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FORCED FREQUENCY LOCKING OF ROTATING WAVES

ВИМУШЕНЕ ЗАХОПЛЕННЯ ЧАСТОТИ ХВИЛЬ, ЩО ОБЕРТАЮТЬСЯ

We describe the frequency locking of an asymptotically orbitally stable rotating wave solutions to autonomous S^1-equivariant differential equations under the forcing of a rotating wave.

Описано захоплення частоти асимптотично орбітально стійких розв'язків типу хвиль, що обертаються, автономних S^1-еквіваріантних диференціальних рівнянь при збуренні хвилею, що обертається.

1. Introduction and the Main Result. Consider the ordinary differential equation

$$\dot{x}(t) = f(x(t)) + \varepsilon y(\omega t). \tag{1}$$

In (1) and in what follows, the vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is supposed to be C^k-smooth. Further, $y\colon \mathbb{R} \to \mathbb{R}^n$ is C^k-smooth and periodic with period one, $\omega > 0$ is a control parameter, and $\varepsilon \geq 0$ is a small perturbation parameter. Suppose that, for $\varepsilon = 0$, there exists an asymptotically orbitally stable periodic solution $x_0(t)$ to (1) with period one. Then, generically, for each pair p and q of relatively prime natural numbers, the following statement is true:

There exists $\varepsilon_0 > 0$ and C^{k-1}-smooth functions ω_- and ω_+, mapping the interval $[0, \varepsilon_0)$ into $\mathbb{R}$, with $\omega_-(0) = \omega_+(0) = p/q$ and $\omega_+(\varepsilon) - \omega_-(\varepsilon) > 0$ for all $\varepsilon \in (0, \varepsilon_0)$, such that, for $\omega \in (\omega_-(\varepsilon), \omega_+(\varepsilon))$, there exists at least one asymptotically stable p/ω-periodic solution to (1) which moves close to the orbit $\{x_0(t)\colon t \in \mathbb{R}\}$. If ω is fixed and ε is changed in such a way that $\omega_-(\varepsilon) < \omega < \omega_+(\varepsilon)$ remains to be satisfied, then this periodic solution changes C^k-smoothly, but its period remains to be equal to p/ω. This phenomenon is the so-called forced frequency locking (or synchronization) of the periodic solution $x_0(t)$ under the small periodic forcing $\varepsilon y(\omega t)$, cf., e.g., [1] (Sec. 7.5.5), [2], [3] (Sec. 11.2), [4] (Chapter 7), [5]. The set

$$\{(\omega, \varepsilon)\colon \omega_-(\varepsilon) < \omega < \omega_+(\varepsilon),\ 0 < \varepsilon < \varepsilon_0\}$$

is the so-called locking region branching off from $(\omega, \varepsilon) = (p/q, 0)$.

In this paper, we consider forced frequency locking in a special situation: First, we suppose the vector field f to be equivariant with respect to an S^1-representation $e^{\gamma A}$ on $\mathbb{R}^n$, i.e., we suppose

$$e^{\gamma A} f(x) = f(e^{\gamma A} x) \quad \text{for all} \quad \gamma \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^n, \tag{2}$$

where $A \neq 0$ is a skew-symmetric real $n \times n$-matrix such that $e^A = I$. And second, we suppose the periodic solution $x_0(t)$ and the forcing $y(t)$ to be so-called rotating waves, i.e.,

$$x_0(t) = e^{tA}\xi_0 \quad \text{and} \quad y(t) = e^{tA}\eta \quad \text{with} \quad \xi_0, \eta \in \mathbb{R}^n.$$

We will show that, generically, all the locking regions, branching off from $(p/q, 0)$ with $p > 1$ and $q > 1$, disappear. In fact, they degenerate to curves. Only the "prime" locking regions, branching off from $(p, 0)$ or $(1/q, 0)$ with natural p and

q, remain to exist. If (ω, ε) belongs to the locking region, for example, branching off from $(\omega, \varepsilon) = (1, 0)$, then there exists at least one asymptotically stable rotating wave solution $e^{\omega t A}\xi(\omega, \varepsilon)$ which moves close to

$$\mathcal{O} := \{ e^{tA}\xi_0 : t \in \mathbb{R} \}.$$

This solution has the same period as the forcing, that is $1/\omega$. Moreover, for (ω, ε) close to $(1, 0)$ but not belonging to the "prime" locking region, there exists an attracting family of quasi-periodic solutions $e^{\omega t A}\xi(t + \tau, \omega, \varepsilon)$ (so-called modulated waves, cf., e.g., [8]) near $\mathcal{O}$. Here, $\tau \in \mathbb{R}$ parametrizes the family and $\xi(t)$ is non-stationary and periodic.

Let us formulate our results more exactly.

Thus, we consider the equation

$$\dot{x}(t) = f(x(t)) + \varepsilon e^{\omega t A}\eta. \tag{3}$$

The assumption that $e^{tA}\xi_0$ is non-stationary periodic solution to (3) with $\varepsilon = 0$ is equivalent to

$$f(\xi_0) = A\xi_0 \quad \text{and} \quad A\xi_0 \neq 0. \tag{4}$$

Without loss of generality, we assume that one is the minimum period of $e^{tA}\xi_0$, i.e.

$$e^{tA}\xi_0 \neq \xi_0 \quad \text{for all} \quad t \in (0, 1). \tag{5}$$

Note that it follows from (2) and (4) that

$$(f'(\xi_0) - A)A\xi_0 = 0.$$

In other words, zero is eigenvalue of $f'(\xi_0) - A$ with eigenvector $A\xi_0$. We assume that, moreover, this eigenvalue is simple and that all other eigenvalues have negative real parts, i.e.,

$$\begin{gathered} \ker(f'(\xi_0) - A) = \operatorname{span}\{A\xi_0\}, \\ \mathbb{R}^n = \ker(f'(\xi_0) - A) \oplus \operatorname{im}(f'(\xi_0) - A) \end{gathered} \tag{6}$$

and

$$\max\{\lambda \in \operatorname{spec}(f'(\xi_0) - A) : \lambda \neq 0\} < 0. \tag{7}$$

It is easy to show (cf. Section 2 of this paper) that (6) and (7) imply that the rotating wave solution $e^{tA}\xi_0$ to (3) with $\varepsilon = 0$ is asymptotically orbitally stable with asymptotic phase. This means that every solution $x(t)$ to (3) with $\varepsilon = 0$, such that $x(0)$ is sufficiently close to $\mathcal{O}$, exists and stays near $\mathcal{O}$ for all $t \geq 0$ and that $x(t) - e^{(t+\tau)A}\xi_0 \to 0$ as $t \to \infty$ for a certain $\tau \in \mathbb{R}$.

Let $\langle \cdot, \cdot \rangle$ be the Euclidean scalar product in $\mathbb{R}^n$ and let $f'(\xi_0)^T$ denote the matrix transposed to $f'(\xi_0)$. Then, because of (6), there exists a unique $v \in \mathbb{R}^n$ such that

$$f'(\xi_0)^T v = -Av \quad \text{and} \quad \langle A\xi_0, v \rangle = 1. \tag{8}$$

We denote

$$\begin{gathered} \Phi(\varphi) := \langle e^{-\varphi A}\eta, v \rangle \quad \text{for} \quad \varphi \in \mathbb{R}, \\ \Phi_+ := \max_{\varphi} \Phi(\varphi), \qquad \Phi_- := \max_{\varphi} \Phi(\varphi), \end{gathered} \tag{9}$$

and assume that the following statement is true:

$$\begin{gathered}\text{There exist } \varphi_+,\ \varphi_- \in [0,1) \text{ such that}\\ \Phi_\pm = \Phi(\varphi_\pm) \text{ and } \Phi''_\pm(\varphi_\pm) \neq 0, \text{ and } \Phi'(\varphi) \neq 0\\ \text{holds for all } \varphi \in [0,1) \text{ with } \varphi \neq \varphi_\pm.\end{gathered} \tag{10}$$

Note that (10) implies that one is the minimum period of the forcing $e^{tA}\eta$.

Now we formulate our main result.

Theorem. *Suppose that* (2), (4), (6), (7), *and* (10) *hold.*

Then there exist $\varepsilon_0 > 0$, $\delta_0 > 0$, *neighborhoods* U *and* V *of* $\mathbb{O}$, C^{k-1}-*functions* ω_+, ω_- and ξ_+, ξ_-, *mapping* $[0,1)$ *into* $\mathbb{R}$ *and* $\mathbb{R}^n$, *respectively, such that the following statements are true*:

(i) $\omega_\pm(0) = 1$, $\omega'_\pm(0) = \Phi_\pm$, $\xi_\pm(0) = e^{\varphi_\pm A}\xi_0$, *and*

$$1 - \delta_0 < \omega_-(\varepsilon) < \omega_+(\varepsilon) < 1 + \delta_0 \quad \textit{for all} \quad \varepsilon \in (0, \varepsilon_0).$$

(ii) *Let* $\omega \in (\omega_-(\varepsilon), \omega_+(\varepsilon))$. *Then there exist two solutions* $x_j(t) = e^{\omega tA}\xi_j(\omega,\varepsilon)$, $j = 1, 2$, *to* (3) *in* U. *The vectors* $\xi_j(\omega,\varepsilon)$ *depend* C^k-*smoothly on* ω *and* ε. *The solution* $x_1(t)$ *is asymptotically stable,* $x_2(t)$ *is unstable. Further, for all* $t_0 \in \mathbb{R}$ *and all solutions* $x(t)$ *to* (3) *with* $x(t_0) \in V$, $x(t)$ *exists and belongs to* U *for all* $t \geq t_0$, *and if* $(t, x(t))$ *does not belong to the stable manifold of* $x_2(t)$, *then*

$$x(t) - e^{\omega tA}\xi_1(\omega,\varepsilon) \to 0 \quad as \quad t \to \infty.$$

(iii) *Let* $\omega \in (1 - \delta_0, \omega_-(\varepsilon))$ *or* $\omega \in (\omega_+(\varepsilon), 1 + \delta_0)$. *Then there exists a family of solutions* $e^{\omega tA}\xi(t+\tau,\omega,\varepsilon)$ *to* (3) *in* U ($\tau \in \mathbb{R}$ *is the family parameter*). *The vectors* $\xi(t,\omega,\varepsilon)$ *are periodic with respect to* t *with period* $T(\omega,\varepsilon)$, $\xi(t,\omega,\varepsilon)$ *and* $T(\omega,\varepsilon)$ *depend* C^k-*smoothly on* ω *and* ε. *Further, for all* $t_0 \in \mathbb{R}$ *and all solutions* $x(t)$ *to* (3) *with* $x(t_0) \in V$, $x(t)$ *exists and belongs to* U *for all* $t \geq t_0$, *and*

$$x(t) - e^{\omega tA}\xi(t+\tau,\omega,\varepsilon) \to 0 \quad as \quad t \to \infty$$

for a certain $\tau \in \mathbb{R}$ (*which depends on* $x(0)$, ω *and* ε).

(iv) $T(\omega,\varepsilon) \to \infty$ *for* $\omega \downarrow \omega_+(\varepsilon)$ *or* $\omega \uparrow \omega_-(\varepsilon)$, $\xi_j(\omega,\varepsilon) \to \xi_+(\varepsilon)$ *for* $\omega \uparrow \omega_+(\varepsilon)$, *and* $\xi_j(\omega,\varepsilon) \to \xi_-(\varepsilon)$ *for* $\omega \downarrow \omega_-(\varepsilon)$ ($j = 1, 2$).

(v) $e^{\omega tA}\xi(t,\omega,\varepsilon) \to e^{(t+\tau)A}\xi_0$ *for* $\varepsilon \downarrow 0$ *and a certain* τ_0 (*which does not depend on* ω).

The motivation for our investigations comes from problems in laser dynamics. The corresponding mathematical models are, as a rule, systems of nonlinear real (for the carrier densities) and complex (for the complex amplitudes of the optical field) differential equations. Moreover, the models are equivariant with respect to an S^1-representation on the state space ($e^{i\gamma}$ works trivially on the carrier densities and by multiplication on the complex amplitudes). In [7], the physical nature of this equivariance is analyzed. The rotating wave solution $e^{tA}\xi_0$ describes the so-called "stationary lasing" state of the laser with frequency normalized to one. For a description of the bifurcations of such states from trivial ones (non-lasing states) see, e.g., [8, 9]. The forcing $\varepsilon e^{\omega tA}\eta$ describes an external optical signalinjected into the laser and coming, for example, from another stationary working laser with frequency ω.

In these applications, the Euclidean norm $\|x(t)\|$ of a solution $x(t)$ is of special interest. Let us discuss the assertions of our theorem in terms of this quantity.

If $x(t)$ is an attracting solution near $\mathbb{C}$, then $\|x(t)\|$ is stationary or periodic depending on whether $\omega \in (\omega_-(\varepsilon), \omega_+(\varepsilon))$ or not. In particular, if $\omega \notin (\omega_-(\varepsilon), \omega_+(\varepsilon))$, then $\|x(t)\|$ oscillates regardless of the fact that the norms of the unforced solution $e^{tA}\xi_0$ and of the forcing $\varepsilon e^{\omega t A}\eta$ do not oscillate.

Often, one is interested in modulated wave solutions $x(t)$ such that the frequency of $\|x(t)\|$, the so-called modulation frequency, and the so-called modulation oscilation

$$\max_t \|x(t)\| - \min_t \|x(t)\|$$

are large. Our theorem states that (in the described situation) the possibilities to come up to these demands are quite limited:

Let, for example, $\Phi_+ > 0$. Further, for $\omega > 1$, let $\varepsilon = \varepsilon_+(\omega)$ be a solution of equation $\omega = \omega_+(\varepsilon)$. Then, for small ε, the modulation oscillation is small because of assertion (v) of our theorem. If one tries to increase the modulation oscillation by increasing ε, one has to pay for this by a decrease of the frequency of $\|x(t)\|$, because this frequency tends to zero for $\varepsilon \uparrow \varepsilon_+(\omega)$.

In other words, if one tries to get modulated wave solutions near a given rotating wave solution of an S^1-equivariant differential equation by forcing this equation by an (unmodulated) rotating wave, then one obtains the following result:

For small forcings (i.e., for $0 < \varepsilon << \varepsilon_+(\omega)$), small modulation oscillations are created and the modulation frequency is close to the difference of the frequencies of the unforced solution on the forcing, that is $1 - \omega$ (because of assertion (v) of our theorem). On the other hand, for large forcing (i.e., for $0 << \varepsilon < \varepsilon_+(\omega)$), large modulation oscillations occur, but the modulation frequencies are small. If ε tends to $\varepsilon_+(\omega)$ from below, then the modulation frequency tends to zero and the modulated wave solution changes into two (if (10) is satisfied) rotating wave solution. These rotating wave solutions are close to certain phase shifts of the unperturbed rotating wave solution and have exactly the same frequency as the forcing. One of them is asymptotically stable, the other is unstable.

If ε is increased further (i.e., $\varepsilon > \varepsilon_+(\omega)$), then the following facts take place:

In the case of $\Phi_- < 0$, the locked rotating wave solutions change quantitatively, only. No modulated wave solutions occur.

But in the case of $\Phi_- > 0$, there exists a positive solution $\varepsilon = \varepsilon_-(\omega)$ of equation $\omega = \omega_-(\varepsilon)$ with $\omega > 1$. Inequality $\varepsilon_-(\omega) > \varepsilon_+(\omega)$ holds and the locking region

$$\{(\omega, \varepsilon)\colon \omega_-(\varepsilon) < \omega < \omega_+(\varepsilon)\} = \{(\omega, \varepsilon)\colon \varepsilon_+(\varepsilon) < \varepsilon < \varepsilon_-(\omega)\}$$

is located above the axis $\omega = 1$. If ε tends to $\varepsilon_-(\omega)$ from below, then the two rotating wave solutions coalesce and disappear, and, again, a family of modulated wave solutions with small modulation frequencies occurs. Hence, if one wants to get a large modulation frequency and a large modulation oscillation, one has further to increase ε (as long as the local description of the solution behavior given by our theorem is valid).

The bifurcation scenarios in the case with $\Phi_+ < 0$ may be described analogously.

2. The Proof. For $x \in \mathbb{R}^n$, we denote $Qx := x - \langle x, v\rangle A\xi_0$. Because of (8), Q is the projector corresponding to the direct sum in (6), i.e., $\ker Q = \ker(f'(\xi_0) - A)$ and $\operatorname{im} Q = \operatorname{im}(f'(\xi_0) - A)$.

Let us introduce new coordinates $\varphi \in \mathbb{R}$ and $h \in \operatorname{im} Q$ in (3) in the following way:

$$x = e^{(\omega t+\varphi)A}(\xi_0 + h). \tag{11}$$

Note that the map $(\varphi, h) \mapsto e^{\varphi A}(\xi_0 + h)$ is injective for $\varphi \in [0, 1)$ and small h (because of (5)), and its image is an open neighborhood of $\mathbb{O}$. Inserting (11) into (3), we get $(\omega + \dot\varphi) A(\xi_0 + h) + \dot h = f(\xi_0 + h) + \varepsilon e^{-\varphi A} y$. Here, we use assumption (2). Hence, near the orbit $\mathbb{O}$, equation (3) may be written in the standard form

$$\begin{aligned} \dot\varphi &= a(\varphi, h, \omega, \varepsilon), \\ \dot h &= P(\varphi, h, \omega, \varepsilon)h + \varepsilon F(\varphi), \end{aligned} \tag{12}$$

where

$$\langle A(\xi_0 + h, v)\rangle a(\varphi, h, \omega, \varepsilon) =$$

$$= \langle -\omega A(\xi_0 + h) + f(\xi_0 + h) + \varepsilon e^{-\varphi A}\eta, v\rangle =$$

$$= \left\langle (1-\omega)A(\xi_0 + h) + \left(\int_0^1 f'(\xi_0 + sh)ds - f'(\xi_0)\right)h + \varepsilon e^{-\varphi A}\eta, v \right\rangle,$$

and, hence,

$$a(\varphi, h, \omega, \varepsilon) =$$

$$= 1 - \omega + \left\langle \left(\int_0^1 f'(\xi_0 + sh)ds - f'(\xi_0)\right)h + \varepsilon e^{-\varphi A}\eta, v \right\rangle (1 + \langle Ah, v\rangle)^{-1}. \tag{13}$$

Here, we use (4) and (8). Further, we have

$$P(\varphi, h, \omega, \varepsilon) = (I - Q)\left\langle \left(\int_0^1 f'(\xi_0 + sh)ds - (\omega + a(\varphi, h, \omega, \varepsilon))A\right)\right\rangle,$$

$$F(\varphi) = e^{-\varphi A}\eta.$$

In particular, $a(\varphi, 0, \omega, 0) = 1 - \omega$ and

$$P(\varphi, 0, \omega, 0) = (I - Q)(f'(\xi_0) - A). \tag{14}$$

Now, we are going to apply Theorem 1 in [10] (Chapter IV.4). In order to verify the assumptions of this theorem, it suffices to show that there exist $\beta > 0$ and a symmetric, positive definite, real $n \times n$ matrix S such that

$$\langle SP(\varphi, 0, \omega, 0)x, x\rangle \le -\beta\langle Sx, x\rangle \text{ for all } \varphi \text{ and } x \ne 0.$$

However, this property follows easily from (7) and (14) (cf., e.g., [11] (Chapter X, Lemma 1.5).

The theorem asserts that, for all ω near one and ε near zero, there exists a one-dimensional invariant manifold

$$h = \varepsilon u(\varphi, \omega, \varepsilon)$$

to (12). The map u is C^{k-1}-smooth and 1-periodic with respect to φ. Moreover, each solution to (12) which moves near $\mathbb{O}$ is asymptotically attracted by a corresponding solution on the invariant circle in the following way: For each solution $\varphi(t)$, $h(t)$ to (12) such that $\min\{|\varphi(0)-\gamma| + \|h(0)-\varepsilon u(\gamma,\omega,\varepsilon)\| : \gamma\in\mathbb{R}\}$ is sufficiently small, there exists a solution $\psi(t)$ to equation

$$\dot{\psi} = a(\psi, \varepsilon u(\psi, \omega, \varepsilon), \omega, \varepsilon) \tag{15}$$

such that

$$\|\varphi(t)-\psi(t)\| + \|h(t)-\varepsilon u(\psi(t),\omega,\varepsilon)\| \to 0 \quad \text{as} \quad t\to\infty.$$

Thus, it remains to investigate Eq. (15).

Obviously, either there exist stationary solutions to (15) or all solutions are periodic.

In order to determine the parameters ω near one and ε near zero such that (15) has stationary solutions, we introduce a new scalar parameter α by

$$\omega = 1 + \varepsilon\alpha.$$

Then (9) and (13) imply

$$a(\psi, \varepsilon u(\psi, 1+\varepsilon\alpha, \varepsilon), 1+\varepsilon\alpha, \varepsilon) = \varepsilon(-\alpha + \Phi(\psi) + b(\psi,\alpha,\varepsilon)) \tag{16}$$

with

$$b(\psi,\alpha,\varepsilon) =$$

$$= \frac{\left\langle \left(\int_0^1 f'(\xi_0 + s\varepsilon u(\psi, 1+\varepsilon\alpha, \varepsilon))ds - f'(\xi_0) \right) u(\psi, 1+\varepsilon\alpha, \varepsilon), v \right\rangle}{1 + \varepsilon\langle Au(\psi, 1+\varepsilon\alpha,\varepsilon), v\rangle}.$$

In particular, equality $b(\psi,\alpha,0) = 0$ holds for all ψ and α.

For $\varepsilon > 0$, ψ is a stationary solution to (15) iff

$$-\alpha + \Phi(\psi) + b(\psi,\alpha,\varepsilon) = 0. \tag{17}$$

Moreover, this solution is asymptotically stable (unstable, respectively) if $\Phi'(\psi) + \partial_\psi b(\psi,\alpha,\varepsilon)$ is negative (positive, respectively).

First, we determine the singular solutions to (17), i.e., the solutions to (17) with

$$\Phi'(\psi) + \partial_\psi b(\psi,\alpha,\varepsilon) = 0. \tag{18}$$

Because of assumption (10), system (17), (18) with $\varepsilon = 0$ has exactly (up to the 1-periodicity with respect to ψ) the solutions $\alpha = \Phi_+$, $\psi = \varphi_+$ and $\alpha = \Phi_-$, $\psi = \varphi_-$, and, in both solutions, the implicit function theorem works (with respect to (ψ,α)). Hence, the solutions to (17), (18) with small ε are $\alpha = \alpha_+(\varepsilon)$, $\psi = \psi_+(\varepsilon)$ and $\alpha = \alpha_-(\varepsilon)$, $\psi = \psi_-(\varepsilon)$, where the maps $\alpha_\pm$ and $\psi_\pm$ are C^{k-1}-smooth, $\alpha_\pm(0) = \Phi_\pm$ and $\psi_\pm(0) = \varphi_\pm$.

Using (10), it is easy to verify that, if α decreases from $\alpha_+(\varepsilon)$ or increases from $\alpha_-(\varepsilon)$, then exactly two regular solutions $\psi_j(\alpha,\varepsilon)$ $(j = 1, 2)$ to (17) grow out of the singular solutions $\psi = \psi_+(\varepsilon)$ or $\psi = \psi_-(\varepsilon)$, respectively. The first one corresponds to an asymptotically stable stationary solution to (15) and, hence, to (12), the second to an unstable one. The first solution attracts all points of the circle with the exception of the second one. Hence, the first stationary solution attracts all points of a neighborhood of the circle with the exception of the points on the stable manifold of the second stationary solution.

The solutions $\psi_j(x,\varepsilon)$ can be smoothly continued (by means of the implicit function theorem) for all small ε and $\alpha\in(\alpha_-(\varepsilon), \alpha_+(\varepsilon))$. For small ε, other solutions to (17) do not exist. Hence, other stationary solutions to (12) with ω near one and h and ε near zero do not exist. Thus, assertions (i) and (ii) of our theorem are proved with

$$\omega_{\pm}(\varepsilon) = 1 + \varepsilon\alpha_{\pm}(\varepsilon),$$

$$\xi_{\pm}(\varepsilon) = e^{\psi_{\pm}(\varepsilon)A}(\xi_0 + \varepsilon u(\psi_{\pm}(\varepsilon), \omega_{\pm}(\varepsilon), \varepsilon)),$$

$$\xi_j(1+\varepsilon\alpha, \varepsilon) = e^{\psi_j(\alpha,\varepsilon)A}(\xi_0 + \varepsilon u(\psi_j(\alpha, \varepsilon), \omega, \varepsilon)).$$

Now, let ε be a near zero and ω near one but $\omega < \omega_-(\varepsilon)$ or $\omega > \omega_+(\varepsilon)$. Then all solutions to (15) are periodic. Denote by $\psi(t, \psi_0, \omega, \varepsilon)$ the solution to (15) with $\psi(0, \psi_0, \omega, \varepsilon) = \psi_0$ and assume that is $T(\omega, \varepsilon)$ is minimum period. Obviously, we have $T(\omega, \varepsilon) \to \infty$ for $\omega \downarrow \omega_+(\varepsilon)$ or $\omega \uparrow \omega_-(\varepsilon)$. Moreover, because of (13), we have $\psi(t, \psi_0, \omega, 0) = \psi_0 + (1-\omega)t$. Hence, the theorem is proved with $\xi(t, \omega, \varepsilon) = e^{\psi(\tau, \psi_0, \omega, \varepsilon)A}(\xi_0 + \varepsilon u(\psi(t, \psi_0, \omega, \varepsilon), \omega, \varepsilon))$ and $\tau_0 = \psi_0$ (where ψ_0 is arbitrarily fixed).

3. Remarks. Let us complete the paper by four remarks.

***Remark* 1.** The first remark concerns assumption (10).

A similar to our theorem but more complicated result holds if one assumes that the map Φ has not only two but $2l$ (with $l \in \mathbb{N}$) local extrema in $[0, 1)$ and that all these local extrema are non-degenerate. In this case, there exist not only two but $2l$ curves $\omega_j(\varepsilon)$ (for $j = 1, \ldots, 2l$) with $\omega_j(0) = 1$,

$$\omega_1'(0) = \max_{\varphi}\langle e^{-\varphi A}\eta, v\rangle \quad \text{and} \quad \omega_{2l}'(0) = \min_{\varphi}\langle e^{-\varphi A}\eta, v\rangle,$$

such that the solution behavior of (3) can be described in the following way:

For $\omega \in (\omega_{2l}(\varepsilon), \omega_1(\varepsilon))$, there exist at least two (but a finite number of) rotating wave solutions to (3) near O. All these solutions have the frequency $1/\omega$. If (ω, ε) intersects one of the curves $\omega = \omega_j(\varepsilon)$, then the number of the rotating wave solutions changes generically by two. If (ω, ε) is not located on one of these curves, then the number of rotating wave solutions is even, half of them are asymptotically stable, the other's are unstable.

For $\omega \notin (\omega_{2l}(\varepsilon), \omega_1(\varepsilon))$, there are no rotating wave solutions near $\mathbb{O}$ but a family of modulated wave solutions as described in our theorem.

***Remark* 2.** Using a more geometric language, the results of our theorem can be formulated as follows:

For all ω near one and ε near zero, there exists an asymptotically stable invariant manifold $\mathcal{M}$ to (1) in the enlarged phase space $\mathbb{R}^n \times S^1$, which is close and diffeomorphic to the two-torus $\mathbb{O} \times S^1$. If (10) is satisfied and if $\omega \in (\omega_-(\varepsilon), \omega_+(\varepsilon))$, then rotating number of the flow on $\mathcal{M}$ is one and there exist one asymptotically stable and one unstable $1/\omega$-periodic rotating wave solution on $\mathcal{M}$. If $\omega \notin (\omega_-(\varepsilon), \omega_+(\varepsilon))$, then the rotation number depends C^k-smoothly on ω and ε, and $\mathcal{M}$ is foliated by periodic (if $T(\omega, \varepsilon)\omega$ is rational) or quasi-periodic (if $T(\omega, \varepsilon)\omega$ is irrational) solutions.

***Remark* 3.** Our theorem implies "implicitly" that there are no locking regions near $(\omega, \varepsilon) = (0, 1)$ others than $\{(\omega, \varepsilon)\colon \omega_-(\varepsilon) < \omega < \omega_+(\varepsilon)\}$.

Let us show explicitly, why our proof does not work if one would try to construct a locking region from $(\omega, \varepsilon) = (p/q, 0)$ with relatively prime natural numbers $p > 1$

and $q > 1$. Indeed, in contrast to (5) and (10) assume that $e^{A/q}\xi_0 = \xi_0$ and $e^{A/p}\eta = \eta$. Then one can easily show that $a(\varphi, h, \omega, \varepsilon)$ and $u(\varphi, \omega, \varepsilon)$ have to be $1/q$-periodic with respect to φ. Therefore, (16) implies that Φ has to be $1/q$-periodic too. On the other hand, by assumption, it has to be $1/p$-periodic (cf. (9)). Hence, Φ must be constant and no arguments being based on the implicit function theorem can work.

***Remark* 4.** The frequency locking of modulated wave solutions under forcings of modulated wave type is of interest in laser dynamics too. Here, one has to distinguish two different phenomena: In the first case, there exist frequency bases of the unforced solution and of the forcing which synchronize among each other "in pairs" (cf. [12]). In the second case, only the modulation frequencies synchronize. This case seems to be the most important from the point of view of the applications, and some work is in preparation.

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