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ON THE IRREVERSIBILITY OF TIME FOR AN ELECTROMAGNETIC HEREDITARY LINEAR SYSTEM *

ПРО НЕЗВОРОТНІСТЬ ЧАСУ ДЛЯ ЕЛЕКТРОМАГНІТНОЇ ЛІНІЙНОЇ СИСТЕМИ СПАДКОВОГО ТИПУ

We give conditions for an electromagnetic linear system of hereditary type under which the time-reversal hypothesis does not hold if the relaxation functions of the electromagnetic field have different behavior at the extremes of the interval of definition. Under the same conditions, it is also possible to prove that these functions are constant if they have the same behavior at the extremes of the interval of definition.

Для лінійної системи теорії електромагнетизму спадкового типу сформульовано умови, при яких гіпотеза про зворотність часу не виконується, якщо релаксаційна функція електромагнітного поля не має однакової поведінки на кінцях інтервалу свого визначення; у противному разі при тих же умовах можна довести, що ці функції є сталими.

1. We formulate conditions for the relaxation functions of an electromagnetic hereditary linear field inside dielectric dissipative materials that do not conduct electric current under which it is possible to trace guidelines for studying hereditary phenomena. An example of the material that is characterized by the properties above is a *holoedric crystal* for which there are no interaction effects of an electric field with magnetic fields and optics phenomena [1].

In a logically well-posed theory that is in agreement with fundamental experiments [2], it is necessary that memory effects for those electromagnetic materials be negligible [3] so that they may be considered as hereditary materials of particular type. Moreover, the time reversibility hypothesis cannot be accepted because it contradicts a strictly dissipative formulation of the second law of thermodynamic according to which it is possible to impose the required thermodynamic restrictions [4, 5] when studying evolution problems of an electromagnetic hereditary field.

The time reversibility hypothesis implies other contradictions [6], when one applies it to demonstrate symmetry properties of the relaxation functions in the problems of visco-elasticity and hereditary electromagnetism. On the basis of the paper [7], we prove that the above problems can be solved with the help of conditions which are physically and mathematically admissible [8]. Actually, it is possible to prove that the assumption on the reversibility of time is not valid. Electromagnetic materials with negligible memory effects can be considered as hereditary materials of particular type, and symmetry property is also valid for electromagnetic relaxation functions [9].

2. Let us consider the constitutive equations of an electromagnetic hereditary field [10] in a rigid and homogeneous dielectric that does not conduct electric current. The following equations determine the value of electric displacement $\mathbf{D}$ and magnetic induction $\mathbf{B}$ as linear functions of the histories, of electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields, respectively:

$$\begin{aligned} \mathbf{D}(x,t) &= \varepsilon(x,0)\mathbf{E}(x,t) + \int_0^{+\infty} \varepsilon'(x,s)\mathbf{E}^t(x,s)\,ds, \\ \mathbf{B}(x,t) &= \mu(x,0)\mathbf{H}(x,t) + \int_0^{+\infty} \mu'(x,s)\mathbf{H}^t(x,s)\,ds, \end{aligned} \tag{1}$$

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where $(\boldsymbol{x}, t) \in \Omega \times [0, d_p]$; $\Omega \subset \mathbf{R}^3$ is a domain with sufficiently regular boundary $\partial\Omega$; the functions $\mathbf{E}^t(\boldsymbol{x}, s) = \mathbf{E}(\boldsymbol{x}, t-s)$ and $\mathbf{H}^t(\boldsymbol{x}, s) = \mathbf{H}(\boldsymbol{x}, t-s)$ for all fixed t $\in \in [0, d_p]$ and $s \in [0, +\infty)$ are the histories of electric and magnetic fields at time t, respectively. Here, ε and μ are the second order tensors defined as follows:

$$\varepsilon(\boldsymbol{x}, s) = \varepsilon(\boldsymbol{x}, 0) + \int_0^s \varepsilon'(\boldsymbol{x}, \tau)\, d\tau, \quad \mu(\boldsymbol{x}, s) = \mu(\boldsymbol{x}, 0) + \int_0^s \mu'(\boldsymbol{x}, \tau)\, d\tau.$$

Suppose that the following conditions are satisfied (below the dependence on x is omitted to simplify notation):

$$\varepsilon'(s),\ s\,\varepsilon'(s),\ \mu'(s),\ s\,\mu'(s) \in L^1(\mathbf{R}^+), \tag{2_i}$$

$$\varepsilon(s) - \varepsilon_\infty = -\int_s^{+\infty} \varepsilon'(\tau)\, d\tau, \quad \mu(s) - \mu_\infty = -\int_s^{+\infty} \mu'(\tau)\, d\tau \in L^1(\mathbf{R}^+),$$

where

$$\varepsilon_\infty = \lim_{s \to +\infty} \varepsilon(s), \quad \mu_\infty = \lim_{s \to +\infty} \mu(s);$$

$$\lim_{s \to +\infty} s^2 \varepsilon'(s) = 0, \quad \lim_{s \to +\infty} s^2 \mu'(s) = 0$$

and for all $s \in [0, +\infty)$

$$\varepsilon(s) = -\varepsilon(-s), \quad \mu(s) = -\mu(-s), \quad \varepsilon'(s) = \varepsilon'(-s), \quad \mu'(s) = \mu'(-s);$$

$$\mathbf{a} \cdot \varepsilon_0 \mathbf{a} > 0 \quad \text{and} \quad \mathbf{a} \cdot \mu_0 \mathbf{a} > 0 \quad \text{for all} \quad \mathbf{a} \in V \backslash \{0\}.$$

Here $\lim_{s \to 0} \varepsilon(s) = \varepsilon_0$, $\lim_{s \to 0} \mu(s) = \mu_0$ and V is a vector space $\mathbf{R}^3$.

Only for the case where $\varepsilon_0 = \varepsilon_\infty$, $\mu_0 = \mu_\infty$, let us assume:

$$\lim_{a \to +\infty} \int_{-at}^{+\infty} a\left[\varepsilon\left(t + \frac{y}{a}\right) - \varepsilon_\infty\right] \frac{\sin y - y\cos y}{y^2}\, dy = 0,$$

$$\lim_{a \to +\infty} \int_{-at}^{+\infty} a\left[\mu\left(t + \frac{y}{a}\right) - \mu_\infty\right] \frac{\sin y - y\cos y}{y^2}\, dy = 0, \tag{2_{ii}}$$

where $t \in [0, +\infty)$, $y = a(s-t)$ and $a > 0$.

Suppose that $\varepsilon'(s)$ and $\mu'(s)$ are continuous and their second derivatives $\varepsilon''(s)$ and $\mu''(s)$ are piecewise continuous. Hence, $\varepsilon''(s)$ and $\mu''(s)$ satyisfy Dini condition at each point of discontinuity and they are bounded in a neighborhood of these points.

Putting $\mathbf{E}(t) = \mathbf{E}(-s)$ $\forall\, t \in (0, -\infty)$ from the first relation in (1), in virtue of Young inequality we get:

$$\|\mathbf{D}(t)\|_{L^p(\mathbf{R})} \le |\varepsilon_0|\, \|\mathbf{E}(t)\|_{L^p(\mathbf{R})} + \|\varepsilon'(s) * \mathbf{E}^t\|_{L^p(\mathbf{R})} \le$$

$$\le |\varepsilon_0|\, \|\mathbf{E}(t)\|_{L^p(\mathbf{R})} + \|\varepsilon'(s)\|_{L^1(\mathbf{R})} \|\mathbf{E}^t\|_{L^p(\mathbf{R})}, \tag{3}$$

$$\|\mathbf{D}(t)\|_{L^p(\mathbf{R})} \le |\varepsilon_\infty|\, \|\mathbf{E}(t)\|_{L^p(\mathbf{R})} + \|[\varepsilon(s) - \varepsilon_\infty] * \dot{\mathbf{E}}^t\|_{L^p(\mathbf{R})} \le$$

$$\le |\varepsilon_\infty|\, \|\mathbf{E}(t)\|_{L^p(\mathbf{R})} + \|\dot{\varepsilon}(s) - \varepsilon_\infty\|_{L^1(\mathbf{R})} \|\dot{\mathbf{E}}(t)\|_{L^1(\mathbf{R})},$$

where $\mathbf{E}(t) \in H^{1,p}(-\infty, +\infty)$ and $\varepsilon(s) - \varepsilon_\infty \in H^{1,1}(-\infty, +\infty)$, $p \ge 1$.

To verify that the convolution integral of the functional given by the first relation in (1) does not contradict the principle of cause effect on $(t, +\infty)$, we rewrite the first relation in (1), putting $\tau = t - s$, in the following form:

$$\mathbf{D}(t) = \varepsilon_0 \mathbf{E}(t) + \int_{-\infty}^{0} \dot{\varepsilon}(\tau - t)\mathbf{E}(\tau)d\tau + \int_{0}^{t} \dot{\varepsilon}(t-\tau)\mathbf{E}(\tau)d\tau =$$

$$= \varepsilon_\infty \mathbf{E}(t) + \int_{-\infty}^{0} [\varepsilon_{-\infty} - \varepsilon(\tau - t)]\mathbf{E}'(\tau)d\tau + \int_{0}^{t} [\varepsilon(\tau - t) - \varepsilon_{-\infty}]\mathbf{E}'(\tau)d\tau. \quad (4)$$

By Young inequality we get

$$\|\mathbf{D}(t)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})} \le |\varepsilon_0(\mathbf{x})|\,\|\mathbf{E}(t)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})} + \|\dot{\varepsilon}^t * \mathbf{E}(\tau)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})} \le$$

$$\le |\varepsilon_0|\,\|\mathbf{E}(t)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})} + \|\dot{\varepsilon}^t\|_{L^p(\mathbf{R})}\,\|\mathbf{E}(\tau)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})}, \quad (5)$$

$$\|\mathbf{D}(t)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})} \le |\varepsilon_\infty|\,\|\mathbf{E}(t)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})} + \|[\varepsilon^t - \varepsilon_\infty] * \dot{\mathbf{E}}(\tau)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})} \le$$

$$\le |\varepsilon_\infty|\,\|\mathbf{E}(t)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})} + \|\varepsilon(t) - \varepsilon_\infty\|_{L^p(\mathbf{R})}\,\|\dot{\mathbf{E}}(\tau)\|_{L^1(\mathbf{R})\cap L^p(\mathbf{R})},$$

where $\mathbf{E}(\tau) \in H^{1,1}(-\infty, +\infty)$ and $\varepsilon(t) - \varepsilon_\infty \in H^{1,p}(-\infty, +\infty)$, $p \ge 1$; $\varepsilon^t = \varepsilon^t(x, t) = \varepsilon(x, t - s)$.

Having compared (3) and (5) we have

$$\mathbf{E}(t),\ \varepsilon(t) - \varepsilon_\infty \in H^{1,1}(-\infty, +\infty) \cap H^{1,p}(-\infty, +\infty), \qquad p \ge 1. \quad (6_1)$$

Using the method that was already applied we can also show the following:

$$\mathbf{H}(t),\ \mu(t) - \mu_\infty \in H^{1,1}(-\infty, +\infty) \cap H^{1,p}(-\infty, +\infty), \qquad p \ge 1. \quad (6_2)$$

Hence, the work of the electromagnetic field

$$L(-\infty, +\infty) = \int_{-\infty}^{+\infty} [\mathbf{D}(t) \cdot \dot{\mathbf{E}}(t) + \mathbf{B}(t) \cdot \dot{\mathbf{H}}(t)]\, dt$$

is bounded because the power of the electromagnetic field $W(t)$ satisfies, in virtue of (3) and (5), the following inequality:

$$\|W(t)\|_{L^1(-\infty, -\infty)} \le \|\mathbf{D}(t)\|_{L^1(-\infty, +\infty)\cap L^p(-\infty, +\infty)}\,\|\dot{\mathbf{E}}(t)\|_{L^\infty(-\infty, +\infty)\cap L^{p/(p-1)}(-\infty, +\infty)} +$$

$$+ \|\mathbf{B}(t)\|_{L^1(-\infty, +\infty)\cap L^p(-\infty, +\infty)}\,\|\dot{\mathbf{H}}(t)\|_{L^\infty(-\infty, +\infty)\cap L^{p/(p-1)}(-\infty, +\infty)}. \quad (7)$$

Inequality (7) and conditions (6) imply

$$\varepsilon(t) - \varepsilon_\infty,\ \mu(t) - \mu_\infty,\ \mathbf{E}(t),\ \mathbf{H}(t) \in H^{1,1}(-\infty, +\infty) \cap H^{1,p}(-\infty, +\infty),$$

$$\dot{\mathbf{E}}(t),\ \dot{\mathbf{H}}(t) \in H^{1,1}(-\infty, +\infty) \cap L^{p/(p-1)}(-\infty, +\infty), \qquad p \ge 1. \quad (8)$$

As to physics, it is very interesting to consider (8) for $p = 2$ and $p = \infty$, i.e., for

$$\varepsilon(t) - \varepsilon_\infty,\ \mu(t) - \mu_\infty,\ \mathbf{E}(t),\ \mathbf{H}(t) \in H^{1,1}(-\infty, +\infty) \cap H^{1,2}(-\infty, +\infty),$$

$$\dot{\mathbf{E}}(t),\ \dot{\mathbf{H}}(t) \in L^\infty(-\infty, +\infty), \quad (9)$$

and

$$\varepsilon(t) - \varepsilon_\infty,\ \mu(t) - \mu_\infty,\ \mathbf{E}(t),\ \mathbf{H}(t) \in H^{1,1}(-\infty, +\infty) \cap H^{1,\infty}(-\infty, +\infty), \quad (10)$$

respectively.

Taking conditions (3) into account and integrating by parts, we express constitutive functionals (1) as follows

$$\mathbf{D}(t) = \varepsilon_\infty \mathbf{E}(t) + \int_0^{+\infty} [\varepsilon(s) - \varepsilon_\infty] \dot{\mathbf{E}}^t(s) ds,$$
$$\mathbf{B}(t) = \mu_\infty \mathbf{H}(t) + \int_0^{+\infty} [\mu(s) - \mu_\infty] \dot{\mathbf{H}}^t(\boldsymbol{x}, s) ds. \quad (11)$$

Having compared (1) and (11) one can evaluate the dissipation of the electromagnetic field

$$[\varepsilon_0 - \varepsilon_\infty]\mathbf{E}(t) = -\int_0^{+\infty} d\{[\varepsilon(s) - \varepsilon_\infty]\mathbf{E}^t(s)\},$$
$$[\mu_0 - \mu_\infty]\mathbf{E}(t) = -\int_0^{+\infty} d\{[\mu(s) - \mu_\infty]\mathbf{E}^t(s)\}.$$

The given hypotheses imply microscopic irreversibility and agree with the strictly dissipative formulation of the second law of thermodynamics. Using the hypotheses presented above and the method used in [7], [8], we obtain the following result:

Theorem 1. *Under hypotherses* (3) *and* $\varepsilon_0 \neq \varepsilon_\infty$, $\mu_0 \neq \mu_\infty$, *it is not possible to formulate, in general, the hypothesi of time reversibility according to which the work of the electromagnetic field along the whole closed circuit is invariant with respect to an inversion of the orientation of the time scale*:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{t} [\dot{\overline{\mathbf{E}}}(t)\cdot\varepsilon(t-s)\overline{\mathbf{E}}'(s) + \dot{\overline{\mathbf{H}}}(t)\cdot\mu(t-s)\overline{\mathbf{H}}'(s)]\, ds\, dt =$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{t} [\dot{\mathbf{E}}(t)\cdot\varepsilon(t-s)\mathbf{E}'(s) + \dot{\mathbf{H}}(t)\cdot\mu(t-s)\mathbf{H}'(s)]\, ds\, dt,$$

where $\overline{\mathbf{E}}(t) = \mathbf{E}(-t)$, $\overline{\mathbf{H}}(t) = \mathbf{H}(-t)$.

Proof. It follows from (9) or (10) that the functions $\varepsilon(s) - \varepsilon_\infty$, $\varepsilon(s) - \varepsilon_0$ and $\mu(s) - \mu_\infty$, $\mu(s) - \mu_0$ are primitive functions of $\varepsilon'(s)$ and $\mu'(s)$, respectively, and, in general, have properties different from regularity when $s \to +\infty$. Therefore, the functions $\varepsilon(s) - \varepsilon_\infty$ and $\mu(s) - \mu_\infty$ can be regarded as functions summable on $\mathbf{R}^+$. Thus, in general, this property cannot be formulated for $\varepsilon(s) - \varepsilon_0$, $\mu(s) - \mu_0$ if $\varepsilon_0 \neq \varepsilon_\infty$, $\mu_0 \neq \mu_\infty$. The inversion of the orientation of the time scale implies different behavior of functions $\varepsilon(s)$ and $\mu(s)$ at infinity. This observation and hypotheses (9) or (10) finally imply the statement of the theorem.

Theorem 2. *Only for the case where* $\varepsilon_0 = \varepsilon_\infty$, $\mu_0 = \mu_\infty$ *and under hypotheses* (2), *the functions* $\varepsilon(s)$ *and* $\mu(s)$ *are constant for all* $s \in [0, +\infty)$. *Hence, the electromagnetic field in the considered dielectric is characterized by the following constitutive equations*:

$$\mathbf{D}(t) = \varepsilon_0 \mathbf{E}(t), \quad \mathbf{B}(t) = \mu_0 \mathbf{H}(t), \quad t \in [0, T_c], \quad (12)$$

where T_c *is called the critical time of memory.*

Proof. By the admitted hypotheses, we obtain

$$\hat{\varepsilon}'_c(w) = w\hat{\varepsilon}_s(w), \quad \hat{\varepsilon}'_s(w) = -w\hat{\varepsilon}_c(w), \quad (13)$$

where $\hat{\varepsilon}'_c(w)$, $\hat{\varepsilon}_c(w)$ denote, respectively, the Fourier cosine-transforms of the functions $\varepsilon'(s)$ and $\varepsilon_s - \varepsilon_\infty$, whereas $\hat{\varepsilon}'_s(w)$, $\hat{\varepsilon}_s(w)$ denote, respectively, Fourier sine-transforms of the functions $\varepsilon'(s)$ and $\varepsilon_s - \varepsilon_\infty$.

Multiplying the first relation in (13) and the second relation in (13) by $\cos wt$ and $\sin wt$, respectively, where $t \geq 0$, and integrating with respect to w, we obtain

$$\int_0^a [\hat{\varepsilon}_c'(w) - w\hat{\varepsilon}_s(w)] \cos wt\, dw = 0, \tag{14}$$

$$\int_0^a [\hat{\varepsilon}_s'(w) + w\hat{\varepsilon}_c(w)] \sin wt\, dw = 0. \tag{15}$$

Under hypotheses (2) it is possible to change the order of integration of the variables s and w in (14) and (15). Summing (14) and (15), we come to the following formula:

$$\int_0^{+\infty} \varepsilon'(s) \frac{\sin a(s-t)}{s-t} ds =$$

$$= \int_0^{+\infty} [\varepsilon(s) - \varepsilon_\infty] \frac{\sin a(s-t) - a(s-t)\sin(s-t)}{(s-t)^2} ds. \tag{16}$$

Formula (16) can also be written as follows:

$$\int_{-at}^{+\infty} \frac{\sin y}{y} \varepsilon'\left(t + \frac{y}{a}\right) dy = \int_{-at}^{+\infty} a\left[\varepsilon\left(t + \frac{y}{a}\right) - \varepsilon_\infty\right] \frac{\sin y - y\cos y}{y^2} dy, \tag{17}$$

where $y = a(s-t)$.

Passing here to the limit as $a \to +\infty$, we obtain the following integro-differential equation:

$$\pi\varepsilon'(t) = \lim_{a \to +\infty} \int_{-at}^{+\infty} a\left[\varepsilon\left(t + \frac{y}{a}\right) - \varepsilon_\infty\right] \frac{\sin y - y\cos y}{y^2} dy. \tag{18}$$

Taking into account formula (1), and the fact that the zero function $\varepsilon(t) - \varepsilon_0 = \varepsilon(t) - \varepsilon_\infty = 0$, $t \geq 0$, is the unique nonsingular solution of equation (18) under the admitted hypotheses we get the first relation in (12).

Applying the already used method, one can show that the function $\mu(s)$ is also constant.

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