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TO THE PROBLEM ON PERIODIC SOLUTIONS OF ONE CLASS OF SYSTEMS OF DIFFERENCE EQUATIONS

ДО ПИТАННЯ ПРО ПЕРІОДИЧНІ РОЗВ'ЯЗКИ ОДНОГО КЛАСУ СИСТЕМ РІЗНИЦЕВИХ РІВНЯНЬ

The scheme of the Samoilenko numerical-analytic method for finding periodic solutions in the form of a uniformily convergent sequence of periodic functions is applied to one class of difference equations.

Схема чисельно-аналітичного методу А. М. Самойленка знаходження періодичних розв'язків у вигляді рівномірно збіжної послідовності періодичних функцій застосована до одного класу різницевих рівнянь.

In [1], the scheme of the Samoilenko numerical method for finding periodic solutions was applied to the following system difference equations with $\lambda > 0$:

$$x_n = \lambda x_n + f(x_n, y_n),$$
$$y_{n+1} = y_n + g_n(x_n, y_n). \tag{1}$$

In this paper, we consider similar questions concerning the system

$$\Delta x_n = x_{n+1} - x_n = \Delta x_n + f_n(x_n, y_n),$$
$$\Delta y_n = y_{n+1} - y_n = g_n(x_n, y_n) \tag{2}$$

for all real λ, but $\lambda \neq -2, -1, 0$.

Note that the case $\lambda = 0$ was considered in [2].

We consider system (2) in the domain $x \in [\alpha, \beta]$, $x \in \Re^1$, $n \in Z$, where $f_n(x, y)$ and $g_n(x, y)$ are scalar numeric sequences periodic in n with period p and satisfying the following conditions:

$$|f_n(x, y)| \leq M, \qquad |g_n(x, y)| \leq M,$$
$$|f_n(x', y') - f_n(x'', y'')| \leq K_1 |x' - x''| + K_2 |y' - y''|, \tag{3}$$
$$|g_n(x', y') - g_n(x'', y'')| \leq K_1 |x' - x''| + K_2 |y' - y''|$$

for $x, x', x'' \in [\alpha, \beta]$, $y, y', y'' \in \Re^1$, $n \in Z$.

By

$$\overline{g}_n = \frac{1}{p} \sum_{n=0}^{p-1} g_n, \qquad \hat{f}_n = \frac{\lambda}{(\lambda+1)^p - 1} \sum_{i=0}^{p-1} (\lambda+1)^{p-i-1} f_i, \tag{4}$$

we denote the mean values in n calculated over the period.

We search periodic solutions of system (2).

Consider the sequence of functions

$$x_n^{m+1}(x_0, y_0) = x_0 +$$
$$+ \sum_{i=0}^{n-1} (\lambda+1)^{n-i} \left[f_i\big(x_i^m(x_0, y_0), y_i^m(x_0, y_0)\big) - \widehat{f_i\big(x_i^m(x_0, y_0), y_i^m(x_0, y_0)\big)} \right],$$
$$y_n^{m+1}(x_0, y_0) = y_0 + \tag{5}$$
$$+ \sum_{i=0}^{n-1} \left[g_i\big(x_i^m(x_0, y_0), y_i^m(x_0, y_0)\big) - \overline{g_i\big(x_i^m(x_0, y_0), y_i^m(x_0, y_0)\big)} \right].$$

Each function in (5) is p-periodic in n and if we assume that sequences (5) uniformly converge to functions $x_n^\infty(x_0, y_0)$ and $y_n^\infty(x_0, y_0)$, then it is easy to see that the limit functions will be periodic solutions of system (2) which pass, for $n = 0$, through the point (x_0, y_0) if (x_0, y_0) are solutions of the system

$$\overline{g_n\big(x_n^\infty(x_0, y_0), y_n^\infty(x_0, y_0)\big)} = 0, \tag{6}$$
$$f_n\big(\widehat{x_n^\infty(x_0, y_0), y_n^\infty(x_0, y_0)}\big) + \lambda x_0 = 0.$$

So the problem of the existence and determination of p-periodic solutions of system (2) is reduced to finding conditions that imply the uniform convergence of sequences (5) and make system (6) solvable.

As already mentioned, the functions $x_n^\infty(x_0, y_0)$ and $y_n^\infty(x_0, y_0)$ are p-periodic in n.

For $n \in [0, p-1]$, we have

$$\left| x_n^1(x_0, y_0) - x_0 \right| =$$
$$= \left| \sum_{i=0}^{n-1} (\lambda+1)^{n-i} \left[f_i(x_0, y_0) - \frac{\lambda}{(\lambda+1)^p - 1} \sum_{i=0}^{p-1} (\lambda+1)^{p-i-1} f_i(x_0, y_0) \right] \right| =$$
$$= \left| \sum_{i=0}^{n-1} (\lambda+1)^{n-i} f_i(x_0, y_0) - \right.$$
$$\left. - \sum_{i=0}^{n-1} (\lambda+1)^{n-i} \frac{\lambda}{(\lambda+1)\big(1-(\lambda-1)^p\big)} \sum_{i=0}^{n-1} (\lambda+1)^{p-i} f_i(x_0, y_0) \right| =$$
$$= \left| \sum_{i=0}^{n-1} (\lambda+1)^{n-i} f_i(x_0, y_0) + \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1} \sum_{i=0}^{p-1} (\lambda+1)^{p-i} f_i(x_0, y_0) \right| \le$$
$$\le \left| \sum_{i=0}^{n-1} (\lambda+1)^{n-i} f_i(x_0, y_0) - \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1} \sum_{i=0}^{n-1} (\lambda+1)^{p-i} f_i(x_0, y_0) \right| +$$
$$+ \left| \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1} \sum_{i=n}^{p-1} (\lambda+1)^{p-i} f_i(x_0, y_0) \right| \le \left| (\lambda+1)^n \sum_{i=0}^{n-1} (\lambda+1)^{-i} f_i(x_0, y_0) - \right.$$
$$\left. - \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1} (\lambda+1)^p \sum_{i=0}^{n-1} (\lambda+1)^{-i} f_i(x_0, y_0) \right| +$$

$$+ \left| \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1} \sum_{i=n}^{p-1} (\lambda+1)^{p-i} f_i(x_0, y_0) \right| \le$$

$$\le \left| \left[(1+\lambda)^n - \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1} (\lambda+1)^p \right] \sum_{i=0}^{n-1} (\lambda+1)^{-i} f_i(x_0, y_0) \right| +$$

$$+ \left| \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1} (\lambda+1)^p \sum_{i=n}^{p-1} (\lambda+1)^{-i} f_i(x_0, y_0) \right| \le$$

$$\le \left| \left[(1+\lambda)^n - \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1} (\lambda+1)^p \right] \sum_{i=0}^{n-1} (\lambda+1)^{-i} M \right| +$$

$$+ \left| \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1} \right| \sum_{i=n}^{n-1} (\lambda+1)^{p-i} M \le$$

$$\le M \frac{\left|(1+\lambda)^n - (\lambda+1)^p\right| \left(|\lambda+1|^{-n} - 1 \right) + \left|1 - (1+\lambda)^n\right| \left(1 - |\lambda+1|^{p-n} \right)}{\left|1 - (1+\lambda)^p\right| \left(|\lambda+1|^{-1} - 1 \right)} =$$

$$= M\alpha_n(\lambda), \tag{7}$$

where

$$\alpha_n(\lambda) = \frac{\left|(1+\lambda)^n - (\lambda+1)^p\right| \left(|\lambda+1|^{-n} - 1 \right) + \left|1 - (1+\lambda)^n\right| \left(1 - |\lambda+1|^{p-n} \right)}{\left|1 - (1+\lambda)^p\right| \left(|\lambda+1|^{-1} - 1 \right)}. \tag{8}$$

In the same way, we can get, for $n \in [0, p-1]$, the following estimate

$$\left| y_n^1(x_0, y_0) - y_0 \right| \le M\alpha_0(\lambda),$$

where $\alpha_0(\lambda) = \lim\limits_{\lambda \to 0} \alpha_n(\lambda)$.

Since the collection of numbers (8) is finite we choose among them the maximal number and denote it by d_λ.

Since the functions in sequences (5) are p-periodic, we can use induction to show that, for all $m = 0, 1, 2, \ldots$, all $n \in Z$, and $x_0 \in [\alpha + d_x M, \beta + d_x M]$, the functions $x_n^m(x_0, y_0)$ belong to $[\alpha, \beta]$.

Let us denote

$$L_\lambda(f_n) = \left| (\lambda+1)^n - \frac{1-(\lambda+1)^n}{1-(\lambda+1)^p} (\lambda+1)^p \right| \sum_{i=0}^{n-1} |\lambda+1|^{-i} f_i +$$

$$+ \left| \frac{1-(\lambda+1)^n}{1-(\lambda+1)^p} \right| \sum_{i=n}^{p-1} |\lambda+1|^{p-i} f_i. \tag{9}$$

By direct verification, we can show that $L_\lambda(1) = \alpha_n(\lambda)$.

A straightfor ward calculation shows that

$$L_\lambda(\alpha_n(\lambda)) \le \alpha_n(\lambda) \left[\frac{\alpha_n(\lambda)}{3} + r_n \right], \tag{10}$$

where

$$\max_{0 \le n \le p-1} r_n \le \frac{1}{2} \alpha_n(\lambda), \tag{11}$$

$$\alpha_n(0) \le \frac{p}{4} \left| |\lambda+1| - 1 \right| \left| \frac{|\lambda+1|^{p/2} + 1}{|\lambda+1|^{p/2} - 1} \right| \alpha_n(\lambda), \tag{12}$$

$$\alpha_n(\lambda) \le \alpha_n(0), \quad n \in [0, p-1]. \tag{13}$$

Consequently,

$$L_\lambda(\alpha_n(\lambda)) \le \alpha_n(\lambda)\Big[\frac{\alpha_n(\lambda)}{3} + \frac{1}{2}\hat{\alpha}_n(\lambda)\Big], \tag{14}$$

$$L_0(\alpha_n(0)) \le \alpha_n(0)\Big[\frac{1}{3}\alpha_n(0) + \frac{1}{2}\overline{\alpha_n(0)}\Big], \tag{15}$$

$$L_0(\alpha_n(\lambda)) \le L_0(\alpha_n(0)) \le \alpha_n(0)\Big[\frac{1}{3}\alpha_n(0) + \frac{1}{2}\overline{\alpha_n(0)}\Big], \tag{16}$$

$$L_\lambda(\alpha_n(0)) \le \frac{p}{4}||\lambda+1|-1|\frac{|\lambda+1|^{p/2}+1}{\big||\lambda+1|^{p/2}-1\big|}L_\lambda(\alpha_n(\lambda)). \tag{17}$$

Let us estimate the difference

$$\big|x_n^2(x_0, y_0) - x_n^1(x_0, y_0)\big| \le$$

$$\le \left|\Big[(1+\lambda)^n - \frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1}(\lambda+1)^p\Big]\sum_{i=0}^{n-1}(\lambda+1)^{-i}\big[K_1\big|x_n^1 - x_0\big| + K_2\big|y_n^1 - y_0\big|\big]\right| +$$

$$+ \left|\frac{(\lambda+1)^n - 1}{(\lambda+1)^p - 1}\right|\sum_{i=0}^{n-1}(\lambda+1)^{p-i}\big[K_1\big|x_n^1 - x_0\big| + K_2\big|y_n^1 - y_0\big|\big] \le$$

$$\le [K_1 L_\lambda(\alpha_n(\lambda)) + K_2 L_\lambda(\alpha_n(0))]M. \tag{18}$$

In the same way, we can get the estimate

$$\big|y_n^2(x_0, y_0) - y_n^1(x_0, y_0)\big| \le [K_1 L_0(\alpha_n(\lambda)) + K_2 L_0(\alpha_n(0))]M. \tag{19}$$

In view of (14) and (17), it follows from inequalities (18) and (19) that

$$\begin{gathered}
\big|x_n^2(x_0, y_0) - x_n^1(x_0, y_0)\big| \le \Big[K_1\alpha_n(\lambda)\Big[\frac{1}{3}\alpha_n(\lambda) + \frac{1}{2}\hat{\alpha}_n(\lambda)\Big]\Big] + \\
+ K_2\frac{p}{4}||\lambda+1|-1|\frac{|\lambda+1|^{p/2}+1}{\big||\lambda+1|^{p/2}-1\big|}\alpha_n(\lambda)\Big[\frac{1}{3}\alpha_n(\lambda) + \frac{1}{2}\hat{\alpha}_n(\lambda)\Big] = \\
= \Big[K_1 + K_2\frac{p}{4}||\lambda+1|-1|\frac{|\lambda+1|^{p/2}+1}{\big||\lambda+1|^{p/2}-1\big|}\Big]\alpha_n(\lambda)N, \\
\big|y_n^2(x_0, y_0) - y_n^1(x_0, y_0)\big| \le \Big[K_1\alpha_n(0)\Big[\frac{1}{3}\alpha_n(0) + \frac{1}{2}\overline{\alpha_n(0)}\Big]\Big] + \\
+ \Big[K_2\alpha_n(0)\Big[\frac{1}{3}\alpha_n(0) + \frac{1}{2}\overline{\alpha_n(0)}\Big]\Big] \le (K_1 + K_2)\alpha_n(0)\frac{p}{3},
\end{gathered} \tag{20}$$

where

$$N = \max_{n\in[0,p-1]}\Big[\frac{1}{3}\alpha_n(\lambda) + \frac{1}{2}\hat{\alpha}_n(\lambda)\Big].$$

The induction implies that

$$\big|x_n^{m+1}(x_0, y_0) - x_n^m(x_0, y_0)\big| \le q_1^m M\alpha_n(\lambda) \le q_1^m M d_\lambda,$$

$$\big|y_n^{m+1}(x_0, y_0) - y_n^m(x_0, y_0)\big| \le q_2^m M\alpha_n(0) \le q_2^m M\frac{p}{2}, \quad m = 0, 1, 2, \ldots, \tag{21}$$

where

$$q_1 = \left[K_1 + K_2 \frac{p}{4} \big| |\lambda+1| - 1 \big| \frac{|\lambda+1|^{p/2}+1}{\big||\lambda+1|^{p/2}-1\big|} \right] N, \qquad q_2 = [K_1 + K_2] \frac{p}{3}.$$

Thus, for sequences (5) to be convergent, it is sufficient that the following inequalities hold:

$$q_1 = q_1(\lambda) < 1, \qquad q_2 < 1. \tag{22}$$

It follows from (21) that the limit functions $x_n^{(\infty)}(x_0, y_0)$, $y_n^{(\infty)}(x_0, y_0)$ satisfy the inequalities

$$\begin{aligned} \left| x_n^{(\infty)}(x_0, y_0) - x_n^m(x_0, y_0) \right| &\le q_1^m \left(1 - q_1^m\right)^{-1} M d_\lambda, \\ \left| y_n^{\infty}(x_0, y_0) - y_n^m(x_0, y_0) \right| &\le q_2^m \left(1 - q_2^m\right)^{-1} M \frac{p}{2}. \end{aligned} \tag{23}$$

Consequently, the problem of finding a periodic solution of system (2) is reduced to the calculation of function (5) if it is known that such a solution exists and if we know the point (x_0, y_0) through which it passes for $n = 0$. But, as has been noted, for limit functions of sequences (5) to be solutions of system (2), it is necessary that equations (6) be solvable with respect to x_0, y_0.

A solution of system (6) is the point through which a p-periodic solution of system (2) passes for $n = 0$. Then the number of periodic solutions of system (2) is determined by the number of solutions of system (6).

Let us denote the left-hand sides of equations (6) by $\Delta^x(x_0, y_0)$ and $\Delta^y(x_0, y_0)$, respectively, and let $\Delta_m^x(x_0, y_0)$ and $\Delta_m^y(x_0, y_0)$ denote the expressions

$$\begin{aligned} \Delta_m^x(x_0, y_0) &= \widehat{f_n\left(x_n^{(m)}(x_0, y_0), y_n^{(m)}(x_0, y_0)\right)} + \lambda x_0, \\ \Delta_m^y(x_0, y_0) &= \overline{g_n\left(x_n^{(m)}(x_0, y_0), y_n^{(m)}(x_0, y_0)\right)}. \end{aligned}$$

Rewrite system (6) in the form

$$\Delta^x(x_0, y_0) = 0, \qquad \Delta^y(x_0, y_0) = 0. \tag{24}$$

Consider the equations

$$\Delta_m^x(x_0, y_0) = 0, \qquad \Delta^y(x_0, y_0) = 0. \tag{25}$$

It is not possible, generally speaking, to solve system (24) because it is not always possible to find the limit functions $x_n^\infty(x_0, y_0)$ and $y_n^\infty(x_0, y_0)$. But it can be shown that the functions Δ_m^x, Δ_m^y, Δ^x, Δ^y are continuous in (x_0, y_0), $\alpha + d < x_0 < B - \alpha$, $-\infty < y_0 < \infty$, and, by using relations (23), one can obtain the estimates

$$\begin{aligned} \left| \Delta^x - \Delta_m^x \right| &\le \left[K_1 q_1^m (1 - q_1)^{-1} d_\lambda + K_2 q_2^m (1 - q_2)^{-1} d_\lambda \right] M, \\ \left| \Delta^y - \Delta_m^y \right| &\le \left[K_1 q_1^m (1 - q_1)^{-1} d_\lambda + K_2 q_2^m (1 - q_2)^{-1} d_\lambda \right] M, \end{aligned} \tag{26}$$

which imply that $\Delta_m^x \to \Delta^x$, $\Delta_m^y \to \Delta^y$ as $m \to \infty$.

Here, we encounter the following problem: Prove that system (24) has solutions if system (25) has a solution for some m. This problem is solved by the following theorem:

Theorem 1. *Let system* (2) *be such that*

(i) *inequalities* (3) *and* (22) *hold and the interval* $[\alpha, \beta]$ *is such that* $(\beta - \alpha)/2 > d_\lambda$,

(ii) *for a certain integer* m, *system* (25) *has an isolated solution* (x^0, y^0),

(iii) *at the singular point* (x^0, y^0), *the index of equations* (25) *is different from zero,*

(iv) *there exists a closed convex region* D_0 *belonging to the domain* $D_0 = \{(x, y),\ \alpha + d_{\lambda+1} < x < B - d,\ -\infty < y < \infty\}$ *and having the point* (x^0, y^0) *as a unique solution of system* (24) *such that the inequalities*

$$\inf_{x_0, y_0 \in \Gamma_{D_0}} \left|\Delta_m^x(x_0, y_0)\right| \geq \left[K_1 q_1^m (1-q_1)^{-1} d_\lambda + K_2 q_2^m (1-q_2)^{-1} d_\lambda\right] M,$$

$$\inf_{x_0, y_0 \in \Gamma_{D_0}} \left|\Delta_m^y(x_0, y_0)\right| \geq \left[K_1 q_1^m (1-q_1)^{-1} d_\lambda + K_2 q_2^m (1-q_2)^{-1} d_\lambda\right] M \tag{27}$$

hold on its boundary Γ_{D_0}.

Then system (2) *has a* p*-periodic solution* $x = x_n$, $x = y_n$ *for which* $(x(0), y(0)) \in D_0$.

This solution is the limit of the uniformly convergent sequence (5). An estimate for the difference between an exact solution and its m-th approximation is given by inequalities (23).

The proof of Theorem 1 is based on estimates (26) and can be carried out similarly to [3].

It is not always easy to check conditions (iv) in Theorem 1 because this requires a suitable choice of the domain D_0. However, for many systems, this condition is satisfied for a more or less arbitrary domain.

This is the case, for example, for of the form

$$\Delta x_n = \lambda_x + \varepsilon f_n(x, y),$$
$$\Delta y_n = \varepsilon g_n(x, y). \tag{28}$$

Here, ε is a small parameter.

For such systems, it is possible to find ε_0 such that, for all $0 < \varepsilon < \varepsilon_0$, the conditions of Theorem 1 are satisfied and inequalities (27) with $m = 0$ hold for a small disk centered at the point the coordinates of which are solutions of system (25). Taking this into account, we get the following theorem for systems of the form (28):

Theorem 2. *Let the functions* $f_n(x, y)$, $g_n(x, y)$ *satisfy inequalities* (3). *Then there exists* $\varepsilon_0 > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$, *system* (28) *has a periodic solution whenever the averaged system*

$$\Delta x_n = \lambda x_n + \widehat{\varepsilon f_n(x, y)}, \qquad \Delta y_n = \overline{\varepsilon g_n(x, y)} \tag{29}$$

has an isolated singular point (x^0, y^0),

$$\lambda x^0 + \widehat{\varepsilon f_n(x^0, y^0)} = 0, \qquad \overline{\varepsilon g_n(x^0, y^0)} = 0,$$

and the index is different from zero.

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