase points x^* and x are considered as the states after collision at the time $t = +0$. This means that, for $t > 0$, particles have the momenta p^* in the first expression $S(t)f(x^*) = f(X(t, x^*))$ and p in the second expression $S(t)f(x) = f(X(t, x))$ until new collisions take place (see Appendix I.2).

Formulas (2.6)–(2.11), (2.12)–(2.14) have been obtained as a derivative of the generalized function $S_N(t)f(x)$ regarded as a function of time t (see Appendix I.1, 2).

The case of negative time can be considered in a completely analogous way.

Consider the function $S_N(-t)f(x)$ for $t > 0$ and define its derivative at $t = 0$ by the formula

$$\left.\frac{dS(-t)f(x)}{dt}\right|_{t=0} = \lim_{\Delta t \to 0} \frac{f(X(-\Delta t, x)) - f(x)}{\Delta t}. \tag{2.12}$$

By repeating the calculation performed above almost word for word, we get

$$\left.\frac{dS_N(-t)f(x)}{dt}\right|_{t=0} = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f(x) +$$

$$+ \sum_{i<j=1}^{N} \delta(q_i - q_j)\theta(\eta_{ij}\cdot(p_i - p_j))\,\eta_{ij}\cdot(p_i - p_j)[f(x^*) - f(x)] =$$

$$= \mathcal{H}_N^- f(x). \tag{2.13}$$

For $q_i = q_j$, $\eta_{ij} \in S_+^2$, one should replace p_i, p_j in the second term on the right-hand side of (2.13) by p_i^*, p_j^*.

For arbitrary negative time, one obtains

$$\frac{dS(-t)f(x)}{dt} = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} S(-t)f(x) +$$

$$+ \sum_{i<j=1}^{N} \delta(q_i - q_j)\theta(\eta_{ij}\cdot(p_i - p_j))\,\eta_{ij}\cdot(p_i - p_j)$$

$$\times [S(-t)f(x^*) - S(-t)f(x)] = \mathcal{H}_N^- S(-t)f(x),$$

$$\frac{dS(-t)f(x)}{dt} = -\sum_{i=1}^{N} P_i(-t)\frac{\partial}{\partial Q_i(-t)} f(X(-t)) + \sum_{i<j=1}^{N} \delta(Q_i(-t) - Q_j(-t)) \times$$

$$\times\, \theta(\eta_{ij}\cdot(P_i(-t) - P_j(-t)))\,[f(X^*(-t)) - f(X(-t))]\left|\eta_{ij}\cdot(P_i(-t) - P_j(-t))\right| =$$

$$= S(-t)\,\mathcal{H}_N^- f(x), \tag{2.14}$$

We stress again that, in the expressions $S(-t)f(x^*)$ and $S(-t)f(x)$, the initial phase points x^* and x are considered as the states after collisions at the time $t=-0$. The motivation is completely the same as for $t>0$. We denote the infinitesimal operator of the semigroup $S_N(-t)$ by $\mathcal{H}_N^-$.

Note that the difference $S(-t)f(x) - S^\circ(t)f(x)$ is different from zero in a certain set D_{-t} of Lebesgue measure zero (see Appendix I.1). Here, $S^\circ(t)$ is the evolution operator of free particles.

3. Sequences of reduced distribution functions and the stochastic hierarchy. In the classical statistical mechanics, the states of a system of N particles are described by the distribution functions $D_N(t, x_1, \dots, x_N)$, which are defined by the formula

$$D_N(t,x) = D_N(t, x_1, \dots, x_N) = S_N(-t)D_N(0, x_1, \dots, x_N), \quad t>0, \tag{3.1}$$

where $D_N(0, x_1, \dots, x_N) = D_N(0, x)$ is an initial state (an initial distribution function).

We define the state of the system of N particles with the stochastic dynamics by the same formula (3.1). But in the case of the system with the stochastic dynamics, the distribution function $D_N(t,x)$ depends not only on time t and phase point x but also on the random vectors $\eta = (\eta_{l_2}, \dots, \eta_{l_N})$ that determine the momenta after pair collisions during the interval $[0, t]$.

We denote by η a collection of all random vectors η_{ij} and, for a sake of simplicity, renumerate them $\eta_{l_2}, \dots, \eta_{l_N}$, $l_2 < \dots < l_3 < \dots < l_N$, $l_2 = 1$. Here, l_s, $2 \le s \le N$, are equal to the numbers of collisions of the s-th particle with all particles with numbers less than s.

In what follows, we consider the function $D_N(t, x_1, \dots, x_N)$ for arbitrary fixed random vectors η, i.e., for fixed realization of η.

The function $D_N(t, x_1, \dots, x_N)$ is symmetric, i.e., invariant under permutations

$$D_N(t, x_1, \dots, x_N) = D_N(t, x_{i_1}, \dots, x_{i_N}), \tag{3.2}$$

where $i_1, \dots, i_N$ is an arbitrary permutation of numbers $(1, \dots, N)$. This property is fulfilled if the initial distribution function $D_N(0, x_1, \dots, x_N)$ is symmetric. In what follows, we consider the initial distribution functions $D_N(0, x_1, \dots, x_N)$ that belong to the space of summable function L_N, are symmetric, positive, and normalized by unity

$$\int D_N(0, x_1, \ldots, x_N)\, dx_1 \ldots dx_N = 1.$$

By using formulas (2.6)–(2.14), we obtain an analog of the Liouville equation for $D_N(t, x_1, \ldots, x_N)$

$$\frac{\partial}{\partial t} D_N(t, x_1, \ldots x_N) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} D_N(t, x_1, \ldots, x_N) +$$

$$+ \sum_{i<j=1}^{N} \delta(q_i - q_j)\, \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times [D_N(t, x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_N) - D_N(t, x_1, \ldots, x_N)], \tag{3.3}$$

where the condition of collision $\eta_{ij} \cdot (p_i - p_j) > 0$ was taken into account by the θ-function. It is necessary to add the boundary condition to Poisson bracket and to replace p_i, p_j by p_i^*, p_j^* for $q_i = q_j$.

Introduce a sequence of reduced distribution functions (or simply the distribution functions) for a system of N particles as follows

$$F_s(t, x_1, \ldots, x_N) =$$

$$= N(N-1) \ldots (N-s+1)\, M \int D_N(t, x_1, \ldots, x_s, x_{s+1}, \ldots, x_N)\, dx_{s+1} \ldots dx_{s+n}, \tag{3.4}$$

$$s = 1, 2, \ldots, N,$$

where M means the operation of averaging over the random variables $\eta_{l_s+1}, \ldots, \eta_{l_N}$ which correspond to the collisions of each of the particles with numbers $s+1, \ldots, N$, with the other $N-1$ particles, or to all possible collisions of all N particles except the collisions between the particles with numbers $1, 2, \ldots, s$. We fix the random vectors η related to the collision of the particles with the numbers $1, 2, \ldots, s$, i.e., we consider some realization of $\eta = (\eta_{l_2}, \ldots, \eta_{l_s})$.

The s-particles distribution functions, which do not depend on any random vectors, is defined as follows

$$N(N-1) \ldots (N-s+1)\, M \times$$

$$\times \int D_N(t, x_1, \ldots, x_s, x_{s+1}, \ldots, x_N)\, \varphi_s(x_1, \ldots, x_s)\, dx_1 \ldots dx_N =$$

$$= \int F_s(t, x_1, \ldots, x_s)\, \varphi(x_1, \ldots, x_s)\, dx_1 \ldots dx_s, \tag{3.4'}$$

where $\varphi_s(x_1, \ldots, x_s)$ is a test function and the averaging operation M corresponds to all the random vectors.

In order to obtain the distribution functions (3.4), which depend on random vectors $\eta_{l_2}, \ldots, \eta_{l_s}$, one should perform the averaging procedure in (3.4′) only with respect to the random vectors of the N-particles system, excluding the random vectors of the s-particle system $\eta_{l_2}, \ldots, \eta_{l_s}$.

We stress that this new conception of distribution functions takes into account the set of lower dimension where particles interact and contribution of this set is different from zero (see Appendix I.1, 2).

For justification of the operation of averaging M see Appendix I.1, 2.

We denote the sequence of the distribution functions (3.4) by

$$F(t) = (F_1(t, x_1), \dots, F_s(t, x_1, \dots, x_s), \dots, F_N(t, x_1, \dots, x_N)).$$

The sequence $F(t)$ is the state of a system of N particles at time t in the framework of canonical ensemble.

Let us proceed to derivation of an equation for the state $F(t)$. For this purpose, we differentiate the left- and right-hand sides of (3.4) with respect to time. By differentiating under the integral sign and using (3.3), we obtain

$$\frac{\partial F_s(t, x_1, \dots, x_N)}{\partial t}$$

$$= N(N-1)\dots(N-s+1)\, M \int \frac{\partial}{\partial t} D_N(t, x_1, \dots, x_s, x_{s+1}, \dots, x_N)\, dx_{s+1} \dots dx_N =$$

$$= \frac{N!}{(N-s)!} M \int \Bigg\{ -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} D_N(t, x_1, \dots, x_s, x_{s+1}, \dots, x_N) +$$

$$+ \sum_{i<j=1}^{N} \delta(q_i - q_j)\, \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times [D_N(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_N) - D_N(t, x_1, \dots, x_s, x_{s+1}, \dots, x_N)] \Bigg\}\, dx_{s+1} \dots dx_N. \quad (3.5)$$

It is easy to see that the integration of the expressions

$$-\sum_{i=1}^{S} p_i \frac{\partial}{\partial q_i} D_N(t, x_1, \dots, x_s, x_{s+1}, \dots, x_N) +$$

$$+ \sum_{i<j=1}^{S} \delta(q_i - q_j)\, \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j)) [D_N(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots$$

$$\dots, x_s, x_{s+1}, \dots, x_N) - D_N(t, x_1, \dots, x_s, x_{s+1}, \dots, x_N)]$$

on the right-hand side of (3.5) yields

$$-\sum_{i=1}^{S} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \dots, x_s) + \sum_{i<j=1}^{S} \delta(q_i - q_j)\, \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times [F_s(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_s) - F_s(t, x_1, \dots, x_s)]. \quad (3.6)$$

Further, the integration of the expression

$$\sum_{i=1}^{s} \sum_{j=s+1}^{N} \delta(q_i - q_j)\, \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j)) [D_N(t, x_1, \dots, x_i^*, \dots$$

$$\dots, x_s, x_{s+1}, \dots, x_j^*, \dots, x_N) - D_N(t, x_1, \dots, x_s, x_{s+1}, \dots, x_N)]$$

yields

$$\sum_{i=1}^{s} \int dp_{s+1}\, d\eta_{is+1}\, \eta_{is+1} \cdot (p_i - p_{s+1})\, \theta(\eta_{is+1} \cdot (p_i - p_{s+1})) \times$$

$$\times [F_{s+1}(t, x_1, \dots, x_i^*, \dots, x_s, x_{s+1}^*) - F_{s+1}(t, x_1, \dots, x_i, \dots, x_s, x_{s+1})]_{q_{s+1} = q_i}. \quad (3.7)$$

Here, we have used the invariance of the function $D_N(t, x_1, \dots, x_N)$ under the permutation.

The contribution of the term

$$-\sum_{i=s+1}^{N} p_i \frac{\partial}{\partial q_i} D_N(t, x_1, \ldots, x_N) \tag{3.8}$$

is zero because the function $D_N(t, x_1, \ldots, x_N)$ tends to zero as $|q_i| \to \infty$.

The contribution of the term

$$\sum_{i<j=s+1}^{N} \delta(q_i - q_j)\, \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j)) \text{ ¥}$$

$$\times \left[D_N(t, x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_N) - D_N(t, x_1, \ldots, x_N)\right]$$

is equal to the integral

$$\int dx_{s+1}\, dx_{s+2}\, d\eta\, \delta(q_{s+1} - q_{s+2})\, \eta \cdot (p_{s+1} - p_{s+2})\, \theta(\eta \cdot (p_{s+1} - p_{s+2})) \times$$

$$\times \left[F_{s+2}(x_1, \ldots, x_s, x_{s+1}^*, x_{s+2}^*) - F_{s+2}(x_1, \ldots, x_s, x_{s+1}, x_{s+2})\right], \tag{3.9}$$

which is zero. This fact can be proved, using new variables $p_{s+1}^* = p'_{s+1}$, $p_{s+2}^* = p'_{s+2}$ and taking into account that the Jacobian of this transformation is equal to unity.

Summing up the above performed calculation, we obtain the following chain of equations:

$$\frac{\partial F_s(t, x_1, \ldots, x_N)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \ldots, x_s) +$$

$$+ \sum_{i<j=1}^{s} \delta(q_i - q_j)\, \eta_{ij} \cdot (p_i - p_j)\, \theta(\eta_{ij} \cdot (p_i - p_j)) \times$$

$$\times \left[F_s(t, x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_s) - F_s(t, x_1, \ldots, x_s)\right] +$$

$$+ \sum_{i=1}^{s} \int dp_{s+1}\, d\eta_{is+1}\, \eta_{is+1} \cdot (p_i - p_{s+1})\, \theta(\eta_{is+1} \cdot (p_i - p_{s+1})) \times$$

$$\times \left[F_{s+1}(t, x_1, \ldots, x_i^*, \ldots, x_s, x_{s+1}^*) - F_{s+1}(t, x_1, \ldots, x_s, x_{s+1})\right]_{q_{s+1} = q_i}, \tag{3.10}$$

$$s = 1, 2, \ldots, N.$$

We call equations (3.10) the *stochastic hierarchy* or *the stochastic Boltzmann hierarchy.*

The obtained stochastic hierarchy (3.10) is equivalent to the Liouville equation (3.3); moreover, the last equation for $F_N(t, x_1, \ldots, x_N)$ exactly coincides with the Liouville equation (3.3). (Strictly speaking, in hierarchy (3.10), we have used the renormalized distribution functions $F'_s = (4\pi)^s F_N$.)

Hierarchy (3.10) describes the evolution of the state $F(t)$ of a finite system. In order to describe the state of infinite system within the framework of canonical ensemble it is necessary to perform the thermodynamic limit procedure and to let the number of particles tends to infinity $N \to \infty$.

Or, more exactly, one must consider the system of N particles at initial time $t = 0$ situated in some bounded domain Λ of the configurational space (this means that $F_s(0, x_1, \ldots, x_s) = 0$ if $q_i \notin \Lambda$ at least for one particle) and then let the number of particles and the volume $V(\Lambda)$ of the domain Λ tend to infinity ($N \to \infty$, $V(\Lambda) \to \to \infty$) so that the density is constant

$$\frac{1}{v} = \frac{N}{V(\Lambda)} = \text{const.}$$

The stochastic hierarchy of infinite systems in the thermodynamic limit has form (3.10) but with $1 \le s \le \infty$. The problem of mathematical justification of the thermodynamic limit procedure is discussed below in Section 7.

Let us introduce a sequence of distribution functions within the framework of grand canonical ensemble, when a system consists of arbitrary number of particles with certain probability. The s-particle distribution functions are defined according to the formula

$$F_s(t, x_1, \dots, x_s) =$$

$$= \frac{1}{\Xi} \sum_{N=0}^{\infty} \frac{1}{N!} M \int D_{s+N}(t, x_1, \dots, x_s, x_{s+1}, \dots, x_{s+N}) \, dx_{s+1} \dots dx_{s+N}, \quad (3.11)$$

where the functions $D_{s+N}(t, x_1, \dots, x_{s+N})$ are defined by (3.1) and the integration with respect to the variables $\eta_{l_s+1}, \dots, \eta_{l_N}$ is carried out over the random variables that correspond to all collisions of the particles with the numbers $1, \dots, s$, $s+1, \dots, s+N$, except the collisions of the particles with the numbers $1, \dots, s$. The random vectors η related to collisions of the particles with the numbers $1, \dots, s$ are fixed. Note that, in this case, the functions $D_{s+N}(0, x_1, \dots, x_{s+N})$ are not normalized by unity.

In (3.11), Ξ is the grand partition function,

$$\Xi = \sum_{N=0}^{\infty} \frac{1}{N!} M \int D_N(t, x_1, \dots, x_N) \, dx_1 \dots dx_N =$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} M \int D_N(0, x_1, \dots, x_N) \, dx_1 \dots dx_N,$$

$$D_N(t, x) = S_N(-t) D_N(0, x).$$

Here, we have used the Liouville theorem (see Appendix I.2).

It is easy to prove that the sequence of the functions

$$F(t) = (F_1(t, x_1), \dots, F_s(t, x_1, \dots, x_s), \dots) \quad (3.12)$$

satisfies the chain of equations (3.10) [8]. The chain of equations (3.10) is called the *stochastic hierarchy*.

Sequence (3.12) describes the state of a system with the stochastic dynamics within the framework of canonical and grand canonical ensemble.

In order to perform the thermodynamic limit procedure within framework of grand canonical ensemble, it is necessary to let the average number of particles $\overline{N}$ tend to infinity or, if the system of particles are situated in a bounded domain Λ of the configurational space, let the volume $V(\Lambda)$ also tend to infinity so that the density

$$\frac{1}{v} = \frac{\overline{N}}{V(\Lambda)} = \text{const.}$$

The problem of justification of the thermodynamic limit is discussed in Section 7.

1. *Bogoliubov N. N.* Problems of dynamical theory in statistical physics. – M.: Gostekhizdat, 1946 (also in Bogoliubov N. N., Selected Works. – Kiev: Naukova Dumka, 1970. – Vol. 2. – P. 99–196.
2. *Cercignani C.* The Grad limit for a system of soft spheres // Communs Pure and Appl. Math. – 1983. – **36**. – P. 479–494.

3. *Lanford O. E.* Time evolution of large classical systems // Lect. Notes Phys. - 1975. – № 38..
4. *Gerasimenko V. I., Petrina D. Ya.* Existence of the Boltzmann–Grad limit for infinite system of hard sphere // Teoret. i Mat. Fiz. – 1990. – **83**. – P. 92–114.
5. *Petrina D. Ya., Gerasimenko V. I.* Mathematical problems of statistical mechanics of hard sphere particle system // Usp. Mat. Nauk. – 1990. – **45**. – P. 192–214.
6. *Petrina D. Ya., Gerasimenko V. I.* Evolution of states of infinite systems in classical statistical mechanics // Sov. Sci. Rev. Ser. C Math. Phys. – 1984. – **5**. – P. 1–60.
7. *Petrina D. Ya., Gerasimenko V. I., Malyshev P. V.* Thermodynamic limit for solutions of Bogoliubov's equations // Ibid. - 1987. – **7**. –P. 280–335.
8. *Petrina D. Ya., Gerasimenko V. I., Malyshev P. V.* Mathematical foundations of classical statistical mechanics. – Continuous systems. – New York: Gordon and Breach, 1989.
9. *Tanaka H.* Probabilistic treatment of the Boltzmann equation of Maxwellian molecules // Z. Warscheinlichkeitstheor. und verw. Geb. – 1978. – **46**. – P. 67–105.
10. *Sznitman A. S.* Equations de type de Boltzmann, spatialement homogenes // Ibid. – 1984. – **66**. – P. 559–592.
11. *Arsen'ev A. A.* On the approximation of the solution of the Boltzmann equation by solution of Itô stochastic differential equations // Z. Vychisl. Mat. i Mat. Fiz. – 1987. – **27**. – P. 400–410.
12. *Arsen'ev A. A.* Approximation of the Boltzmann equation by stochastic equations // Ibid. - 1988. – **28**. – P. 560–567.
13. *Skorokhod A. V.* Stochastic equation for complex systems. – Moscow: Nauka, 1983.
14.. *Lachowicz M., Pulvirenti M.* A stochastic particle system modeling the Euler equation // Arch. Ration. Mech. and Anal. – 1990. – **109**. – P. 81–93.
15. *Caprino S., De Masi A., Presutti E., and Pulvirenti M.* A stochastic particle system modeling the Carleman equation // J. Statist. Phys. – 1989. – **55**. – P. 625–638.
16.. *Lachowicz M.* A system of stochastic differential equations modeling the Euler and the Navier–Stokes hydrodynamic equations // Jap. J. Industr. and Appl. Math. – 1993. – **10**. – P. 109–131.
17. *Albeverio S., Gesztesy S., Høegh-Krohn R., Holden H.* Solvable models in quantum mechanics. - New York; Berlin: Springer, 1988.
18. *Koshmanenko V. D.* Singular quadratic forms in the theory of perturbations of self-adjoint operators. – Kiev: Naukova Dumka, 1993.
19. *Gelfand I. M., Graev M. I., Vilenkin N. Ya.* Generalized functions. – Moscow: Nauka, 1962.
20. *Krein S. G.* Linear differential equations in Banach space. - Moscow: Nauka, 1967.
21. *Cercignani C., Gerasimenko V. I., Petrina D. Ya.* Many-particle dynamics and kinetic equations.- Dordrecht: Kluwer, 1997.
22. *Cercignani C., Illner R., Pulvirenti M.* The mathematical theory of dilute gases. – New York: Springer-Verlag, 1997.
23. *Bogolyubov N. N., Bogolyubov N. N. (jr.)* Introduction to statistical mechanics. – Moscow: Nauka, 1984.
24. *Cercignani C.* On the master equation in the space in homogeneous case / Pichon G. (ed.) Theories cinetiques classiques et relativistes // Colloq. Int. CNRS. – 1975. – № 236.– P. 209–221.
25. *Dorfman J. R., Ernst M. H.* Hard-sphere binary-collision operators // J. Statist. Phys. – 1989. – **57**. – P. 581 – 593.
26. *Di Perna R. J., Lions P. L.* Global solutions of Boltzmann's equation and the entropy inequality // Arch. Ration. Mech. and Anal. – 1991. – P. 47 – 55.
27. *Di Perna R. J., Lions P. L.* On the cauchy problem for Boltzmann equations: Global existence and weak stability // Ann. Math. – 1989. – **130**. – P. 321 – 366.

Received 06.06.96