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# STABILITY OF STOCHASTIC SYSTEMS IN DIFFUSION APPROXIMATION SCHEME

# СТІЙКІСТЬ СТОХАСТИЧНИХ СИСТЕМ У СХЕМІ ДИФУЗІЙНОЇ АПРОКСИМАЦІЇ

By using the solution of singular perturbation problem, we obtain the sufficient conditions of stability of a dynamical system with rapid Markov switchings under the condition of exponential stability of the averaged diffusion process.

Одержано достатні умови стійкості динамічної системи зі швидкими марковськими перемиканнями при умові експоненціальної стійкості усередненого дифузійного процесу з використанням розв'язку проблеми сингулярного збурення.

**1. Introduction.** The asymptotic analysis of stochastic systems is developed mainly in two directions:

I. The analysis of stochastic systems in the series scheme — average and diffusion approximation.

II. The investigation of system on increasing time intervals — stability problem.

The first type of problems leads to the average theory created by N. N. Bogolyubov [1] and devolved by Yu. A. Mitropolskii and A. M. Samoilenko [2], I. I. Gikhman [3, 4], A. V. Skorokhod [5] and many others.

The second type of problems leads to the stability theory founded by A. M. Lyapunov [6] for deterministic systems and by H. J. Kushner [7], N. N. Krasovskii and J. Ya. Kac [8], R. Z. Khasminskii [9, 10], A. V. Skorokhod [11], E. F. Tsarkov [12, 13] among others for stochastic systems.

Naturally, one can consider a mixed stability problem, the analysis of stochastic system on increasing time intervals under averaging or diffusion approximation conditions.

This is the problem of stability of an initial stochastic system under the stability condition for an averaged system.

A problem of this sort was first stated by N. N. Bogolyubov for deterministic evolutional systems. The best possible result was obtained by A. M. Samoilenko [2]. The stability of stochastic system with wide band noise disturbances under diffusion approximation conditions was stated by G. L. Blankenship and G. C. Papanicolaou [14] and, later on, by E. F. Tsarkov [12, 13] for stochastic systems with delay. The mixed stability problem for dynamical system with rapid Markov switchings was considered in [15] and with semi-Markov switchings in [16]. It is natural that, for sufficiently small values of the parameter series , the stability of the averaged system provides the stability of an initial system. The basic approach consists of Lyapunov's perturbed function is considered by using a solution of singular perturbation problem for a reducibly invertible operator.

The remarkable fact is that every solution of singular perturbation problem can be utilized in the stability mixed problem under average, double average and diffusion approximation conditions.

**2. The method of Lyapunov's functions.** The method of Lyapunov's functions is widely used in the analysis of stability problems. As is well-known, the trajectory of dynamical system processes the semigroup property which in an abstract form can be formulated as follows. Let the trajectory of dynamical system $U(t)$ be determined by a solution of the autonomous differential equation

$$dU(t)/dt = C(U(t)). \tag{1}$$

The semigroup of operators $C_t$ is defined by

$$C_t V(u) := C(U_t), \qquad U_0 = u, \tag{2}$$

in a Banach space $B_U$ of measurable real-valued bounded functions $V(u)$, $u \in U$, with sup-norm: $\| V(u) \| := \sup_{u \in U} | V(u) |$. The generator of the semigroup (2) acts in the following manner:

$$CV(u) := \lim_{t \to 0} t^{-1} [C_t - I] V(u) = C(u) dV(u)/du.$$

The integral equation for the semigroup

$$V(U_t) = V(u) + \int_0^t CV(U_s) ds$$

can be rewritten with the derivative along a path of a dynamical system

$$\dot{V}(u) := CV(u) = C(u) dV(u)/du \tag{3}$$

in the following form:

$$V(U_t) = V(u) + \int_0^t \dot{V}(U_s)\, ds. \tag{4}$$

In the stability theory, derivative (3) is called the Lyapunov operator. Now the stability conditions for the deterministic dynamical system (1) can be formulated in the explicit form. There is existence of Lyapunov's functions $V(u)$ satisfies the following two conditions:

(C1) $V(u) > 0, \quad u \neq 0; \quad V(0) = 0;$
(C2) $V(u) \leq 0.$

The source inequality

$$V(U_t) \leq V(u) \tag{5}$$

can be obtained from the integral equation (4) for the semigroup. The stability of the trajectory is the simple corollary of this inequality (5).

The same approach can be used in the stability problem for a Markov process $\eta_t$, $t \geq 0$ [10]. Now semigroup of operators $P_t$ is given by a well-known relation with a conditional expectation

$$P_t V(u) := E[V(\eta_t)/\eta_0 = u]$$

using the generator

$$QV(u) := \lim_{t \to 0} t^{-1} [P_t - I] V(u)$$

of the Markov semigroup and the martingale characterization of the Markov process $\eta_t$, $t \geq 0$, in the form [7]

$$V(\eta_t) - V(u) - \int_0^t QV(\eta_s) ds = \mu_t.$$

We can conclude that the derivative along a path of Markov process is defined as

$$\dot{V}(u) = QV(u).$$

Again, the stability conditions for the Markov process can be formulated for Lyapunov's function $V(u)$ in the following form:

(C1) $V(u) > 0, \quad u \neq 0; \quad V(0) = 0;$
(C2) $\dot{V}(u) \leq 0.$

The source inequality has the form

$$V(\eta_t) \le V(u) + \mu_t.$$

The last step in the stability problem for the Markov process is realized by using the well-known inequality for the supremum of supermartingales.

**3. Ergodic Markov switchings.** The initial stochastic system is considered in the following form:

$$dU^\varepsilon(t)/dt = \varepsilon^{-1} C_1(U^\varepsilon(t), \kappa(t/\varepsilon^2)) + C_0(U^\varepsilon(t), \kappa(t/\varepsilon^2)). \tag{6}$$

The velocity functions $C_k(u, x)$, $u \in R^d$, $x \in X$, $k = 1, 0$, satisfy the conditions which provide the existence of a global solution of the deterministic dynamical equations for every $\varepsilon > 0$:

$$dU_x^\varepsilon(t)/dt = \varepsilon^{-1} C_1(U_x^\varepsilon(t), x) + C_0(U_x^\varepsilon(t), x), \quad x \in X. \tag{7}$$

The switching process $\kappa(t)$, $t \ge 0$, is a Markov jump process on a measurable phase space $(X, \mathcal{X})$ with the generator

$$Q\varphi(x) = q(x) \int_X P(x, dy)[\varphi(y) - \varphi(x)]. \tag{8}$$

The stochastic kernel $P(x, B)$, $x \in X$, $B \in \mathcal{X}$ is supposed to define the imbedded Markov chain $\kappa_n$, $n \ge 0$, uniformly ergodic with the stationary distribution $\rho(B)$, $B \in \mathcal{X}$. The stationary distribution of the Markov process $\kappa(t)$, $t \ge 0$, is defined by the relation

$$\pi(dx)q(x) = q\rho(dx), \qquad q = \int_X \pi(dx)q(x).$$

The intensity function is supposed to be bounded:

$$\sup_{x \in X} q(x) \le C < +\infty.$$

The main property of the generator $Q$ is the *reducible invertibility* [17]. The stationary distribution $\pi(dx)$ defines the projector $\Pi$ on the null-space of the generator $Q$ as

$$\Pi\varphi(x) := \int_X \pi(dx)\varphi(x) =: \hat{\varphi}1,$$

where

$$\hat{\varphi} := \int_X \pi(dx)\varphi(x), \qquad 1 := 1(x) \equiv 1, \quad x \in X.$$

There exists a solution of the equation

$$Q\varphi(x) = \psi(x), \qquad \Pi\psi(x) = 0$$

which can be represented as

$$\varphi(x) = R_0\psi(x), \qquad \Pi\psi(x) = 0,$$

where the potential operator $R_0$ is determined by

$$R_0 := \int_0^\infty [\Pi - P_t]dt = [Q + \Pi]^{-1} - \Pi; \tag{9}$$

here, $P_t$, $n \geq 0$, is a semigroup of operator determined by the transition probabilities of the Markov process $\kappa(t)$, $t \geq 0$. The potential operator $R_0$ is a reducible inverse to the generator $Q$:

$$QR_0 = R_0 Q = I - \Pi.$$

The solution $U^\varepsilon(t)$ of stochastic system (6) coupled with the switching process form the two component Markov process

$$U_t^\varepsilon := U^\varepsilon(t), \quad \kappa_t^\varepsilon := \kappa(t/\varepsilon), \quad t \geq 0,$$

which is determined by the generator

$$L^\varepsilon \varphi(u, x) = [\varepsilon^{-2} Q + \varepsilon^{-1} C_1(x) + C_0(x)] \varphi(u, x), \tag{10}$$

where the generators $C_k(x)$, $x \in X$, $k = 1, 0$, associated with deterministic system (7) are acting in the following way:

$$C_k(x)\varphi(u) := C_k(u, x)\varphi'(u), \quad k = 1, 0.$$

Note that, in the case of $u \in R^d$,

$$C(u, x)\varphi(u) := \sum_{i=1}^{d} C^i(u, x)\partial\varphi(u)/\partial u_i$$

by definition. The generator $L^\varepsilon$ in (10) has a singular perturbed form with the reducible invertible operator $Q$ and the perturbing operators $C_k(x)$, $x \in X$, $k = 1, 0$. The stability mixed problem for stochastic system (6) is realized in the following statement:

**Theorem 1.** A (Average). *Let* $\kappa(t)$, $t \geq 0$, *be the Markov jump process uniformly ergodic on a measurable phase space* $(X, \mathcal{X})$ *with the stationary distribution* $\pi(B)$, $B \in \mathcal{X}$, *and the potential operator* $R_0$ *defined by* (9). *Let the balance condition*

$$\int_X \pi(dx) C_1(u, x) \equiv 0 \tag{11}$$

*be satisfied. Then the solution* $U^\varepsilon(t)$ *of stochastic system* (6) *converges weakly to the diffusion process* $\zeta(t)$ *as* $\varepsilon \to 0$, *which is determined by the generator*

$$L\varphi(u) = a(u)\varphi'(u) + B^2(u)\varphi''(u),$$

*where*

$$a(u) = \int_X \pi(dx) a(u, x),$$

$$a(u, x) = C_0(u, x) + C_1(u, x) R_0 C'_{1u}(u, x),$$

$$B^2(u) = \int_X \pi(dx) C_0(u, x) R_0 C_0(u, x).$$

S (Stability). *Let* $V(u)$ *be Lyapunov's function for the limit diffusion satisfying the exponential stability condition*

$$LV(u) \leq -cV(u), \quad c > 0.$$

*Let the velocity functions* $C_k(u, x)$, $k = 1, 0$, *satisfy the following inequalities*:

(i) $|C_k(u, x)V'(u)| \leq c_1 V(u)$, $k = 1, 0$;

(ii) $| C_k(u,x) R_0 [C_r(u,x) V'(u)]' | \le c_2 V(u), \ k, r = 1, 0;$

(iii) $| C_k(u,x) | R_0 [C_1(u,x) R_0 [C_1(u,x) V'(u)]']' | \le c_3 V(u), \ k = 1, 0.$

*Then, for every sufficiently small* $\varepsilon \le \varepsilon_0$, *the stochastic system* $U^{\varepsilon}(t)$ *is asymptotically stable with probability one*:

$$\mathcal{P}\left\{ \lim_{t \to \infty} \| U^{\varepsilon} \| = 0 \right\} = 1.$$

**4. Diffusion approximation with merging and averaging.** The initial stochastic system is considered in the form

$$dU^{\varepsilon}(t)/dt = \varepsilon^{-1} C_1 ( U^{\varepsilon}(t), \kappa_{\varepsilon}(t/\varepsilon^3)) + C_0 ( U^{\varepsilon}(t), \kappa_{\varepsilon}(t/\varepsilon^3)), \tag{12}$$

where the Markov jump process $\kappa_{\varepsilon}(t), \ t \ge 0$, satisfies the *phase merging condition* [18]. Therefore, we have the generator

$$Q_{\varepsilon} \varphi(x) = q(x) \int_X P_{\varepsilon}(x, dy) [\varphi(y) - \varphi(x)].$$

The stochastic kernel $P_{\varepsilon}(x, B)$ is represented as

$$P_{\varepsilon}(x, B) = P(x, B) + \varepsilon P_1(x, B).$$

The stochastic kernel $P(x, B)$ is coordinated with the splitting of phase space

$$X = \bigcup_{v \in V} X_v, \quad X_v \cap X_{v'} = \varnothing, \quad v \ne v', \tag{13}$$

as follows:

$$P(x, X_v) = \begin{cases} 1, & x \in X_v; \\ 0, & x \notin X_v. \end{cases}$$

The support Markov process $\kappa(t), \ t \ge 0$, given by generator (8) is supposed to be uniformly ergodic in every class $X_v, \ v \in V$, with stationary distributions $\pi_v(dx), \ v \in \in V$.

Under the condition

$$p_v = \int_{X_v} \pi_v(dx) P_1(x, X_v) < 0, \quad v \in V,$$

the Markov process $\kappa_{\varepsilon}(t)$ spends a long time in every class $X_v$ for a small $\varepsilon > 0$ and sooner or later leaves each class $X_v$.

The asymptotic behaviour of the Markov process $\kappa_{\varepsilon}(t), \ t \ge 0$, as $\varepsilon \to 0$ can be investigated by using the martingale characterization of Markov process $\kappa^{\varepsilon}(t) := := \kappa_{\varepsilon}(t/\varepsilon)$ in the form [19]

$$\mu_t^{\varepsilon} = \varphi(\kappa^{\varepsilon}(t)) - \int_0^t L^{\varepsilon} \varphi(\kappa^{\varepsilon}(s))\, ds,$$

$$L^{\varepsilon} \varphi(x) = [\varepsilon^{-1} Q + Q_1] \varphi(x),$$

where

$$Q_1 \varphi(x) := q(x) \int_X P_1(x, dy) \varphi(y).$$

The phase merging effect is realized by Proposition 1 (Section 5) for test functions

$$\varphi^\varepsilon(x) = \varphi(v(x)) + \varepsilon\varphi_1(x),$$

where $v(x) = v$, $x \in X_v$, is a merging function on $X$ corresponding to splitting (13).

The martingale characterization for the limit Markov process

$$\hat{\kappa}(t) = P\text{-}\lim_{\varepsilon\to 0} v(\kappa^\varepsilon(t))$$

on the merged phase space $(V, \mathcal{V})$ is as follows:

$$\mu_t = \hat{\varphi}(\hat{\kappa}(t)) - \int_0^t \hat{Q}_1\hat{\varphi}(\hat{\kappa}(s))\,ds$$

with the generator

$$\hat{Q}_1\hat{\varphi}(v) := \hat{q}(v)\int_V \hat{P}(v, dv')[\hat{\varphi}(v') - \hat{\varphi}(v)],$$

where

$$\hat{P}(v, \Gamma) := \int_{X_v} \pi_v(dx)P_1(x, X_\Gamma), \qquad X_\Gamma := \bigcup_{v\in\Gamma} X_v,$$

$$\hat{q}(v) := q_v \cdot p_v,$$

$$q_v^{-1} := \int_{X_v} \pi_v(dx)/q(x).$$

The analysis of stochastic system (12) in diffusion approximation scheme with merging and averaging is realized by

**Theorem 2.** A (Average). *Assume that the switching Markov jump process* $\kappa_\varepsilon(t)$, $t \ge 0$, *satisfies the merging conditions and the merged Markov process* $\hat{\kappa}(t)$, $t \ge 0$, *on the merged phase space* $(V, \mathcal{V})$ *is uniformly ergodic with the stationary distribution* $\pi(dv)$. *Let the velocity function* $C_1(u, x)$ *satisfy the balance condition*

$$\int_V \pi(dv)\int_{X_v} \pi_v(dx)C_1(u, x) \equiv 0.$$

*Then the solution* $U^\varepsilon(t)$ *of the stochastic system* (12) *converges weakly to the diffusion process* $\zeta(t)$, $t \ge 0$, *determined by the generator*

$$\hat{L}\varphi(u) = \hat{a}(u)\varphi'(u) + \hat{B}^2(u)\varphi''(u), \tag{14}$$

*where*

$$\hat{a}(u) = \int_V \pi(dv)a_v(u),$$

$$a_v(u) = \int_{X_v} \pi_v(dx)a(u, x),$$

$$a(u, x) = C_0(u, x) + C_1(u, x)R_0C'_{1u}(u, x),$$

$$\hat{B}(u) = \int_V \pi(dv)\int_{X_v} \pi_v(dx)C_0(u, x)R_0C_0(u, x).$$

S (Stability). *Let $V(u)$ be Lyapunov's function for the limit diffusion process $\zeta(t)$ satisfying the exponential stability condition*

$$\hat{L}V(u) \le -cV(u), \quad c > 0.$$

*Let the velocity functions $C_k(u,x)$, $k = 1, 0$, satisfy the following inequalities*:

(i) $|C_0(u,x)V'(u)| \le c_1 V(u)$;

(ii) $|C_k(u,x)\hat{R}_0[\hat{C}_r(u,k)V'(u)]'| \le c_2 V(u)$;

(iii) $|C_k(u,x)\hat{R}_0[\hat{C}_1(u,k)\hat{R}_0[\hat{C}_1(u,k)V'(u)]']'| \le c_3 V(u)$.

*Then a solution of stochastic system* (12) *is asymptotically stable with probability one*:

$$\mathcal{P}\left\{\lim_{t\to\infty}\|U^{\varepsilon}(t)\| = 0\right\} = 1.$$

***Remark* 1.** The inequalities in Theorems 1 and 2 can be simplified with an additional conditions of the velocity functions $C_k(u,x)$, $k = 1, 0$. However, the combination on the left-hand side of Lyapunov's function and the velocity functions is essential. That is, for the linear stochastic system, these inequalities are valid automatically for a square defined Lyapunov's function.

**5. Problems of singular perturbation.** The diverse scheme of asymptotical analysis of stochastic systems can be reduced to the problem of singular perturbation of a reducible invertible operator, which can be formulated in the following way: For a given vector $\psi \in B$, the asymptotic solution

$$\varphi^{\varepsilon} = \varphi + \varepsilon\varphi_1$$

of the equation

$$[\varepsilon^{-1}Q + Q_1]\varphi^{\varepsilon} = \psi + \theta^{\varepsilon}$$

is constructed with the asymptotically negligible term $\theta^{\varepsilon}$:

$$\|\theta^{\varepsilon}\| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

A problem of this sort arises due to an asymptotic inversion of singular perturbed operator:

$$[\varepsilon^{-1}Q + Q_1]^{-1} = Q^0 + \varepsilon Q^1 + \ldots.$$

There exist many situations which cannot be classified (see, for example, [18]). Meanwhile, it is possible to extract some logically complete variants [16, 19].

The classification of problems of singular perturbation is based on properties of a *contracted* operator $\hat{Q}_1$ determined by the relation

$$\hat{Q}_1\Pi = \Pi Q_1 \Pi. \tag{15}$$

The contracted operator $\hat{Q}_1$ acts on the contracted null-space $\hat{N}_Q$.

***Example*** [19]. Let $Q$ be a generator of the Markov ergodic process with a finite number of ergodic classes $X = \bigcup_{k=1}^{N} X_k$ and let $\pi_k(dx)$, $1 \le k \le N$, be stationary distributions on $X_k$, $1 \le k \le N$. The projector $\Pi$ onto null-space $N_Q$ acts as follows:

$$\Pi\varphi(x) = \sum_{k=1}^{N}\hat{\varphi}_k I_k(x), \qquad \hat{\varphi}_k := \int_{X_k}\varphi(x)\pi_k(dx);$$

here,

$$I_k(x) = \begin{cases} 1, & x \in X_k; \\ 0, & x \notin X_k. \end{cases}$$

The contracted null-space $\hat{N}_Q$ is an $N$-dimensional Euclidean space of vectors $\hat{\varphi} = (\hat{\varphi}_k,\ 1 \le k \le N)$.

Let a perturbing operator $Q_1$ get as follows:

$$Q_1 \varphi(x) = \int_X Q_1(x, dy)\varphi(y), \qquad x \in X.$$

Then the contracted operator $\hat{Q}_1$ on $\hat{N}_Q$ is defined according to relation (15) by the matrix

$$\hat{Q}_1 = [q_{kr}\colon 1 \le k, r \le N],$$

where

$$q_{kr} := \int_{X_k} \pi_k(dx) Q_1(x, X_r)$$

and

$$\hat{Q}_1 \hat{\varphi} := \left( \sum_{r=1}^{N} q_{kr}\hat{\varphi}_r,\ 1 \le k \le N \right).$$

*There are three logically complete variants*:

(i) $\hat{Q}_1$ is invertible: there exists $\hat{Q}_1^{-1}$;

(ii) $\hat{Q}_1$ is zero-operator: $\hat{Q}_1\hat{\varphi} = 0$ for all $\hat{\varphi} \in \hat{N}_Q$;

(iii) $\hat{Q}_1$ is reducibly invertible: there exists null-space $\hat{N}_{\hat{Q}_1} \subset \hat{N}_Q$ such that

$$\hat{N}_Q = \hat{N}_{\hat{Q}_1} \oplus \hat{R}_{\hat{Q}_1}.$$

There exists also the potential operator $\hat{R}_0 := [\hat{Q}_1 + \hat{\Pi}]^{-1} - \hat{\Pi}$, where $\hat{\Pi}$ is a projector onto $\hat{N}_{\hat{Q}_1}$ which is defined by the relation

$$\hat{\Pi}\hat{\varphi} = \hat{\varphi}\hat{1}, \qquad \hat{\varphi} \in \hat{N}_{\hat{Q}_1}.$$

Here, $\hat{1}$ is a unit vector in $\hat{N}_{\hat{Q}_1}$.

The solutions of singular perturbation problems in these three variants are given in the following three propositions (see [16, 19]):

**Proposition 1.** *Let the contracted operator* $\hat{Q}_1$ *be invertible*: $\exists\, \hat{Q}_1^{-1}$. *Then the asymptotic representation*

$$[\varepsilon^{-1} Q + Q_1](\varphi + \varepsilon\varphi_1) = \psi + \theta^{\varepsilon}$$

*can be realized by the following relations*:

$$\hat{Q}_1\hat{\varphi} = \hat{\psi},$$

$$\varphi_1 = R_0(\psi - Q_1\varphi),$$

$$\theta^{\varepsilon} = \varepsilon Q_1 R_0(\psi - Q_1\varphi).$$

**Proposition 2.** *Let the contracted operator* $\hat{Q}_1$ *be a zero-operator*:

$$\hat{Q}_1\hat{\varphi} = 0 \qquad \forall \hat{\varphi} \in \hat{N}_Q.$$

*Let, in addition, the operator* $Q_0 = Q_2 - Q_1 R_0 Q_1$ *after contraction on the space* $\hat{N}_Q$ *have the inverse operator* $\hat{Q}_0^{-1}$.

*Then the asymptotic representation*

$$[\varepsilon^{-2} Q + \varepsilon^{-1} Q_1 + Q_2](\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2) = \psi + \theta^\varepsilon$$

*can be realized by the following relations*:

$$\hat{Q}_0\hat{\varphi} = \hat{\psi},$$

$$\varphi_1 = -R_0 Q_1 \varphi,$$

$$\varphi_2 = R_0(\psi - Q_0\varphi),$$

$$\theta^\varepsilon = \varepsilon[Q_2\varphi_1 + [Q_1 + \varepsilon Q_2]\varphi_2].$$

**Proposition 3.** *Let the contracted operator* $\hat{Q}_1$ *be reducible invertible with null-space* $\hat{N}_{\hat{Q}_1} \subset \hat{N}_Q$ *defined by the projector* $\hat{\Pi}$. *Let the twice contracted operator* $\hat{\hat{Q}}_2$ *on* $\hat{N}_{\hat{Q}_1}$, *defined by the relations*

$$\hat{\hat{Q}}_2\hat{\Pi} = \hat{\Pi}\hat{Q}_2\hat{\Pi}, \qquad \hat{Q}_2\Pi = \Pi Q_2 \Pi,$$

*be invertible*: $\exists\, \hat{\hat{Q}}_2^{-1}$.

*Then the asymptotical representation*

$$[\varepsilon^{-2} Q + \varepsilon^{-1} Q_1 + Q_2](\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2) = \psi + \theta^\varepsilon$$

*can be realized by the following relations*:

$$\hat{\hat{Q}}_2\hat{\varphi} = \hat{\psi},$$

$$\hat{\varphi}_1 = \hat{R}_0(\hat{\psi} - \hat{Q}_2\hat{\varphi}),$$

$$\varphi_2 = R_0(\psi - Q_2\varphi - Q_1\varphi_1),$$

$$\theta^\varepsilon = \varepsilon[Q_2\varphi_1 + [Q_1 + \varepsilon Q_2]\varphi_2].$$

Moreover, there exists a more complicated situation of singular perturbation established on the combination of the already considered facts.

**Proposition 4.** *Let the contracted operator* $\hat{Q}_1$ *be reducible invertible with null-space* $\hat{N}_{\hat{Q}_1} \subset \hat{N}_Q$. *Let the operator* $\hat{\hat{Q}}_2$ *which is defined by the relations*

$$\hat{\hat{Q}}_2\hat{\Pi} = \hat{\Pi}\hat{Q}_2\hat{\Pi}, \qquad \hat{Q}_2\Pi = \Pi Q_2 \Pi,$$

*be a zero-operator*: $\hat{\hat{Q}}_2\varphi = 0 \;\; \forall\, \varphi \in N_Q$.

*Let, in addition, the operator* $\hat{Q}_0 = \hat{Q}_3 - \hat{Q}_2\hat{R}_0\hat{Q}_2$ *after contraction on the space* $\hat{N}_{\hat{Q}_1}$ *have the inverse operator* $\hat{\hat{Q}}_0^{-1}$.

*Then the asymptotic representation*

$$[\varepsilon^{-3} Q + \varepsilon^{-2} Q_1 + \varepsilon^{-1} Q_2 + Q_3](\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3) = \psi + \varepsilon\theta^\varepsilon \tag{16}$$

*can be realized by the relations*

$$\hat{\hat{Q}}_0\hat{\varphi} = \hat{\psi},$$

$$\hat{\varphi}_1 = -\hat{R}_0\hat{Q}_2\varphi,$$
$$\varphi_2 = \hat{R}_0(\hat{\psi} - \hat{Q}_0\hat{\varphi}),$$
$$\varphi_3 = R_0^{\cdot}(\psi - Q_3\varphi - Q_2\varphi_1 - Q_1\varphi_2),$$
$$\theta^\varepsilon = [Q_1\varphi_3 + Q_2(\varphi_2 + \varepsilon\varphi_3) + Q_3(\varphi_1 + \varepsilon\varphi_2 + \varepsilon^2\varphi_2)]. \tag{17}$$

***Proof of Proposition* 4.** As usual, let us consider the expansion of the left-hand side of (16) with respect to degrees of parameter $\varepsilon$ and comparing the results with the right-hand side of (16). We get the following relations:

$$Q\varphi = 0,$$
$$Q\varphi_1 + Q_1\varphi = 0,$$
$$Q\varphi_2 + Q_1\varphi_1 + Q_2\varphi = 0,$$
$$Q\varphi_3 + Q_1\varphi_2 + Q_2\varphi_1 + Q_3\varphi = \psi,$$
$$Q_1\varphi_3 + Q_2(\varphi_2 + \varepsilon\varphi_3) + Q_3(\varphi_1 + \varepsilon\varphi_2 + \varepsilon^2\varphi_3) = \theta^\varepsilon. \tag{18}$$

The first equality in (18) means that $\varphi \in N_Q$, i.e., $\Pi\varphi = \varphi$. Moreover, the vector $\varphi$ can be chosen in $N_{\hat{Q}_1}$, i.e., $\hat{\Pi}\varphi = \varphi$.

That is $Q_1\varphi = 0$. Hence, the second equality in (18) gives $Q\varphi_1 = 0$, i.e., $\Pi\varphi_1 = \varphi_1$.

Now, the solvability condition for the third equality in (18) has the form

$$\hat{Q}_1\varphi_1 + \hat{Q}_2\varphi = 0. \tag{19}$$

However under condition of Proposition 4, the following solvability condition for Eq. (19) takes place:

$$\hat{\Pi}\hat{Q}_2\hat{\Pi}\varphi = \hat{\hat{Q}}_2\varphi = 0.$$

The solution of Eq. (19) is represented by the second relation in (17).

For the forth equation in (18), the solvability condition has the following form:

$$\hat{Q}_1\varphi_2 + \hat{Q}_2\varphi_1 + \hat{Q}_3\varphi = \hat{\psi}. \tag{20}$$

Using the representation $\varphi_1 = -\hat{R}_0\hat{Q}_2\varphi$ and the definition of the operator $\hat{Q}_0 := \hat{Q}_3 - \hat{Q}_2\hat{R}_0\hat{Q}_2$, Eq. (20) is transformed as follows:

$$\hat{Q}_1\varphi_2 = \hat{\psi} - \hat{Q}_0\varphi. \tag{21}$$

The solvability condition for Eq. (21) gives the first relation in (17). The solution of Eq. (21) is represented by the third relation in (17). The last two relations in (17) are evident.

**6. Proof. of Theorem 2.** The scheme of the proof for both Theorems 1 and 2 is the same. Therefore, let us consider the proof of Theorem 2. The starting points is the martingale characterization of the coupled Markov process

$$U_t^\varepsilon := U^\varepsilon(t), \qquad \kappa_t^\varepsilon := \kappa(t/\varepsilon^3), \quad t \geq 0,$$

in the form

$$\mu_t^\varepsilon = \varphi^\varepsilon(U_t^\varepsilon, \kappa_t^\varepsilon) - \varphi^\varepsilon(u, x) - \int_0^t L^\varepsilon\varphi^\varepsilon(U_s^\varepsilon, \kappa_s^\varepsilon)\,ds, \tag{22}$$

where the generator $L^\varepsilon$ is given by

$$L^{\varepsilon} = \varepsilon^{-3} Q + \varepsilon^{-2} Q_1 + \varepsilon^{-1} C_1(x) + C_0(x). \tag{23}$$

The operators $C_k(u, x)$, $k = 1, 0$, are given by

$$C_k(x)\varphi(u) = C_k(u, x)\varphi'(u), \quad k = 1, 0.$$

The problem of singular perturbation for generator (23) is considered under the conditions of Proposition 4 (Section 5) in the following form:

$$L^{\varepsilon}[V(u) + \varepsilon\varphi_1(u, v(x)) + \varepsilon^2\varphi_2(u, v(x)) + \varepsilon^3\varphi_3(u, x)] =$$
$$= \hat{L}\, V(u) + \varepsilon\, \theta_{\varepsilon}(u, x),$$

where $V(u)$ is Lyapunov's function for the limit diffusion process determined by generator (14). The contracted operator $\hat{L}$ is defined by the following relations (Section 5):

$$\hat{L}\hat{\Pi} = \hat{\Pi}\hat{Q}_0\hat{\Pi}, \qquad \hat{Q}_0 = \hat{C}_0 - \hat{C}_1\hat{R}_0\hat{C}_1.$$

The operator $\hat{R}_0$ is the potential of the merged Markov process $\hat{x}(t)$, $t \geq 0$, and the projector $\hat{\Pi}$ acts as follows:

$$\hat{\Pi}\varphi(v) := \int_V \hat{\pi}(dv)\varphi(v).$$

Now we have to calculate the operator

$$\hat{L} = \hat{\hat{C}}_0 - \widehat{\hat{C}_1\hat{R}_0\hat{C}_1}.$$

First we obtain

$$\hat{\hat{C}}_0\, \varphi(u) = a_0(u)\varphi'(u), \qquad a_0(u) := \int_V \hat{\pi}(dv)\hat{C}_0(u, v).$$

The following composite calculation gives us

$$\hat{C}\varphi(u) := \widehat{\hat{C}_1\hat{R}_0\hat{C}_1} = -\int_V \hat{\pi}(dv)\hat{C}_1\hat{R}_0\hat{C}_1\varphi(u) = -\int_V \hat{\pi}(dv)\hat{C}_1\hat{R}_0\hat{C}_1(u, v)\varphi'(u) =$$
$$= \int_V \hat{\pi}(dv)\hat{C}_1 \int_V \hat{R}_0(v, v')\hat{C}_1(u, v')\varphi'(u) =$$
$$= \int_V \hat{\pi}(dv)\hat{C}_1(u, v) \int_V \hat{R}_0(v, dv')[\hat{C}_1(u, dv')\varphi'' + \hat{C}'_{1u}(u, v')\varphi'(u)] =$$
$$= \sigma^2(u)\varphi''(u) + a_1(u)\varphi'(u),$$

where

$$a_1(u) := \int_V \hat{\pi}(dv)\hat{C}_1(u, v) \int_V \hat{R}_0(v, dv')\hat{C}'_{1u}(u, v').$$

Therefore,

$$a(u) = a_0(u) + a_1(u).$$

By Proposition 4, the martingale characterization (22) is transformed to the form

$$\mu_t^{\varepsilon} = V(U_t^{\varepsilon}) - V(t) - \int_0^t \hat{L}V(U_s^{\varepsilon})ds + \psi_t^{\varepsilon} \tag{24}$$

with a negligible term $\psi_t^\varepsilon$ satisfying the pattern limit theorem [19, Section 3.4]. To prove the stability part of Theorem 2, the negligible term in (24) is calculated in an explicit form, which provides the inequality

$$V(U_t^\varepsilon) \leq cV(u) + \mu_t^\varepsilon$$

under condition (i) – (iii) of Theorem 2.

The standard scheme from [14] can be applied to complete the proof of Theorem 2.

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