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NORMALIZATION AND AVERAGING ON COMPACT LIE GROUPS IN NONLINEAR MECHANICS*

НОРМАЛІЗАЦІЯ ТА УСЕРЕДНЕННЯ НА КОМПАКТНИХ ГРУПАХ ЛІ У НЕЛІНІЙНІЙ МЕХАНІЦІ

We consider the method of normal forms, the Bogolyubov averaging method, and the method of asymptotic decomposition proposed by Yu. A. Mitropol'skii and the author of this paper. Under certain assumptions about group-theoretical properties of a system of zero approximation, the results obtained by the method of asymptotic decomposition coincide with the results obtained by the method of normal forms or the Bogolyubov averaging method. We develop a new algorithm of asymptotic decomposition by a part of variables and its partial case — the algorithm of averaging on a compact Lie group. For the first time, it became possible to consider asymptotic expansions of solutions of differential equations on noncommutative compact groups.

Розглянуто метод нормальних форм, метод усереднення за Боголюбовим та метод асимптотичної декомпозиції, запропонований Ю. О. Митропольським та автором цієї статті. Якщо зробити певні припущення щодо теоретико-групових властивостей системи нульового наближення, то метод асимптотичної декомпозиції приводить до результатів, що здобуваються за методом нормальних форм або за методом усереднення за Боголюбовим. Розвинуто новий метод асимптотичної декомпозиції за частиною змінних та його частинний випадок — алгоритм усереднення на компактних групах Лі. Це дало змогу вперше отримати асимптотичне представлення розв'язків системи нелінійних диференціальних рівнянь на компактних некомутативних групах.

The idea of introducing coordinate transformations for the simplification of the analytic expression of a general problem is a powerful one. Symmetry and differential equations were close partners since the time of the founding masters, namely, Sophus Lie (1842–1899) and his disciples. Till now, symmetry plays a very important role. The ideas of symmetry penetrated deep into various branches of science: mathematical physics, mechanics, etc.

The role of symmetry in perturbation problems of nonlinear mechanics, which was already used by many investigators since 70s (J. Mozer, G. Hori, A. Kamel, and U. Kirchgraber), has been considerably developed in recent years to gain further understanding and development of such constructive and powerful methods as the averaging method and method of normal forms.

Normalization techniques within the framework of the averaging method were considered in the works of A. M. Molchanov [2], A. D. Bryuno [3], S. N. Chow and J. Mallet–Paret [4], Yu. A. Mitropol'skii and A. M. Samoilenko [5], and J. A. Sanders and F. Verhulst [6].

The group-theoretic approach in the problem of quasiperiodic vibration was used by J. Mozer [7]. An approach where Lie series in a parameter were used as a transformation was considered in the works of G. Hori [8, 9], A. Kamel [10], U. Kirchgraber [11],

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U. Kirchgraber and E. Stiefel [12], V. N. Bogaevskii and A. Ya. Povzner [13], and V. F. Zhuravlev and D. N. Klimov [14].

The asymptotic method of nonlinear mechanics developed by N. M. Krylov, N. N. Bogolyubov, and Yu. A. Mitropol'skii and known as the KBM method (see, for example, the monograph of N. N. Bogolyubov and Yu. A. Mitropol'skii [15]) is a powerful tool for investigation of nonlinear vibrations.

The present paper deals with the development of new normalization procedures and averaging algorithms in problems of nonlinear vibrations. Namely, we develop asymptotic methods of perturbation theory with the wide use of group-theoretic techniques. Various assumptions about specific group properties are investigated, and it is shown that they lead to modifications of existing methods (such as the Bogolyubov averaging method and the Poincaré–Birkhoff normal form) as well as to the formulation of a new method. We also develop normalization techniques on Lie groups.

1. Mathematical background. Below, we give a short survey of two methods, namely, the Bogolyubov averaging method and the method of normal forms.

1.1. *The standard system and Bogolyubov averaging.* The new normalization technique was developed by Yu. A. Mitropol'skii and A. K. Lopatin [16, 17] and A. K. Lopatin [18, 19]. In their works, a new method was proposed for the investigation systems of differential equations with small parameters. It was a further development of the Bogolyubov averaging method referred to by the authors as "the method of asymptotic decomposition". The idea of a new approach originates from the Bogolyubov averaging method [15], but its realization requires essentially new apparatus — the theory of continuous transformation groups.

Let us explain the idea of the new approach. As is known, the starting point of the investigation by the averaging method is a system in the standard form

$$\frac{dx}{dt} = \varepsilon X(x, t, \varepsilon) \ , \tag{1}$$

where $x = \mathrm{col}[x_1, \dots, x_n]$, $X(x, t, \varepsilon)$ is an n-dimensional vector.

System (1), upon averaging

$$X_0(\xi, \varepsilon) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(\xi, t, \varepsilon) dt$$

and with a special change of variables, is reduced to the averaged system

$$\frac{d\bar{x}}{dt} = \varepsilon X_0^{(1)}(\bar{x}) + \varepsilon^2 X_0^{(2)}(\bar{x}) + \dots, \tag{2}$$

which does not explicitly contain the argument t. (To guarantee the existence of the average we impose special conditions on the functions $X_j(x, t, \varepsilon)$, $j = \overline{1, n}$. We omit the explicit form of these conditions). Let us rewrite the initial system (1) in the equivalent form

$$\frac{dx}{dt} = \varepsilon X(x, y, \varepsilon), \qquad \frac{dy}{dt} = 1 \tag{3}$$

and the averaged system (2), correspondingly, in the form

$$\frac{d\bar{x}}{dt} = \varepsilon X_0(\bar{x}), \qquad \frac{d\bar{y}}{dt} = 1, \tag{4}$$

where $X_0(\bar{x}) = X_0^{(1)}(\bar{x}) + \varepsilon X_0^{(2)}(\bar{x}) + \dots$. The integration of (4) is simpler than that of (3) because the variables are separated: The system for slow variables $\bar{x}$ does not contain the fast variable $\bar{y}$ and is integrated independently.

Everything stated above allows us to interpret the averaging method in the following way: *The averaging method transforms* (3) *with nonseparated variables into* (4) *with separated fast and slow variables.*

The described property of the separation of variables with the help of the averaging method has group-theoretic characteristics: *The averaging method transforms* (3), *which is not invariant under the action of the one-parameter transformation group generated by the vector field*

$$W = \frac{\partial}{\partial y},$$

associated with the system of zero approximation (3), *into the averaged system* (4), *which is invariant under the action of the one-parameter transformation group generated by the vector field*

$$U = \frac{\partial}{\partial \bar{y}},$$

associated with the system of zero approximation (4). This statement can easily be proved.

1.2. *The method of normal forms.* Consider a system of differential equations with coefficients analytic in a neighborhood of zero

$$\begin{gathered} \dot{y}_1 = a_{11}y_1 + \ldots + a_{1n}y_n + \sum f^1_{m_1 \ldots m_n}\, y_1^{m_1} \ldots y_n^{m_n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{y}_n = a_{n1}y_1 + \ldots + a_{nn}y_n + \sum f^n_{m_1 \ldots m_n}\, y_1^{m_1} \ldots y_n^{m_n}. \end{gathered} \tag{5}$$

Nonlinear terms on the right-hand sides of system (5) are started with terms of order not lower than two.

We consider the problem of finding an analytic change of variables

$$y = f(z), \qquad z = [z_1, \ldots, z_n] \tag{6}$$

that turns the maximal number of coefficients with nonlinear terms into zero. The limiting case is the linearization of system (5), i.e., the transformation of it into

$$\dot{z} = \mathcal{A}z, \qquad \mathcal{A} = [a_{ij}], \quad i, j = \overline{1, n}, \tag{7}$$

under the action of the change of variables (6). Since the indicated procedure is ultimately reduced to the solvability of linear inhomogeneous algebraic equations, it turns out that the reduction

$$(5) \overset{y=f(z)}{\longrightarrow} (7) \tag{8}$$

is not always possible. In the general case, we obtain a system of nonlinear differential equations $\dot{z} = \mathcal{A}z + F(z)$. This system is called a normal form.

We call the corresponding nonlinear term in the equation presented above a *resonance term.* It is clear that only resonance terms remain in the normal form. In particular, linearization by (8) is possible only if there are no resonance terms (see, e.g., [14]).

1.3. *Generalization of the Bogolyubov averaging method by using the symmetry of the standard system.* The method of asymptotic decomposition is based on the group-theoretic interpretation of the averaging method. Consider the system of ordinary differential equations

$$\frac{dx}{dt} = \omega(x) + \varepsilon\tilde{\omega}(x), \tag{9}$$

where

$$\omega(x) = \mathrm{col}\,[\omega_1(x), \ldots, \omega_n(x)]; \qquad \tilde{\omega}(x) = \mathrm{col}\,[\tilde{\omega}_1(x), \ldots, \tilde{\omega}_n(x)].$$

The differential operator associated with the perturbed system (9) can be represented as

$$U_0 = U + \varepsilon\tilde{U},$$

where

$$U = \omega_1 \frac{\partial}{\partial x_1} + \ldots + \omega_n \frac{\partial}{\partial x_n}, \qquad \tilde{U} = \tilde{\omega}_1 \frac{\partial}{\partial x_1} + \ldots + \tilde{\omega}_n \frac{\partial}{\partial x_n}.$$

By a certain change of variables in the form of a series in ε

$$x = \varphi(\bar{x}, \varepsilon), \tag{10}$$

system (9) is transformed into a new system

$$\frac{d\bar{x}}{dt} = \omega(\bar{x}) + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} b^{(\nu)}(\bar{x}), \tag{11}$$

which is called *a centralized system*. For this system, $\bar{U}_0 = \bar{U} + \varepsilon\bar{\tilde{U}}$, where

$$\bar{U} = \omega_1(\bar{x}) \frac{\partial}{\partial \bar{x}_1} + \ldots + \omega_n(\bar{x}) \frac{\partial}{\partial \bar{x}_n},$$

$$\bar{\tilde{U}} = \sum_{\nu=1}^{\infty} \varepsilon^{\nu} N_{\nu}, \qquad N_{\nu} = b_1^{(\nu)}(\bar{x}) \frac{\partial}{\partial \bar{x}_1} + \ldots + b_n^{(\nu)}(\bar{x}) \frac{\partial}{\partial \bar{x}_n}. \tag{12}$$

We impose a condition on the choice of transformations (10) requiring that the centralized system (11) should be invariant with respect to the one-parameter transformation group

$$\bar{x} = e^{s\bar{\mathbf{U}}(\bar{x}_0)} \bar{x}_0, \tag{13}$$

where $\bar{x}_0$ is the vector of new variables. Therefore, after the change of variables (13), system (11) turns into

$$\frac{d\bar{x}_0}{dt} = \omega(\bar{x}_0) + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} b^{(\nu)}(\bar{x}_0),$$

which coincides with the original one to within notation. This means that we have the identities $[\bar{U}, N_{\nu}] \equiv 0$ for $\bar{U}$, N_{ν}, $\nu = 1, 2, \ldots$.

Below, we present some material that is necessary for understanding the structure of this paper as a whole. The essential point in realizing the indicated scheme of the algorithm of asymptotic decomposition is that transformations (10) are chosen in the form of a series

$$x = e^{\varepsilon\mathbf{S}} \bar{x}, \tag{14}$$

where

$$S = S_1 + \varepsilon S_2 + \ldots,$$

$$S_j = \gamma_{j1}(\bar{x}) \frac{\partial}{\partial \bar{x}_1} + \ldots + \gamma_{jn}(\bar{x}) \frac{\partial}{\partial \bar{x}_n}.$$

The coefficients of $S_j, \gamma_{j1}(\bar{x}), \ldots, \gamma_{jn}(\bar{x})$ are unknown functions. They should be determined by the recurrent sequence of operator equations

$$[U, S_{\nu}] = F_{\nu}. \tag{15}$$

The operator F_{ν}, $\nu = 1, 2, \ldots$, is a known function of U and $S_1, \ldots, S_{\nu-1}$ obtained at previous steps.

In the case where S depends upon ε, the Lie series (14) is called a Lie transformation. Thus, the application of a *Lie transformation* as a change of variables enables us to use the technique of continuous transformation groups.

It is known from the theory of linear operators that the solvability of the inhomogeneous operator equation (15) depends on the properties of solutions of the homogeneous equation

$$[U, S_\nu] = 0. \tag{16}$$

Operator (12) N_ν is the projection of the right-hand side of the equation onto the kernel of operator (16), which is determined from the condition of solvability in the sense of the inhomogeneous equation

$$[U, S_\nu] = F_\nu - N_\nu, \quad \nu = 1, 2, \ldots . \tag{17}$$

Depending on the way of solving equations (15)–(17), various modifications of the algorithm of the method of asymptotic decomposition are obtained.

The principal conclusion that can be drawn from a comparison of the two methods is the following: In the method of asymptotic decomposition, the operation of averaging that is used in the Bogolyubov averaging method is a certain way of constructing the projection $\operatorname{pr} F$ of the operator F.

In the method of asymptotic decomposition, *the centralized system* is a direct analog *of the averaged* system of the Bogolyubov averaging method.

The operation of averaging used in the method of asymptotic decomposition for construction of the projection of an operator onto an algebra of centralizer is called the *Bogolyubov projector.*

The last statement means the following: Let us apply the method of asymptotic decomposition to the Bogolyubov system in the standard form (3). Let us write the operator F_ν on the right-hand side of (15) as

$$F_\nu = f_{\nu 1}(x, y)\frac{\partial}{\partial x_1} + \ldots + f_{\nu n}(x, y)\frac{\partial}{\partial x_n}.$$

Define the Bogolyubov projection of the operator $\operatorname{pr} F_\nu$ as

$$\operatorname{pr} F_\nu = \langle f_{\nu 1}(x, y)\rangle \frac{\partial}{\partial x_1} + \ldots + \langle f_{\nu n}(x, y)\rangle \frac{\partial}{\partial x_n},$$

where

$$\langle f_{\nu k}(x, y)\rangle =_{\text{def}} f^0_{\nu k}(x) \tag{18}$$

are the average values of the coefficients $f_{\nu k}$. This notion requires exact definition.

In the Bogolyubov averaging method, the average value is understood as

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f_{\nu k}(x, s)ds = f^0_{\nu k}(x) < +\infty, \quad k = 1, \ldots, n.$$

In our further exposition, definition (18) is understood as the average value on the group.

We hope that such a preview will make the main part of the paper easier to understand. We illustrate the further exposition of the material in the next two subsections by two physically motivated examples: nonlinear oscillators in the plane and the motion of a point on a sphere. There are classical results for the first example, and one can compare them with the present approach. The second example is nontrivial as it cannot be considered by existing methods in a similar way.

2. Examples: models connected with nonlinear oscillator in the plane. I.

2.1. *Algorithm of the method of asymptotic decomposition in the space of homogeneous polynomials* (*group* GL(2)). Along with the linear space V over P generated by elements $x_1, \ldots, x_n$, we consider the linear space $V_{\otimes\nu}$ over P, which is equal to the direct product of the spaces V taken ν times. The vector row composed of the basis elements of $V_{\otimes\nu}$ is denoted by $\hat{\hat{x}}_{m_\nu}$. It is evident that $m_1 = n$ and $\hat{\hat{x}}_{m_1} = [x_1, \ldots, x_n]$.

Let $\mathcal{Q}$ be a constant matrix of dimension $m_\nu \times n$ with elements $q_{ij} \in P$, where $i = \overline{1,\ m_\nu}$, $j = \overline{1,\ n}$, and let

$$q = \hat{\hat{x}}_{m_\nu} \mathcal{Q}, \qquad q =_{\text{def}} \|q_1, \ldots, q_n\|.$$

For an arbitrary sequence of matrices $\mathcal{Q}$, the totality of differential operators

$$X = q_1 \frac{\partial}{\partial x_1} + \ldots + q_n \frac{\partial}{\partial x_n}, \quad q_i \in V_{\otimes\nu},$$

yields the linear space over P, which is denoted by $\mathcal{B}(V_{\otimes\nu})$. The matrix $\mathcal{Q}$ is called *the matrix of the operator* X.

Consider the system of two equations of the first order

$$\dot{x}_1' = x_2', \qquad \dot{x}_2' = -x_1' + \varepsilon\,(1 - x_1'^2)\, x_2'. \tag{19}$$

The differential operator associated with system (19) is

$$U_0' = U' + \varepsilon\, \widetilde{U}',$$

where

$$U' = x_2' \frac{\partial}{\partial x_1'} - x_1' \frac{\partial}{\partial x_2'}, \qquad \widetilde{U}' = \left(x_2' - x_1'^2\, x_2'\right) \frac{\partial}{\partial x_2'}.$$

Write these operators in the form

$$U' = \hat{\hat{x}}_{m_1}' \mathcal{F} \partial', \qquad \mathcal{F} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Represent the operator $\widetilde{U}$ as the sum

$$\widetilde{U}' = \widetilde{U}_{\otimes 1}' + \widetilde{U}_{\otimes 3}', \qquad \widetilde{U}_{\otimes i}' \in \mathcal{B}(V_{\otimes i}), \quad i = 1,\ 3,$$

where

$$\widetilde{U}_{\otimes 1}' = \hat{\hat{x}}_{m_1}'\, \mathcal{Q}_{m_1,1}\, \partial, \qquad \widetilde{U}_{\otimes 3}' = \hat{\hat{x}}_{m_3}'\, \mathcal{Q}_{m_3,1}\, \partial.$$

Calculate two approximations in the transformed operator (12)

$$U_0' = U' + \varepsilon\, N_1' + \varepsilon^2\, N_2'.$$

Calculate the operators S_1 and S_2, which can be obtained from the equations

$$[U,\ S_1] = \widetilde{U} - \operatorname{pr} \widetilde{U},$$

$$[U_1,\ S_2] = \left\{ -[\widetilde{U},\ S_1] - \frac{1}{2}\, [S_1,\ [U, S_1]] \right\} - \operatorname{pr} \{\ldots\} \tag{20}$$

upon the change of variables (14). Solve these equations in two steps. First, we find S_1. We have

$$S_1 \equiv S_{\otimes 11} + S_{\otimes 31}, \quad S_{\otimes i1} \in \mathcal{B}\,(V_{\otimes i}), \quad i = 1, 3,$$

where $S_{\otimes i1} \equiv \hat{\hat{x}}_{m_i}\, \Gamma_{1i}\, \partial$, $i = 1; 3$, and Γ_{1i} are the rectangular matrices of dimensions $m_i \times n$ which are solutions of the system of independent algebraic equations

$$\mathcal{F}_i\, \Gamma_{1i} - \Gamma_{1i}\, \mathcal{F} = \mathcal{Q}_{m_i,1} - \operatorname{pr} \mathcal{Q}_{m_i,1}, \qquad \mathcal{F} = \mathcal{A}^T, \quad i = 1, 3. \tag{21}$$

At the second step, we find S_2. We can see that $S_2 \in \mathcal{B}\,(V_{\otimes 5})$ implies the structure of the right-hand sides of equation (20). We have to find a solution in the form of the sum

$$S_2 = \sum_{i=1}^{5} S_{\otimes i2}, \qquad S_{\otimes i2} = \hat{\hat{x}}_{m_i}\, \Gamma_{2i}\, \partial, \quad i = \overline{1,\ 5},$$

where Γ_{2i} are solutions of the system of algebraic equations

$$\mathcal{F}_i \Gamma_{2i} - \Gamma_{2i} \mathcal{F} = \mathcal{Q}_{m_i,2} - \mathrm{pr}\, \mathcal{Q}_{m_i,2}, \quad i = \overline{1,\ 5}.$$

Let us perform necessary calculations for the first approximation. Consider equation (21). Here, $\mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{F}_3$ are the matrices of the representation of the operator U in the subspaces $V_{\otimes 1},\ V_{\otimes 2},\ V_{\otimes 3}$.

Pass from equations (21) to the equations in the spaces $\hat{R}^{(m_1,n)},\ \hat{R}^{(m_2,n)}$

$$G_{\mathcal{F}}^{(i)} \hat{\Gamma}_{i1} = \hat{\mathcal{Q}}_{m_i,1} - \mathrm{pr}\, \hat{\mathcal{Q}}_{m_i,1},$$

where

$$G_{\mathcal{F}}^{(i)} = \mathcal{F}_i \otimes \mathcal{E}_2 - \mathcal{E}_{m_i} \otimes \mathcal{F}^T, \quad i = 1,\ 3,$$

and $\hat{\Gamma}_{1i},\ \hat{\mathcal{Q}}_{m_i,1}$ are vector columns composed of rows of the matrices $\Gamma_{1i},\ \mathcal{Q}_{m_i,1}$.

Taking into account that the difference $\hat{\mathcal{Q}}_{m_i,1} - \hat{\mathcal{Q}}_{m_i,1N}$ belongs to the image $T_{\mathcal{F}}^{(i)}$ of the operator $G_{\mathcal{F}}^{(i)}$ and is orthogonal to the kernel of the operator $G_{\mathcal{F}}^{(i)T}$, we obtain the system of linear algebraic equations for finding $\mathrm{pr}\, \hat{\mathcal{Q}}_{m_i,1}$.

Finally, we get the operator U_0 in the first approximation:

$$U_0 = U + \varepsilon N_1,$$

where

$$N_1 = \mathrm{pr}\tilde{U} = N_{\otimes 11} + N_{\otimes 13};$$

$$N_{\otimes 11} = \hat{\hat{x}}_{m_1} \mathcal{Q}_{m_1,1N}\, \partial = \frac{1}{2} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right);$$

$$N_{\otimes 31} = \hat{\hat{x}}_{m_3} \mathcal{Q}_{m_3,1N}\, \partial = -\frac{1}{8} \left((x_1^2 + x_2^2)\, x_1 \frac{\partial}{\partial x_1} + (x_1^2 + x_2^2)\, x_2 \frac{\partial}{\partial x_2} \right).$$

After similar calculations, we find the centralized system in the second approximation

$$\frac{dx_1}{dt} = x_2 + \varepsilon \left(\frac{1}{2} - \frac{1}{8} \left(x_1^2 + x_2^2\right) \right) x_1 +$$

$$+ \varepsilon^2 \left(-\frac{1}{4} + \frac{3}{8} \left(x_1^2 + x_2^2\right) - \frac{11}{128} \left(x_1^2 + x_2^2\right)^2 \right) x_2;$$

$$\frac{dx_2}{dt} = -x_1 + \varepsilon \left(\frac{1}{2} - \frac{1}{8} \left(x_1^2 + x_2^2\right) \right) x_2 -$$

$$- \varepsilon^2 \left(-\frac{1}{4} + \frac{3}{8} \left(x_1^2 + x_2^2\right) - \frac{11}{128} \left(x_1^2 + x_2^2\right)^2 \right) x_1.$$

We can easily see that, upon the transformation of the variables according to the formulas

$$y_1 = \sqrt{x_1^2 + x_2^2}, \qquad y_2 = \mathrm{arctg}\, \frac{x_1}{x_2},$$

the centralized system takes the form

$$\frac{dy_1}{dt} = \varepsilon \left(\frac{1}{2} - \frac{1}{8} y_1^2 \right) y_1;$$

$$\frac{dy_2}{dt} = 1 - \varepsilon^2 \left(\frac{1}{4} - \frac{3}{8} y_1^2 + \frac{11}{128} y_1^4 \right).$$

To pass to the solution of the initial equations (19) in the second approximation, we have to know the operator S_2. The calculation of S_2 is analogous to that of S_1.

2.2. *Procedures of normalization in the spaces of representations of the groups* GL(2) *and* SO(2). Consider the nonlinear oscillator (1). All considerations of Subsection

2.1 were based on the invariance property of the subspaces $V_{\otimes 1},\ V_{\otimes 2},\dots$ which are associated with the system of zero approximation. The fact of invariance is expressed by the relation

$$U\hat{\hat{x}}_{m_j} = \hat{\hat{x}}_{m_j}\mathcal{F}_j, \quad j = 1, 2, \dots,$$

where $\mathcal{F}_j$ is the matrix of representation of U in the subspace $V_{\otimes j}$.

A natural question arises: Are the subspaces $V_{\otimes 1},\ V_{\otimes 2},\dots$ unique invariant subspaces in the linear space of homogeneous polynomials? It turns out that they are not.

Consider the linear space $T_\otimes$ that is the direct sum of the subspaces with the bases

$$\begin{gathered} f^{(m_1)} = [\ x_1,\ x_2\ ], \\ f^{(m_2)} = [\ 2x_1x_2,\ x_2^2 - x_1^2\ ], \\ f^{(m_3)} = [\ 3\ (x_1^2 + x_2^2)x_1 - 4x_1^3,\ 4x_2^3 - 3\ (x_1^2 + x_2^2)x_2\ ], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \end{gathered} \tag{22}$$

It is easy to verify that each subspace $T_{\otimes j}$ turns into itself under the action of U, e.g., is invariant with respect to it. To do so, it is sufficient to find the matrices of representation of U in these subspaces

$$Uf^{(m_j)} = f^{(m_j)}\mathcal{F}_j, \qquad \mathcal{F}_j = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}.$$

For a better understanding of the structure of the space $T_\otimes$, let us introduce new variables ρ and φ by the formula

$$x_1 = \rho \sin \varphi, \qquad x_2 = \rho \cos \varphi.$$

In new variables, the basis vectors (22) are written as follows:

$$\hat{\hat{\varphi}}_{m_k} = [\rho^k \sin k\varphi,\ \rho^k \cos k\varphi], \quad k = 1, 2, \dots .$$

So, passing to the space $T_\otimes \subset T(V)$ means passing from the space of homogeneous polynomials in two variables to the space of trigonometric functions (Fourier series).

The described process of choosing a new representation space for the operator U has deep group-theoretic background. Let us consider this process in detail.

Consider the set of four linearly independent operators

$$V_{11} = x_1\frac{\partial}{\partial x_1}, \qquad V_{12} = x_1\frac{\partial}{\partial x_2}, \qquad V_{21} = x_2\frac{\partial}{\partial x_1}, \qquad V_{22} = x_2\frac{\partial}{\partial x_2}, \tag{23}$$

which generate a complete linear finite-dimensional Lie algebra $gl(2)$ of order four. From (23), a general linear group $GL(2)$ is restored. To write the elements of this group in explicit form, let us write its general element in terms of a Lie series

$$x' = \exp\ Vx, \tag{24}$$

where

$$V = s_{11}V_{11} + s_{12}V_{12} + s_{21}V_{21} + s_{22}V_{22},$$

and $s_{11}, s_{12}, s_{21}, s_{22}$ are group parameters which range in a neighborhood of zero.

We write the series (24) in the finite form

$$x' = xe^{\mathcal{F}_1(s)}.$$

The matrix $\mathcal{G}(s) = e^{\mathcal{F}_1(s)}$, where $\mathcal{F}_1$ is the representation matrix of V in the subspace $V_{\otimes 1}$, determines the general element of $GL(2)$.

In view of the above considerations, we can say that *the linear space of homogeneous polynomials $T(V)$ is the representation space for the general linear group $GL(n),\ n=2$.*

The operator U of the system of zero approximation generates the rotation group $SO(2)$ in the plane. To find the explicit form of the elements of this group, we also use a Lie series

$$x' = \exp(\varphi U)x.$$

After the corresponding computations, we arrive at the result

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus, the linear space of trigonometric functions $T_{\otimes}$ is the representation space for the rotation group $SO(2)$ in the plane. Let us denote this space by $T_{SO(2)}$.

In the method of normal forms, the representation space for the general linear group $GL(n)$ is chosen as a representation space. In the method of asymptotic decomposition, the representation space for the subgroup of the same group $GL(n)$ is chosen as a representation space.

So, the method of normal forms, which uses the universal representation space of the general linear group, does not consider the true algebraic structure of the system of zero approximation.

Contrary to this, the method of asymptotic decomposition is essentially based on the deep connection between the representation theory for continuous groups and special functions of mathematical physics. This theory has been extensively developed for the last decades (see Vilenkin N. Ya. [20] and Barut A. and Roczka R. [21]).

2.3. *Algorithm of asymptotic decomposition for a perturbed motion on $SO(2)$.* Let us consider the Van der Pol system

$$\dot{x}_1 = x_2; \qquad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)\,x_2. \tag{25}$$

as perturbed motion on $SO(2)$ (see Section 3). The system of zero approximation (25) yields the group $SO(2)$. Pass to the polar coordinates in (25)

$$x_1 = \rho'\sin\varphi', \qquad x_2 = \rho'\cos\varphi'. \tag{26}$$

Finally, we obtain

$$\begin{gathered} \frac{d\rho'}{dt} = \varepsilon\frac{\rho'}{2}\left(1 - \frac{\rho'^2}{4} + \cos 2\varphi' + \frac{\rho'^2}{4}\cos 4\varphi'\right), \\ \frac{d\varphi'}{dt} = 1 - \varepsilon\frac{1}{2}\left(\sin 2\varphi' - \frac{\rho'^2}{2}\sin 2\varphi' + \frac{\rho'^2}{4}\sin 4\varphi'\right). \end{gathered} \tag{27}$$

Write the operator U'_0 associated with system (27)

$$U'_0 = U'_1 + \varepsilon\widetilde{U}',$$

where

$$\begin{gathered} U'_0 = \frac{\partial}{\partial\varphi'}, \qquad \widetilde{U'} = b_1(\rho',\varphi')\frac{\partial}{\partial\rho'} + b_2(\rho',\varphi')\frac{\partial}{\partial\varphi'}, \\ b_1(\rho',\varphi') = \frac{\rho'}{2}\left(1 - \frac{\rho'^2}{4} + \cos 2\varphi' + \frac{\rho'^2}{4}\cos 4\varphi'\right), \\ b_2(\rho',\varphi') = 1 - \varepsilon\frac{1}{2}\left(\sin 2\varphi' - \frac{\rho'^2}{2}\sin 2\varphi' + \frac{\rho'^2}{4}\sin 4\varphi'\right). \end{gathered} \tag{28}$$

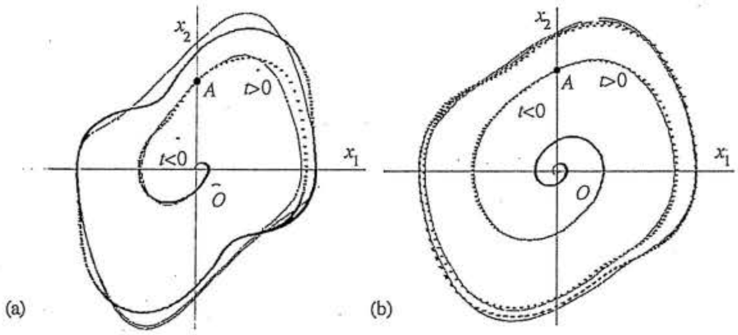

Fig. 1. The solution in the phase plane for the Van der Pol system (solid lines correspond to the exact solution; dotted lines correspond to the approximate solution (a centralized system of the first approximation)): $\varepsilon = 1$ (a), $\varepsilon = 0.5$ (b).

The operator U has the representation matrix $\mathcal{F}_{m_n}$ in the subspace $T_{\otimes n}$. This matrix can be calculated by

$$Uf^{(n)} = f^{(n)}\mathcal{F}_{m_n}, \quad \mathcal{F}_{m_n} = \begin{bmatrix} 0 & -n \\ n & 0 \end{bmatrix}.$$

Let us illustrate the application of the method of asymptotic decomposition to system (1) in the representation space of $T_{\otimes}$. Calculate only the first approximation. Let the single term S_1 be in transformation (14) and let the transformed operator be represented by the sum

$$U_0 = U + \varepsilon N_1.$$

According to the general theory, we should consider the equation

$$[U,\ S_1] = F_1, \quad F_1 =_{\mathrm{def}} \widetilde{U}. \tag{29}$$

After the change of variables (26), $\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}$ turn into $L_1,\ L_2$, respectively, where

$$L_1 = \sin\varphi \frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}, \qquad L_2 = \cos\varphi \frac{\partial}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}.$$

Rewrite $U, \widetilde{U}$ in the new variables by using L_1 and L_2. We have

$$U \equiv \frac{\partial}{\partial \varphi} \equiv [\rho\sin\varphi,\ \rho\cos\varphi]\,\mathcal{F}L,$$

$$\widetilde{U} = [\rho\sin\varphi,\ \rho\cos\varphi]\,\mathcal{Q}_{11}L + [\rho^3\sin 3\varphi,\ \rho^3\cos 3\varphi]\,\mathcal{Q}_{31}L.$$

Write the operator of the transformation S_1, following the structure of the right-hand side of equation (29), in the form

$$S_1 = S_{11} + S_{31},$$

where

$$S_{11} = [\rho\sin\varphi,\ \rho\cos\varphi]\,\Gamma_{11}L, \qquad S_{31} = [\rho^3\sin 3\varphi,\ \rho^3\cos 3\varphi]\,\Gamma_{31}L,$$

and Γ_{11}, Γ_{31} are unknown second-order square matrices. In the general case, they depend on the variable ρ.

Substituting U, $\widetilde{U}$ and S_1 into equation (29), we obtain two independent subsystems of linear algebraic equations

$$\mathcal{F}_1\Gamma_{j1} - \Gamma_{j1}\mathcal{F} = \mathcal{Q}_{j1}, \quad j = 1, 2.$$

All further calculations are similar to those in the previous subsection. We give only final results.

Thus, the operator N_1 defined by the matrix can be written in the final form

$$N_1 \equiv \left(\frac{1}{2}-\frac{1}{8}\rho^2\right)\rho\sin\varphi L_1+\left(\frac{1}{2}-\frac{1}{8}\rho^2\right)\rho\cos\varphi L_2 \equiv \rho\left(\frac{1}{2}-\frac{1}{8}\rho^2\right)\frac{\partial}{\partial\rho}.$$

By the operator $U_0 = U + \varepsilon N_1$, we restore the centralized system of the first approximation

$$\dot{\rho} = \frac{\varepsilon}{2}\left(1-\frac{1}{4}\rho^2\right)\rho, \qquad \dot{\varphi} = 1.$$

The comparison of the algorithm of asymptotic decomposition in the representation space $T_\otimes$ of trigonometric functions described in this subsection with an analogous algorithm in the space of polynomials $T(V)$ considered in the previous subsection shows a substantial decrease in calculating efforts. This fact is explained by lowering the order of the representation matrices $\mathcal{F}_j$ of the operator U in the subspaces $T_{\otimes j}$ as compared with the subspace $V_{\otimes j}$. Indeed, in the first case, the order of the matrices $\mathcal{F}_j$ is equal to 2 and does not change. In the second case, it grows proportionally to the index j.

Finally, let us compare the algorithm of asymptotic decomposition with existing methods. If the representation space $T_\otimes$ of the group $SO(2)$ is chosen, then we obtain the results of the Krylov–Bogolyubov asymptotic method. If the representation space $V_\otimes$ of the general linear group $GL(2)$ (the space of homogeneous polynomials) is chosen, then we obtain the results of the method of normal forms.

3. Averaging on compact Lie groups.

3.1. *The construction of a quadratic-form integral of a linear system.* Let us consider a linear system with constant coefficients

$$\frac{dy}{dt} = \tilde{\mathcal{A}}y, \tag{30}$$

where $y = [y_1, \ldots, y_n]^T$, (T denotes transposition), $\tilde{\mathcal{A}} = \text{const}$, $\tilde{\mathcal{A}} \in M_n(R)$ ($M_n(R)$ is the set of all real $n \times n$ matrices). We want to know when system (30) has the quadratic-form integral

$$F(y) = y^T\mathcal{Y}y, \quad \mathcal{Y} \in M_n(R). \tag{31}$$

By (30), the derivative of the integral $F(y) = c$

$$\frac{dF(y)}{dt} = y^T(\tilde{\mathcal{A}}^T\mathcal{Y} + \mathcal{Y}\tilde{\mathcal{A}})y \equiv 0, \tag{32}$$

is identically equal to zero. This implies the matrix equation

$$\tilde{\mathcal{A}}^T\mathcal{Y} + \mathcal{Y}\tilde{\mathcal{A}} = 0. \tag{33}$$

It is equivalent to

$$\bar{\mathcal{G}}\hat{Y} = 0, \tag{34}$$

where $\bar{\mathcal{G}} = \tilde{\mathcal{A}}^T \otimes \mathcal{E} + \mathcal{E} \otimes \tilde{\mathcal{A}}^T$, $\otimes$ is the sign of direct product of matrices, and $\hat{Y}$ is the vector formed of the rows of $\mathcal{Y}$.

The matrix equation (33) (or (34)) has a zero solution if and only if the matrices $\mathcal{A}$ and $-\mathcal{A}$ have common eigenvalues. Matrices with at least two purely imaginary eigenvalues form the most important class of solutions of (33). It is clear that this class of matrices is the most important for vibration theory.

Suppose that (33) has a solution $\mathcal{Y} = \mathcal{B}$, where $\mathcal{B}$ is a nonsingular symmetric matrix which can be reduced by a nonsingular matrix $\mathcal{Q}$ to the diagonal form $\mathcal{Q}^T\mathcal{B}\mathcal{Q} = \mathcal{E}$.

***Example* 2.** Let $n = 3$ in (39). The linear system of zero approximation

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3 - x_1, \\ \dot{x}_3 &= -x_2, \end{aligned} \tag{42}$$

has the integral

$$F(x) = {x_1}^2 + {x_2}^2 + {x_3}^2.$$

Matrix (40) has the form

$$\begin{bmatrix} \dfrac{\partial}{\partial x_1}, & \dfrac{\partial}{\partial x_2}, & \dfrac{\partial}{\partial x_3} \\ 2x_1, & 2x_2, & 2x_3 \end{bmatrix}.$$

The base of the algebra $so(3)$ of the rotation plane group $SO(3)$ is formed by the operators

$$U_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}, \qquad U_1 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \qquad U_1 = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}.$$

We pass to the spherical coordinates in R^n

$$\begin{aligned} x_1 &= \rho' \sin\theta'_{n-1} \ldots \sin\theta'_2 \sin\theta'_1, \\ x_2 &= \rho' \sin\theta'_{n-1} \ldots \sin\theta'_2 \cos\theta'_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-1} &= \rho' \sin\theta'_{n-1} \ldots \cos\theta'_{n-2}, \\ x_n &= \rho' \cos\theta'_{n-1}. \end{aligned} \tag{43}$$

The variables $\theta'_1, \ldots, \theta'_{n-1}$ vary within the limits $0 \le \theta'_1 \le 2\pi$, $0 \le \theta'_k \le \pi$, $k \ne 1$.

The transformations inverse to (43) are given by the relations

$$\theta'_1 = \frac{r_1}{x_2}, \ldots, \theta'_{n-1} = \frac{r_{n-1}}{x_n},$$

$$\rho' = r_n, \qquad r_k = \sqrt{x_1^2 + \ldots + x_k^2}.$$

It is easy to show that operators (41) of the base do not contain the derivative $\dfrac{\partial}{\partial \rho}$ and do not depend on ρ'.

Let us rewrite system (39) in the new variables. We have

$$\frac{d\rho'}{dt} = \varepsilon \tilde{\omega}_1(\rho', \theta'_1, \ldots, \theta'_{n-1}), \tag{44}$$

$$\frac{d\theta'}{dt} = f(\theta') + \tilde{\omega}^{(n-1)}(\rho', \theta'_1, \ldots, \theta'_{n-1}),$$

where $\theta' = \left[\theta'_1, \ldots, \theta'_{n-1}\right]^T$, $f = [f_2, \ldots, f_n]^T$, and $\tilde{\omega}^{(n-1)} = [\tilde{\omega}_2, \ldots, \tilde{\omega}_n]^T$ are known vectors.

The operator associated with (44) is

$$U'_0 = U' + \varepsilon \tilde{U}', \tag{45}$$

where

$$U = f_2(\theta) \frac{\partial}{\partial \theta_1} + \ldots + f_{n-1}(\theta) \frac{\partial}{\partial \theta_{n-1}},$$

$$\tilde{U} = \tilde{\omega}_1(\rho, \theta) \frac{\partial}{\partial \rho} + \tilde{\omega}_2(\rho, \theta) \frac{\partial}{\partial \theta_1} + \ldots + \tilde{\omega}_n(\rho, \theta) \frac{\partial}{\partial \theta_{n-1}}.$$

According to the algorithm of asymptotic decomposition (see Section 1), we change the variables

$$\rho' = \exp(\varepsilon S)U'\rho, \theta' = \theta,$$

where

$$\exp(\varepsilon S) = 1 + \frac{\varepsilon}{1!}S + \frac{\varepsilon^2}{2!}S^2 + \dots ,$$

$$S = S_1 + \varepsilon S_2 + \dots , \qquad S_j = \gamma_j(\rho, \theta)\frac{\partial}{\partial \rho}, \quad j = 1, 2, \dots ,$$

in system (45).

Here, unlike the general algorithm of asymptotic decomposition, only the variable ρ, which is slow for system (44), is transformed.

According to the Campbell–Hausdorff formula, we have

$$\dot{U}_0 = U + \varepsilon(-[U, S_1] + F_1) + \dots + \varepsilon^n(-[U, S_n] + F_n) + \dots ,$$

where

$$F_1 = \tilde{U} \equiv f_{11}\frac{\partial}{\partial \rho} + \sum_{j=1}^{n-1} f_{1j}\frac{\partial}{\partial \theta_j},$$

$$F_1 = -[F_1, S_1] - \frac{1}{2}[S_1, [U, S_1]] \equiv f_{n1}\frac{\partial}{\partial \rho} + \sum_{j=1}^{n-1} f_{nj}\frac{\partial}{\partial \theta_j}.$$

The operator S_j is determined by the sequence of recurrent operator equations

$$[U, S_j] = \tilde{F}_j - \mathrm{pr}\tilde{F}_j, \tag{46}$$

where $\tilde{F}_j = f_{j1}\frac{\partial}{\partial \rho}$, $j = 1, 2, \dots$.

Let us consider the equation

$$[U, S] = \tilde{F} - \mathrm{pr}\tilde{F}, \tag{47}$$

which is called the *representative equation of system* (45), to show the technique of its solving.

By virtue of the commutativity of $U, \frac{\partial}{\partial \rho}$, the operator equation (47) is reduced to the differential equation

$$U\gamma = f(\rho, \theta) - \tilde{f}, \tag{48}$$

where $\tilde{f}$ is an unknown function.

The differential equation (48) is easily reduced to an infinite sequence of finite-dimensional linear algebraic equations by using the right-hand sides of the Fourier expansion of (48) in the Hilbert space $H = L^2(G) = H_1 \oplus H_2 \oplus \dots$, where H_l are the subspaces of irreducible representations of the group $SO(n)$ of weight l.

The function $f(\rho, \theta)$ can be written as a uniformly convergent series

$$f = \sum_{l,j,i} b^l_{ij}(\rho)\tau^l_{ij}(\theta). \tag{49}$$

The solution γ and undefined function $\tilde{f}$ are found as the series

$$\gamma = \sum_{l,j,i} \gamma^l_{ij}(\rho)\tau^l_{ij}(\theta), \qquad \tilde{f} = \sum_{l,j,i} \tilde{b}^l_{ij}(\rho)\tau^l_{ij}(\theta). \tag{50}$$

The free term of expansion (49)

$$b^0_{00} = \frac{1}{|G|}\int f(\theta)d\theta$$

is equal to the average value of the function $f(\rho,\theta)$ on the group. It should take $\tilde{b}^0_{00} = b^0_{00}$. After the substitution of series (49) and (50) in equation (48), we arrive at the sequence of systems of linear algebraic equations

$$\mathcal{F}_l \bar{\gamma}_l = \bar{b}_l - \mathrm{pr}\bar{\tilde{b}}_l, \tag{51}$$

where $\mathcal{F}_l$ is the matrix of representation of the operator $\mathbf{U}$ in the subspace H_l and $\bar{\gamma}_l$, $\bar{b}_l$, $\bar{\tilde{b}}_l$ are the vectors of the coefficients of components of expansions in the subspace H_l. The vector $\bar{\tilde{b}}_l$ should be taken from the condition of solvability of the matrix equation (51) (see [17], Appendix).

If we determine S_j from system (46) by the algorithm described above, then the operator U_0 turns into

$$U_0 = U + \varepsilon(\mathrm{pr}\hat{F}_1 + (F_1 - \hat{F}_1)) + \ldots + \varepsilon^k(\mathrm{pr}\hat{F}_k + (F_k - \hat{F}_k)) + \ldots,$$

where

$$\mathrm{pr}\hat{F}_k + (F_k - \hat{F}_k) = \sum_{j=2}^{n} f_{kj}\frac{\partial}{\partial\theta_j},$$

and system (44), correspondingly, turns into

$$\frac{d\rho}{dt} = \varepsilon\big(b^0_{00}(\rho) + \tilde{f}_{11}(\rho,\theta)\big) + \varepsilon^2\big(b^1_{00}(\rho) + \tilde{f}_{21}(\rho,\theta)\big) + \ldots, \tag{52}$$
$$\frac{d\theta}{dt} = f(\theta) + \varepsilon\big(f_1^{(n-1)}(\rho,\theta)\big) + \varepsilon^2\big(f_2^{(n-1)}(\rho,\theta)\big) + \ldots.$$

The algorithm described above is called *the algorithm of asymptotic decomposition by a part of variables*. It is worthy to know when the first equation in system (52) does not depend upon the variable θ, i.e.,

$$\frac{d\rho}{dt} = \varepsilon b^0_{00}(\rho) + \varepsilon^2 b^1_{00}(\rho) + \ldots.$$

In this case, the algorithm of asymptotic decomposition by a part of variables is called *the algorithm of averaging on a group* $SO(n)$. Below, we show that, for the group $SO(2)$, the algorithm of averaging on a group is the only way of realization of the algorithm of asymptotic decomposition by a part of variables. This is not true for the noncommutative group $SO(3)$.

4. Examples: Models connected with nonlinear oscillator in the plane. II.

4.1. *Partial group averaging for perturbed motion on* SO(2). Let us return to the consideration of the perturbed system (27). We change the algorithm for the solution of the operator equation (54). In (14), we set

$$S_j = \gamma_j(\rho,\varphi)\frac{\partial}{\partial\rho}, \ \ j = 1, \ldots.$$

The operator equation (54) can be written in the form

$$[U, S_1] = b_1(\rho,\varphi)\frac{\partial}{\partial\rho} - \langle b_1(\rho,\varphi)\rangle\frac{\partial}{\partial\rho},$$

where $S_1 = \gamma_1(\rho,\varphi)\partial/\partial\rho$. Hence, only the variable ρ is transformed. Obviously,

$$\langle b_1(\rho,\varphi)\rangle = \frac{\rho}{2} - \frac{1}{8}\rho^3.$$

On calculations, the centralized system of the first approximation reduces to

$$\dot{\rho} = \frac{\varepsilon}{2}\left(1 - \frac{1}{4}\rho^2\right)\rho,$$

$$\dot{\varphi} = 1 - \frac{\varepsilon}{2}\left(\sin 2\varphi - \frac{\rho^2}{2}\sin 2\varphi + \frac{\rho^2}{4}\sin 4\varphi\right). \tag{53}$$

For finding $\gamma_1(\rho, \varphi)$, we have the differential equation

$$\frac{\partial \gamma_1}{\partial \varphi} = b_1(\rho, \varphi) - \langle b_1(\rho, \varphi) \rangle,$$

which is easy to solve:

$$\gamma_1(\rho, \varphi) = \frac{\rho}{4}\sin(2\varphi) + \frac{\rho^3}{32}\sin(4\varphi).$$

It is important that the analysis of the first equation in system (53) displays the existence of a stable limit cycle. One can also illustrate this fact graphically. The comparison of the solution of the initial perturbed system (27) and the centralized system of the first approximation (53) (previously reduced to the initial variables x_1 and x_2) is shown in Fig. 1.

The advantage of the partial group averaging lies in the fact that it enables us to obtain approximate equations with much less calculation efforts. Nevertheless, these equations help us to perform qualitative analysis of the initial nonlinear system.

4.2. *Group averaging for perturbed motion on* SO(2). In the case of commutative groups, one can transform all variables, i.e., the algorithm of asymptotic decomposition is applicable. Let us apply it to system (27) with averaging on the group $SO(2)$ defined as

$$\langle f(\rho, \varphi) \rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(\rho, \varphi)\, d\varphi.$$

We restrict ourselves to the first approximation and consider the operator equation

$$[U, S_1] = \widetilde{U} - \operatorname{pr}\widetilde{U}, \tag{54}$$

where $S_1 = \gamma_1(\rho, \varphi)\partial/\partial\rho + \gamma_2(\rho, \varphi)\partial/\partial\varphi$. Let us calculate the average values of the coefficients (28). According to the general theory,

$$\operatorname{pr}\widetilde{U} = \left(\frac{\rho}{2} - \frac{1}{8}\rho^3\right)\frac{\partial}{\partial\rho}.$$

Therefore, the centralized (averaged) system in the first approximation takes the form

$$\dot{\rho} = \varepsilon\left(\frac{\rho}{2} - \frac{1}{8}\rho^3\right), \qquad \dot{\varphi} = 1$$

The operator equation (54) is replaced by the system of differential equations

$$\frac{\partial \gamma_j}{\partial \varphi} = b_j(\rho, \varphi) + \langle b_j(\rho, \varphi) \rangle, \quad j = 1, 2.$$

Such systems are easily integrated in trigonometric functions.

5. Examples: Motion of a point on a sphere.

5.1. *Linear equations.* Consider the system of equations (42) of motion of a point on a sphere.

To show this, note that the system has two integrals

$$v_1(x) = x_1 + x_2 = c_1, \tag{55}$$

$$v_2(x) = x_1^2 + x_2^2 + x_3^2 = c_2. \tag{56}$$

Hence, the motion described by system (42) takes place in the circle which is the intersection of sphere (56) with radius $\rho = \sqrt{c_2}$ and plane (55); see Fig. 2a.

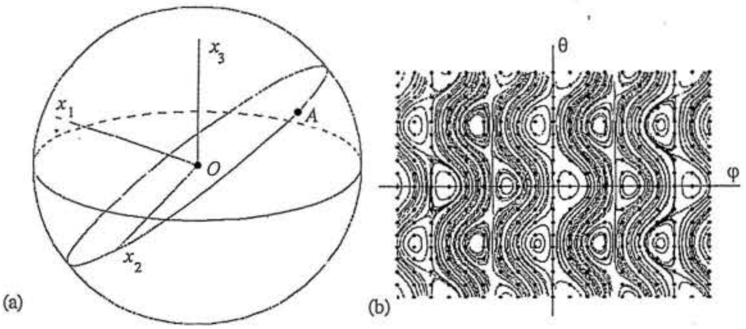

Fig. 2. (a) The solution of a linear system of equations of motion of a point on a sphere.
(b) The solution in the phase plane for angle spherical variables governing the motion of a point on a sphere.

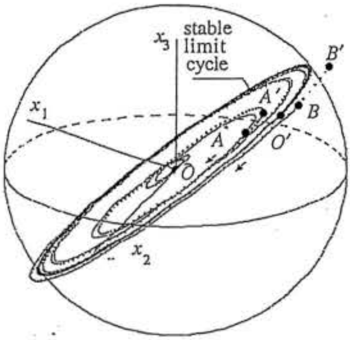

Fig. 3. The solution of nonlinear equations of motion of a point on a sphere: initial points of trajectories in the plane $x_1 + x_3 = 0$.

The motion on a sphere is quite complicated. By introducing spherical coordinates in system (42)

$$x_1 = \rho \sin\theta \cos\varphi, \qquad x_2 = \rho \sin\theta \sin\varphi, \qquad x_3 = \rho \cos\theta, \quad \rho = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

we can clarify this fact. System (42) takes the form

$$\begin{aligned} \dot\rho &= 0, \\ \dot\theta &= \sin\varphi, \\ \dot\varphi &= -1 + \operatorname{ctg}\theta \cos\varphi. \end{aligned} \tag{57}$$

The trajectories in the phase plane of the last two equations of system (57) are depicted in Fig. 2b.

The fact that the solution of system (42) is an element of $SO(3)$ (see Example 2 in Subsection 3.4) is important for what follows.

5.2. *Nonlinear equations.* Now suppose that system (42) is subjected to nonlinear perturbations:

$$\begin{aligned} \dot x_1 &= x_2 + \varepsilon \frac{x_1}{\rho} F(x), \\ \dot x_2 &= x_3 - x_1 + \varepsilon \frac{x_2}{\rho} F(x), \end{aligned} \tag{58}$$

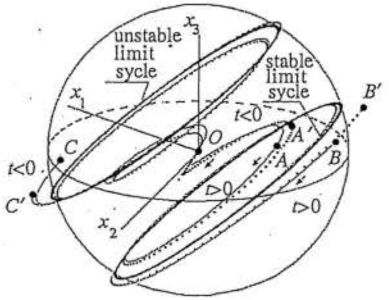

Fig. 4. The solution of nonlinear equations of motion of a point on a sphere: initial points of trajectories in the plane $x_1 + x_3 = \pm c$.

$$\dot{x}_3 = -x_2 + \varepsilon \frac{x_3}{\rho} F(x),$$

where

$$\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad F(x) = h^2 - \rho^2 + \frac{x_3 - x_1}{\rho} \frac{1}{\sqrt{|\rho|}},$$

and ε is a positive parameter.

The nonlinear system (58) has limit cycles. Any trajectories originating in the plane

$$x_1 + x_3 = 0$$

(points A, A', B, B', C, C' in Fig. 3) are winding on a limit cycle. Any trajectories originating in the plane

$$x_1 + x_3 = \pm c, \ \ c = \text{const}$$

are either winding on a limit cycle (points A, A', B, B' in Fig. 4), or unwinding from a limit cycle (points A, A', C, C' in Fig. 4).

5.3. *Group averaging for perturbed motion on* $SO(3)$. Let us introduce spherical coordinates in system (58)

$$x_1' = \rho' \sin\theta' \cos\varphi', \qquad x_2' = \rho' \cos\theta' \sin\varphi', \qquad x_3' = \rho' \cos\theta'.$$

System (58) takes the form

$$\begin{aligned} \dot{\rho}' &= \varepsilon f(\rho', \theta', \varphi'), \\ \dot{\theta}' &= \sin\varphi', \\ \dot{\varphi}' &= -1 + \operatorname{ctg}\theta' \cos\varphi', \end{aligned} \tag{59}$$

where

$$f(\rho', \theta', \varphi') = h^2 - \rho'^2 + (\sin\varphi' - \cos\varphi') \sin\theta' \frac{1}{\sqrt{|\rho'|}}.$$

For the operator associated with this system, we have

$$U_0' = U' + \varepsilon \tilde{U}',$$

where

$$U' = (-1 + \operatorname{ctg}\theta' \cos\varphi') \frac{\partial}{\partial\varphi'} + \sin\varphi' \frac{\partial}{\partial\theta'}$$

$$\tilde{U}' = f(\rho', \theta', \varphi') \frac{\partial}{\partial\rho'}.$$

Now let us apply the algorithm of asymptotic decomposition in the first approximation to system (59), using partial group averaging on $SO(3)$. Performing the transformations

$$\rho' = e^{\varepsilon \mathbf{S}} \rho, \qquad \theta' = \theta, \qquad \varphi' = \varphi,$$

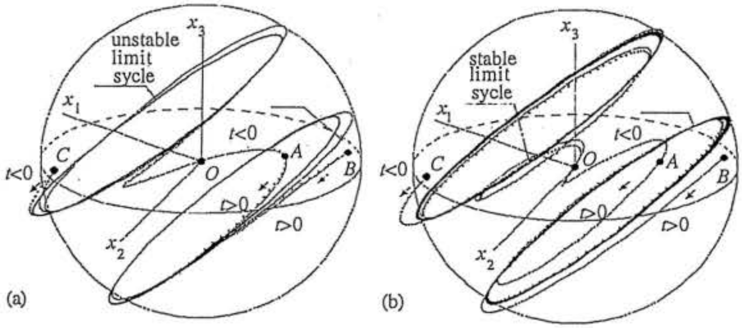

Fig. 5. The solution of nonlinear equations of motion of a point on a sphere (solid lines correspond to the exact solution; dotted lines correspond to the approximate solution (a centralized system of the first approximation)): $\varepsilon = 0.5$ (a), $\varepsilon = 0.1$ (b).

$$S = S_1 = \gamma_1(\rho, \theta, \varphi)\frac{\partial}{\partial \rho},$$

according to the general theory, we get

$$[U, S_1] = \hat{F}_1 - \mathrm{pr}\hat{F}_1,$$

where

$$\hat{F}_1 = f(\rho, \theta, \varphi)\frac{\partial}{\partial \rho}.$$

Recall that $\hat{F}_1$ is obtained from F_1 by omitting terms with all derivatives but $\partial/\partial\rho$.

It now reasonable to use the fact that the coefficients of $\hat{F}_1$ are functions on the group $SO(3)$. For expressions in Fourier series, we use basic spherical functions. After calculation, we get

$$f(\rho, \theta, \varphi) = f_0 + c_1 Y_1^1 + c_{-1}Y_1^{-1} + c_0 Y_1^0,$$

where

$$f_0 = h^2 - \rho^2,$$

$$Y_1^1 = -\frac{\sqrt{3}}{2\sqrt{2\pi}}e^{i\varphi}\sin\theta, \qquad Y_1^{-1} = \frac{\sqrt{3}}{2\sqrt{2\pi}}e^{-i\varphi}\sin\theta, \qquad Y_1^0 = \sqrt{2}\cos\theta.$$

$$c_1 = \frac{\sqrt{2\pi}}{\sqrt{3}}), \qquad c_2 = -\frac{\sqrt{2\pi}}{\sqrt{3}}, \qquad c_0 = \sqrt{2}\frac{\sqrt{2\pi}}{\sqrt{3}}.$$

The free term f_0 in the expressions for f in Fourier series in basic spherical functions is equal to the "average of this function on the group SO(3)." It is calculated by the formula

$$f_0 = \langle f(\rho, \theta, \varphi)\rangle =_{\mathrm{def}} \int_0^{2\pi}\int_0^{\pi} f(\rho, \theta, \varphi)\sin\theta d\theta d\varphi$$

According to the general theory, we have

$$\mathrm{pr}\hat{F}_1 = \mathrm{pr}\left(f(\rho, \theta, \varphi)\frac{\partial}{\partial \rho}\right) = \langle f(\rho, \theta, \varphi)\rangle\frac{\partial}{\partial \rho}.$$

As a result, we obtain the centralized system of the first approximation

$$\dot{\rho} = \varepsilon(h^2 - \rho^2),$$
$$\dot{\theta} = \sin\varphi, \tag{60}$$

$$\dot{\varphi} = -1 + \operatorname{ctg}\theta \cos\varphi.$$

For finding the coefficient $\gamma_1(\rho, \theta, \varphi)$ in S_1, we get the equation

$$U\gamma_1 = c_1 Y_1^1 + c_{-1} Y_1^{-1} + c_0 Y_1^0,$$

which can easily be solved:

$$\gamma_1(\rho, \theta, \varphi) = \frac{\sqrt{2\pi}}{2\sqrt{3}} \sin\varphi \sin\theta \frac{1}{\sqrt{|\rho|}}.$$

The first equation in system (60) displays the existence of two limit cycles $\rho = \pm h$. The trajectories of exact (58) and approximate (60) (reduced to the initial variables) systems are shown in Fig. 5 for different values of ε.

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