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INFINITE SYSTEMS OF STOCHASTIC DIFFERENTIAL EQUATIONS AND SOME LATTICE MODELS ON COMPACT RIEMANNIAN MANIFOLDS

НЕСКІНЧЕННІ СИСТЕМИ СТОХАСТИЧНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ТА ДЕЯКІ ГРАТЧАСТІ МОДЕЛІ НА КОМПАКТНИХ РІМАНОВИХ МНОГОВИДАХ

Stochastic dynamics associated with Gibbs measures on an infinite product of compact Riemannian manifolds is constructed. The probabilistic representations for the corresponding Feller semigroups are obtained. The uniqueness of the dynamics is proved.

Побудовано стохастичну динаміку, асоційовану з гібсівськими мірами на нескінченних добутках компактних ріманових многовидів. Одержано ймовірнісні зображення феллерівських півгруп. Доведено єдиність динаміки.

1. Introduction. Constructions of the stochastic dynamics associated with Gibbs measures are connected with so-called stochastic quantization methods. In the case of a linear single spin space, such constructions can be covered via the general theory of stochastic differential equations (SDE) on infinite dimensional linear spaces. This case has been actively studied, see, e.g., [1] and the review given in [2].

The case of a compact Riemannian manifold as a single space has received a great interest in recent years. The construction of Feller semigroups is given in [3, 4]. L_2-stochastic dynamics has been considered in [5]. These works contain also an overlook of previous results. For an alternative approach, see also [6, 7]. Most results in these papers are devoted to interactions of a finite range.

In the paper [8], the construction of Glauber dynamics is given for some lattice models on compact Lie groups and their homogeneous spaces equipped with invariant Riemannian structure, including the case of infinite range of interaction. In the present work, we extend the approach of [8] to the case of a general compact Riemannian manifold M. In order to be able to investigate an infinite system of SDE on M, we use an embedding of M into a Euclidean space. This gives us a possibility to apply the general theory of SDE in Hilbert spaces.

In the first section, we study a stochastic differential equation on M in terms of the embedding of M into a Euclidean space. This section can be considered as an adaptation of well-known results (see, e.g., [9]) to our framework. In the next section, we investigate a system of SDE on $M^{\mathbf{Z}^d}$, where $\mathbf{Z}^d$ is the d-dimensional integer lattice. In particular, we construct solutions to these SDE and study their dependence on initial data. As a result, we obtain probabilistic representations for corresponding Feller semigroups. In the third section, we apply these results to lattice models associated with Gibbs measures on $M^{\mathbf{Z}^d}$. In the last section, we prove the uniqueness of the corresponding stochastic dynamics.

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2. Stochastic differential equations on compact manifolds via embedding into Euclidean spaces. Let M be a compact complete connected N-dimensional manifold and let TM be its tangent bundle. We will consider the following stochastic differential equation (SDE) of the Stratonovich type on M:

$$d\xi(t) = a(\xi(t))\,dt + B(\xi(t)) \cdot dw(t), \tag{1}$$

where a is a C^1-vector field on M, B is a C^2-mapping $M \times \mathbf{R}^N \to TM$ such that $B(x) \in \mathcal{L}(\mathbf{R}^N, T_x M)$ for any $x \in M$, and w is a Wiener process in $\mathbf{R}^N$.

It is well known that the Cauchy problem for this equation has a unique solution ξ_x for any initial value $x \in M$ (see, e.g., [9, 10]). The aim of this section is to show, following [9], how this result can be obtained by the embedding technique.

Let $\varphi: M \to \mathbf{R}^n$ be a smooth embedding of M into a Euclidean space $\mathbf{R}^n$. It is well known that such embedding exists if $n > 2N+1$. We will identify M with its image $\varphi(M) \subset \mathbf{R}^n$. Then the tangent bundle TM is a submanifold of $\mathbf{R}^n \times \mathbf{R}^n$. Let us define the normal bundle νM with the fibers $\nu_x M$ being the orthogonal complements to the corresponding fibers $T_x M$ in $\mathbf{R}^n$.

Lemma 1 [9]. *There exists $r > 0$ and a neighborhood $U_r \subset \nu M$ of the zero section $(M, 0)$ of νM, $U_r = \{(x, v): |v| < r\}$, which is diffeomorphic to a neighborhood $N_r = \bigcup_{x \in M} \{y \in \mathbf{R}^N: |y - x| < r\}$ of M in $\mathbf{R}^n$.*

N_r will be called the tubular neighborhood of M with radius r.

Let us choose some positive $r_1 < r$ and a smooth function $F: \mathbf{R}^n \to \mathbf{R}^1$ with support in N_r that is equal to 1 on N_{r_1}. Having such a function, for any mapping Φ of M into some linear space P, we define a mapping $\tilde{\Phi}: \mathbf{R}^n \to P$ as follows:

$$\tilde{\Phi}(y) = 0, \quad y \notin N_r, \tag{2}$$

and

$$\tilde{\Phi}(y) = \Phi(x_y) F(y), \quad y \in N_r, \tag{3}$$

where (x_y, v_y) is the image of y in U_r.

Let us consider the SDE

$$d\xi(t) = \tilde{A}(\xi(t))\,dt + \tilde{B}(\xi(t))\, c\, dw(t) \tag{4}$$

in $\mathbf{R}^n$, or in the Itô form,

$$d\xi(t) = \left[\tilde{A}(\xi(t)) + \frac{1}{2}\operatorname{tr} \tilde{B}'(\xi(t))\tilde{B}(\xi(t))\right] dt + \tilde{B}(\xi(t))\,dw(t). \tag{5}$$

This equation obviously has Lipschitz coefficients with compact support and, therefore, it has a unique solution ξ_y for any initial data $y \in \mathbf{R}^n$. The process ξ generates a Markov semigroup T_t, $t \geq 0$, in the space $C_b(\mathbf{R}^n)$ by the formula

$$T_t u(y) = E(u(\xi_y(t))). \tag{6}$$

The generator H of this semigroup is given on the space $C_b^2(\mathbf{R}^n)$ by the expression

$$Hu(y) = -\frac{1}{2}\operatorname{tr} D^2 u(y) + (\tilde{A}(y), \nabla u(y))_{\mathbf{R}^n}, \tag{7}$$

where $Du(y) = \tilde{B}^*(y) \nabla u(y)$ (∇ means the gradient) and $(\cdot,\cdot)_{\mathbf{R}^n}$ is the Euclidean scalar product in $\mathbf{R}^n$.

Theorem 1 [9]. *For any $x \in M$, the process ξ_x does not leave M a. s. and gives the unique solution to the corresponding Cauchy problem for equation* (1).

Proof. Note, first, that any solution of (1) also solves (4) and, conversely, any solution of (4) that does not leave M solves (1). The uniqueness follows from the uniqueness for (4). Therefore, it suffices to prove the first part of the theorem. It is easy to see that it follows from the Itô formula applied to the function $\mathbf{D}(y)=|v_y|^2$ on N_r,

$$\mathcal{D}(\xi_y(t)) = \mathcal{D}(y) + \int_0^t H\mathcal{D}(\xi(\tau))\,d\tau + \int_0^t \tilde{B}(\xi(t))^*\,\nabla\mathcal{D}(\xi(\tau))\,dw(\tau) \qquad (8)$$

and $\mathcal{D}(\xi_y(t))=0$ because the vector fields $\tilde{A}(y)$ and $\tilde{B}(y)X$, $X\in\mathbf{R}^N$, are tangent to the level surface $\{z\in\mathbf{R}^n\colon \mathcal{D}(z)=\mathcal{D}(y)\}$ of the function $\mathcal{D}$.

Corollary. *The formula* $T_t u(x)=E(u(\xi_x(t)))$ *defines a semigroup* T_t, $t\geq 0$, *in the space* $C(M)$. *It is the unique Markov semigroup with the generator coinciding with the restriction of the operator* H *to* $C^2(M)$ *(see, e.g.,* [9, 10]).

Let as assume from now that M is equipped with a Riemannian structure given by the operator field $G(x)\colon T_xM\to T_x^*M$. We denote the corresponding scalar product in T_xM by $(\cdot,\cdot)_G$, $(X,Y)_G(x)=\langle G(x)X,Y\rangle$. Let Δ_G, d_G, and ∂_G be the corresponding Laplace–Beltrami operator, gradient ($d_Gf(x)=G^{-1}(x)f'(x)$), and the covariant derivative, respectively. We will omit the lower index G if possible.

Let us suppose that the diffusion operator B satisfies the equality

$$G^{-1}(x) = B(x)B^*(x). \qquad (9)$$

Then the generator H of the corresponding process ξ_x has the form

$$Hf(x) = \Delta f(x) + (a(x)+b(x), df(x)), \qquad (10)$$

where the vector field b is defined by the expression [9]

$$b(x) = -\frac{1}{2}\operatorname{tr}(B'(x)B(x)). \qquad (11)$$

We understand the expression $B'(x)B(x)$ as a linear operator $A\colon \mathbf{R}^N\to\mathbf{R}^N\otimes T_xM$ defined by $(Ah_1,h_2)=(B(x)h_2)'B(x)h_1$.

Note that, for each metric G, the diffusion operator B satisfying (9) exists for some N.

Let us consider a probability measure μ on M of the following form:

$$d\mu(x) = \frac{1}{Z}e^{E(x)}\,dx, \qquad (12)$$

where dx is the Riemannian volume measure on M, E is a twice differentiable real function on M, and Z is a normalization constant. For this measure, the following formula of integration by parts is true:

$$\int\langle u'(x),X(x)\rangle\,d\mu(x) = -\int[(\Lambda(x),X(x))_G + \operatorname{div}_G X(x)]\,u(x)\,d\mu(x), \qquad (13)$$

where

$$\Lambda(x) = d_GE(x)\in T_xM$$

is the vector logarithmic derivative of μ and $\operatorname{div}_G X(x)=\operatorname{tr}\partial_G X(x)$.

We consider a pre-Dirichlet form $\mathcal{E}$ associated with the measure μ:

$$\mathcal{E}(u,v) = \frac{1}{2}\int(du(x),dv(x))_G\,d\mu(x), \qquad (14)$$

where $u,v\in C^2(M)$. It follows from the formula of integration by parts (13) that

$$\mathcal{E}(u, v) = \int H_\mu u(x) v(x)\, d\mu(x), \tag{15}$$

where

$$H_\mu u = -\frac{1}{2}\Delta u - \frac{1}{2}(\Lambda, du)_G. \tag{16}$$

Let B satisfy (9) and let $a = \Lambda - b$. Then the operator H associated with the process ξ_x coincides with H_μ.

***Remark* 1.** By virtue of the smoothness of the embedding of M into $\mathbf{R}^n$ and the compactness of M, the following estimate holds for any $C^1(\mathbf{R}^n)$:

$$\left\| \frac{\partial}{\partial y} f(x_y) \right\|_{\mathbf{R}^n} \le c_1 \| df(x_y) \|_{T_{x_y}M} \le c_2 \left\| \frac{\partial}{\partial y} f(x_y) \right\|_{\mathbf{R}^n}, \tag{17}$$

where the constants c_1 and c_2 do not depend on f and $y \in N_r$. In particular, for $x \in M$,

$$\| \nabla f(x) \|_{\mathbf{R}^n} \le c_1 \| df(x) \|_{T_x M} \le c_2 \| \nabla f(x) \|_{\mathbf{R}^n}. \tag{18}$$

3. Systems of stochastic differential equations on *M*. The aim of this section is to extend the approach discussed above to the case of an infinite system of SDE of a special type.

Let us consider the integer lattice $\mathbf{Z}^d$, $d \ge 1$, and define the space $M^{\mathbf{Z}^d}$, which is an infinite product of manifolds M:

$$M^{\mathbf{Z}^d} = \times_k M_k, \quad M_k = M, \quad k \in \mathbf{Z}^d. \tag{19}$$

We also introduce the space $M^A = \times_{k \in A} M_k$ for any $A \subset \mathbf{Z}^d$. Let us define the space $(\mathbf{R}^n)^{\mathbf{Z}^d}$ similarly to $M^{\mathbf{Z}^d}$.

The elements of these spaces will be denoted by $x = (x_k)$, $y = (y_k)$, etc., where x_k, y_k, etc. are their k-components, $k \in \mathbf{Z}^d$. We will write Δ_k, d_k, etc. for operators Δ, d, etc. acting in the corresponding spaces with index k. For example,

$$d_k f(x) = G^{-1}(x_k) \frac{\partial}{\partial x_k} f(x), \quad (X(x), Y(x))_k = \langle G(x_k) X_k(x_k), Y(x) \rangle.$$

Let $\mathcal{A}$ be the family of all finite subsets of $\mathbf{Z}^d$. For any $k \in \mathbf{Z}^d$, we will consider a family $\mathcal{V}_k = (V_{A,k})_{A \in \mathcal{A}}$ of C^1-mappings $V_{A,k}: M^A \to TM$ such that $V_{A,k}(x) \in$ $\in T_{x_k} M_k$.

We will assume the following:

$$\sup_{k \in \mathbf{Z}^d} \sup_{x \in M^{\mathbf{Z}^d}} \sum_{A \in \mathcal{A}} \| V_{A,k}(x) \|_{T_{x_k} M_k} |A| < \infty, \tag{20}$$

where $|A|$ is the number of elements of A and $\| V_{A,k}(x) \|_{T_{x_k} M_k}$ is the norm in $T_{x_k} M_k$ associated with our fixed Riemannian structure,

$$\sup_{k \in \mathbf{Z}^d} \sum_{j \in \mathbf{Z}^d} \sup_{x \in M^{\mathbf{Z}^d}} \sum_{A \in \mathcal{A}} \| d_j V_{A,k}(x) \|_{T_{x_j} M \otimes T_{x_k} M} < \infty, \tag{21}$$

where $\| d_j V_{A,k}(x) \|_{T_{x_j} M \otimes T_{x_k} M}$ is the corresponding norm in the space $T_{x_j} M \otimes T_{x_k} M$.

We will consider now a system of SDE of the following form:

$$d\xi_k(t) = a_k(\xi(t))\,dt + B_k(\xi_k(t))\cdot dw_k(t), \quad \xi_k(0) = x_k \in M, \quad k \in \mathbf{Z}^d, \tag{22}$$

where

$$a_k(x) = \sum_{A\in\mathcal{A}} V_{A,k}(x), \tag{23}$$

the mapping $B_k\colon M\times\mathbf{R}^N \to TM$ is C^2 uniformly in k in the sense that

$$\sup_{x\in M,\, k\in\mathbf{Z}^d} \Big(\| B_k(x)\|_{\mathcal{L}(\mathbf{R}^N,\, T_xM)} + \| B_k'(x)\|_{\mathcal{L}(\mathbf{R}^N,\, \mathcal{L}(\mathbf{R}^N,\, T_xM))} + {}$$

$$+ \| B_k''(x)\|_{\mathcal{L}(\mathbf{R}^N,\, \mathcal{L}(\mathbf{R}^N,\, \mathcal{L}(\mathbf{R}^N,\, T_xM)))}\Big) < \infty,$$

and w_k are independent Wiener processes with values in $\mathbf{R}^N$.

In order to use the theory of infinite-dimensional SDE, we will rewrite this system in the form of an SDE in a Hilbert space. For this, let us introduce the Hilbert space $l_{2,p}(\mathbf{Z}^d \to \mathbf{R}^n) \subset (\mathbf{R}^n)^{\mathbf{Z}^d}$ with the norm $\|\cdot\|_p$ given by the expression

$$\| x\|_p^2 = \sum_{k\in\mathbf{Z}^d} |x_k|^2 p_{|k|}, \tag{24}$$

where $p = (p_s)_{s\in\mathbf{Z}_+,\, s\geq 0}$ is some weight sequence, $p\in l_1$. Obviously, $l_{2,p}(\mathbf{Z}^d \to \mathbf{R}^n)$ contains the space of bounded sequences and, therefore, contains $M^{\mathbf{Z}^d}$ (as in the previous section, we identify M with its image $\varphi(M)\subset\mathbf{R}^n$).

Similarly, we define the space $l_{2,p}(\mathbf{Z}^d \to \mathbf{R}^N)$ and assume that the spaces $l_2(\mathbf{Z}^d \to \mathbf{R}^N)$ and $l_2(\mathbf{Z}^d \to \mathbf{R}^n)$ are defined in the usual way (which corresponds to $p_s \equiv 1$). Below, we will use the notation

$$l_{2,p}(\mathbf{Z}^d \to \mathbf{R}^n) = \mathcal{H}_p, \quad l_{2,p}(\mathbf{Z}^d \to \mathbf{R}^N) = \mathcal{X}.$$

As above, let us extend the coefficients a_k and B_k to all $(\mathbf{R}^n)^{\mathbf{Z}^d}$. We set

$$\tilde V_{A,k}(y) = V_A(x_y)\prod_{j\in A} F(y_j), \tag{25}$$

where $x_y = \{(x_j)_y\}_{j\in A} \in M^{\mathbf{Z}^d}$,

$$\tilde A_k = \sum_{A\in\mathcal{A}} \tilde V_{A,k}, \tag{26}$$

$$\tilde B_k(y_k) = B_k((x_k)_y)F(y_k). \tag{27}$$

Let us consider the equation

$$d\xi(t) = \tilde A(\xi(t))\,dt + \tilde B(\xi(t))\cdot dw(t), \tag{28}$$

in the Hilbert space $\mathcal{H}_p$, where $\tilde A(y)$ is an element of $\mathcal{H}_p$ with components $\tilde a_k(y)$, $\tilde B(y)$ is the Hilbert–Schmidt operator $\mathcal{X}\to\mathcal{H}_p$ generated by the block-diagonal matrix with nonzero blocks $\tilde B_{kk}(y) \equiv \tilde B_k(y_k)$ (the space of such Hilbert–Schmidt operators will be denoted by $S_2(\mathcal{X}, \mathcal{H}_p)$), and $w(t)$ is the Wiener process in $\mathcal{X}$.

Lemma 2. 1. *There exists a weight sequence* $p \in l_1$ *such that the mapping*

$$\mathcal{H}_p \ni x \mapsto \tilde{A}(x) \in \mathcal{H}_p \tag{29}$$

is bounded and satisfies the Lipschitz condition.

2. *The mapping*

$$\mathcal{H}_p \ni x \mapsto \tilde{B}(x) \in \mathcal{S}_2(\mathcal{X}, \mathcal{H}_p) \tag{30}$$

is twice continuously differentiable with bounded derivatives for any weight sequence $p \in l_1$.

Proof. A candidate for the derivative $\tilde{a}'(y)$, if it exists, is given by the block matrix with the elements

$$\tilde{a}'_{kj}(y) = \frac{\partial}{\partial y_j}\tilde{a}_k(y) \in \mathcal{L}(\mathbf{R}^n).$$

Then

$$\tilde{a}'_{kj}(y) = \sum_{A \in \mathcal{A}} \frac{\partial}{\partial y_j}\left[V_A^k(x_y)\prod_{i \in A} F((y)_i)\right] = \sum_{A \in \mathcal{A}}\left[\frac{\partial}{\partial y_j} V_{A,k}(x_y)\right]\prod_{i \in A} F((y)_i) +$$

$$+ \sum_{A \in \mathcal{A}} V_{A,k}(x_y)\left[\frac{\partial}{\partial y_j} F((y)_j)\right] \prod_{i \in A,\, i \neq j} F((y)_i), \tag{31}$$

where, for any $P, Q \in \mathbf{R}^n$, the expression $P \cdot Q$ means an $n \times n$ matrix with components $(P \cdot Q)_{ij} = P_i Q_j$.

Then, by virtue of (16), for some constant c and any $y \in (\mathbf{R}^n)^{\mathbf{Z}^d}$,

$$\| \tilde{a}'_{kj}(y) \|_{\mathcal{L}(\mathbf{R}^n)} < c \sum_{A \in \mathcal{A}} \left[\| d_j V_{A,k}(x_y) \|_{T_{x_j}M \otimes T_{x_k}M} + \| V_{A,k}(x_y) \|_{T_{x_k}M} \right]. \tag{32}$$

Let us consider the matrix r with elements $r_{kj} = \sup_{y \in (\mathbf{R}^n)^{\mathbf{Z}^d}} \| \tilde{a}'_{kj}(y) \|$. By virtue of conditions (20) and (21), the sum $\sum_j r_{kj}$ is uniformly bounded in k. This is sufficient for the existence of the positive sequence $(p_s) \in l_1$ such that $\sum_k r_{kj} p_{|k|} < C p_{|j|}$ for some constant C [11]. By Schur's test, the matrix r generates a bounded operator in $l_{2,p}(\mathbf{Z}^d \to \mathbf{R}^1)$ with norm less than C (see, e.g., [12]). It is easy to see that, for any $y \in (\mathbf{R}^n)^{\mathbf{Z}^d}$, the matrix $\tilde{a}'(y)$ generates then a bounded operator in $\mathcal{H}_p$ with norm bounded by C uniformly in y. Thus, mapping (29) is differentiable in any direction $h \in \mathcal{H}_p$ with the derivative $\tilde{a}'(y)h$. This implies that this mapping satisfies the Lipschitz condition with Lipschitz constant C.

The proof of the second statement can be obtained by a similar argument and is, in fact, simpler because both first-order and second-order derivatives of $\tilde{B}$ have a block-diagonal form.

Now let us fix some weight sequence as in Lemma 2 and consider SDE (28) in the space $\mathcal{H}_p$.

Lemma 3. *The Cauchy problem for equation* (28) *is uniquely solvable for any initial data* $y \in \mathcal{H}_p$, *and its solution* ξ_y *continuously depends on* y *in the square mean sense.*

Proof. This statement follows from Lemma 2 and the general theory of SDE in Hilbert spaces (see, e.g., [13]).

We can now introduce semigroups $\tilde{T}$ acting in the space $C_b(\mathcal{H}_p)$:

$$\tilde{T}_t f(y) = E(f(\xi_y(t))). \tag{33}$$

Let us denote by $M_p^{\mathbf{Z}^d}$ the space $M^{\mathbf{Z}^d}$ equipped with the topology induced from $l_{2,p}(\mathbf{Z}^d \to \mathbf{R}^n)$. This topology coincides with the topology generated by the metric

$$\rho_p(x, x') = \left(\sum_{k \in \mathbf{Z}^d} p_{|k|} \rho(x_k, x'_k)^2 \right)^{1/2}, \tag{34}$$

where ρ is the metric on M that corresponds to our fixed Riemannian structure.

We will denote by $C_b\left(M_p^{\mathbf{Z}^d}\right)$ the space of bounded continuous functions on $M_p^{\mathbf{Z}^d}$. Let us fix a weight sequence p from Lemma 3.

Theorem 2. *The solution $\xi_x(t)$ of equation (28) in $\mathcal{H}_p$ with an initial value $x \in M^{\mathbf{Z}^d}$ does not leave $M^{\mathbf{Z}^d}$ a.s. The process ξ_x defines the semigroups T_t,*

$$T_t f(y) = u(t, x) \equiv E(f(\xi_x(t))), \tag{35}$$

acting in the space $C_b\left(M_p^{\mathbf{Z}^d}\right)$.

Proof. Let us prove that the process ξ_x, $x \in M^{\mathbf{Z}^d}$, stays on $M^{\mathbf{Z}^d}$. For this, similarly to the proof of Theorem 1, we can apply the Itô formula to the functions $\mathcal{D}_k(y) = \mathcal{D}(y_k)$ and show that $\mathcal{D}_k(\xi_x(t)) = 0$ for all $k \in \mathbf{Z}^d$.

The process $\xi_y(t)$ is continuous in $y \in \mathcal{H}_p$ a.s. and, therefore, $\xi_x(t)$ is continuous in $x \in M_p^{\mathbf{Z}^d}$. Hence, for $f \in C_b\left(M^{\mathbf{Z}^d}\right)$, the function $u(t, \cdot)$ also belongs to $C_b\left(M^{\mathbf{Z}^d}\right)$.

Our next goal is to establish the invariance of some spaces of smooth functions on $M^{\mathbf{Z}^d}$. Let us suppose that, for any $A \in \mathcal{A}$ and any $k \in \mathbf{Z}^d$, the function $V_{A,k}$ belongs to the class $C^{Q+1}(M^A, TM)$, $Q \in \mathbf{N}$, $Q \geq 2$, and

$$\sup_{k \in \mathbf{Z}^d} \sum_{i_1, \ldots, i_q} \sup_{x \in M^{\mathbf{Z}^d}} \sum_{A \in \mathcal{A}} \| d_{i_1} \ldots d_{i_q} V_{A,k}(x) \| < \infty, \tag{36}$$

for any $q \leq Q$ ($\| \cdot \|$ denotes here the natural norm in the space $T_{x_{i_1}} M \otimes \ldots \otimes T_{x_{i_q}} M \otimes \otimes T_{x_k} M$).

We also assume that the mappings B_k belong to the class C_b^{Q+2} uniformly in k in the similar sense as in (22).

First of all, let us introduce the spaces $RC_b^q\left(M_p^{\mathbf{Z}^d}\right)$, $q \geq 1$, of functions u on $M^{\mathbf{Z}^d}$ which are restrictions of functions $\tilde{u} \in C_b^q(\mathcal{H}_p)$ to $M^{\mathbf{Z}^d}$ (the space of q times continuously differentiable functions on $\mathcal{H}_p$ bounded together with derivatives up to the qth order). Let us remark that $RC_b^q\left(M_p^{\mathbf{Z}^d}\right)$ contains the space of cylinder functions $\mathcal{F}C^q(M^{\mathbf{Z}^d})$.

Lemma 4. *There exists a weight sequence $p \in l_1$ such that the semigroup T_t preserves the spaces $RC_b^q\left(M_p^{\mathbf{Z}^d}\right)$, $q \leq Q$, as well as the space $C_b\left(M_p^{\mathbf{Z}^d}\right)$.*

Proof. Generalizing the construction given in the proof of Lemma 2, one can show that there exists a weight sequence $p \in l_1$ such that the mapping

$$\mathcal{H}_p \ni x \mapsto \tilde{a}(x) \in \mathcal{H}_p \tag{37}$$

is Q times differentiable and its Q-derivative satisfies the Lipschitz condition.

The mapping

$$\mathcal{H}_p \ni x \mapsto \tilde{B}(x) \in \mathcal{S}_2(\mathcal{X}, \mathcal{H}_p) \tag{38}$$

is $Q+2$ times continuously differentiable with bounded derivatives for any weight sequence $p \in l_1$ by virtue of the $Q+2$-differentiability of B_k. It follows then from the general theory of SDE in a Hilbert space that the semigroup $\tilde{T}_t$ preserves the spaces $C_b^q(\mathcal{H}_p)$, $q \le Q$. This obviously implies the statement of the lemma.

4. Stochastic dynamics for lattice models associated with Gibbs measures on $M^{\mathbf{Z}^d}$. Let us consider a family of potentials $\mathcal{U} = (U_A)_{A \in \mathcal{A}}$, where $\mathcal{A}$ is the set of all finite subsets of $\mathbf{Z}^d$, $U_A \in \mathcal{F}C^2(M^{\mathbf{Z}^d})$.

Let $\mathcal{A}(k)$ be the set of all sets $A \in \mathcal{A}$ that contain a point $k \in \mathbf{Z}^d$. We will assume the following:

$$\sum_{A \in \mathcal{A}(k)} |U_A(x)| < \infty \tag{39}$$

for any $x \in M^{\mathbf{Z}^d}$ and $k \in \mathbf{Z}^d$,

$$\sup_{k \in \mathbf{Z}^d} \sup_{x \in M^{\mathbf{Z}^d}} \sum_{A \in \mathcal{A}} \| d_k U_A(x) \| \, |A| < \infty, \tag{40}$$

$$\sup_{k \in \mathbf{Z}^d} \sum_{j \in \mathbf{Z}^d} \sup_{x \in M^{\mathbf{Z}^d}} \sum_{A \in \mathcal{A}} \| d_j d_k U_A(x) \| < \infty. \tag{41}$$

Let $\Gamma(\mathcal{U})$ be the family of Gibbs measures associated with the family of potentials $\mathcal{U}$. Heuristically any $\mu \in \Gamma(U)$ can be given by the expression

$$d\mu(x) = \frac{1}{Z} e^{E(x)} dx, \tag{42}$$

where $dx = \otimes_k dx_k$ is the product of invariant measures on M_k and

$$E(x) = \sum_{A \in \mathcal{A}} U_A(x). \tag{43}$$

For a rigorous definition, see, e.g., [5].

Let us mention that, for $\mu \in \Gamma(U)$, the following formula of integration by parts is true: For any $u \in \mathcal{F}C^2(M^{\mathbf{Z}^d})$ and a finite number of vector fields $X_k \in TM_k$,

$$\int \sum_k (d_k u(x), X_k(x))_k \, d\mu(x) =$$

$$= -\int \sum_k \left[(\Lambda_k(x), X_k(x))_k + \operatorname{div}_G X_k(x) \right] u(x) \, d\mu(x), \tag{44}$$

where $\Lambda_k(x) = d_k U_k(x)$, $U_k(x) = \sum_{A \in \mathcal{A}(k)} U_A(x)$, ($\Lambda_k(x)$ and $U_k(x)$ exist by virtue of conditions (39)–(40) [5]).

We will call $\Lambda = (\Lambda_k)$ the logarithmic derivative of μ.

The set $\Gamma(\mathcal{U})$ is not empty under conditions (39)–(40) (see, e.g., [5]). Let us fix some $\mu \in \Gamma(\mathcal{U})$.

For $u, v \in \mathcal{F}C^2(M^{\mathbf{Z}^d})$ let us define a pre-Dirichlet form

$$\mathcal{E}(u, v) = \frac{1}{2}\int \sum_k (d_k u(x), d_k v(x))_k d\mu(x). \tag{45}$$

Obviously, it has a generator H_μ acting in $L_2(M^{\mathbf{Z}^d}, \mu)$ on the domain $\mathcal{F}C^2(M^{\mathbf{Z}^d})$ as

$$H_\mu u(x) = -\frac{1}{2}\sum_k \Delta_k u(x) - \frac{1}{2}\sum_k (\Lambda_k(x), d_k u(x))_k. \tag{46}$$

***Remark* 2.** In the case of finite range of interactions, conditions (39)–(41) are obviously satisfied.

Theorem 3. *There exists a weight sequence $p \in l_1$ and a Markov process ξ_x with values in $M_p^{\mathbf{Z}^d}$ such that the associated semigroup $T_t u(x) = E(u(\xi_x(t)))$ acts in the space $C_b(M_p^{\mathbf{Z}^d})$ and its generator coincides with H_μ on $\mathcal{F}C^2(M_p^{\mathbf{Z}^d})$.*

Proof. Let us consider system (22) with $B_k = B$ given by (9), $V_{\Lambda,k} = d_k U_\Lambda$ for $\Lambda \neq \{k\}$, and $V_{\Lambda,k}(x) = d_k U_\Lambda(x) - b(x_k)$ for $\Lambda = \{k\}$, s defined by (11). Conditions (20) and (21) are satisfied by virtue of (40) and (41). The statement of the theorem now follows from Theorem 2.

***Remarks.* 3.** It is easy to impose conditions on the derivatives up to the Qth order of the functions U_Λ that ensure the invariance of the classes $RC_b^q(M_p^{\mathbf{Z}^d})$, $q \le Q$, with respect to the semigroup T for some weight sequence $p \in l_2$.

4. In particular, let us suppose that the potentials U_Λ are 4-differentiable and

$$\sup_{k \in \mathbf{Z}^d} \sum_{i_1, i_2} \sup_{x \in M^{\mathbf{Z}^d}} \sum_{\Lambda \in \mathcal{A}} \| d_{i_1} d_{i_2} d_k U_\Lambda(x) \| < \infty, \tag{47}$$

$$\sup_{k \in \mathbf{Z}^d} \sum_{i_1, i_2, i_3} \sup_{x \in M^{\mathbf{Z}^d}} \sum_{\Lambda \in \mathcal{A}} \| d_{i_1} d_{i_2}, d_{i_3} d_k U_\Lambda(x) \| < \infty. \tag{48}$$

Then the semigroup T leaves the space $RC_b^2(M_p^{\mathbf{Z}^d})$ invariant. Hence, the operator H_μ is essentially self-adjoint on $RC_b^2(M_p^{\mathbf{Z}^d})$. It can be shown by analogy with [1] that it is also essentially self-adjoint in $\mathcal{F}C^2(M^{\mathbf{Z}^d})$. The last result has been proved in the case of finite range of interactions in [5].

5. Uniqueness of dynamics. The aim of this section is to prove the essential self-adjointness of the operator H_μ under weaker conditions than in Remark 4.

Theorem 4. *For any family $\mathcal{U}$ of potentials satisfying assumptions* (40), (41), *and any Gibbs measure $\mu \in \Gamma(\mathcal{U})$, the pre-Dirichlet operator H_μ defined on $\mathcal{F}C^2(M^{\mathbf{Z}^d})$ is an essentially self-adjoint operator in $L_2(M^{\mathbf{Z}^d}, \mu)$.*

Proof. We will essentially follow the scheme of [1, 5].

Let us approximate the potentials U_Λ by smooth functions $U_\Lambda^n \in C^\infty(M^\Lambda)$, $n \in \mathbf{N}$, such that

$$\| U_\Lambda^n - U_\Lambda \|_{C^2(Z^\Lambda)} \le e^{-d(\Lambda) - n}, \tag{49}$$

where $d(A) = \max_{x \in A} |k|$. It is easy to see that the potentials U_A^n satisfy conditions (40) and (41) and

$$\sup_{k \in \mathbf{Z}^d} \sup_{x \in M^{\mathbf{Z}^d}} \sum_{A \in \mathcal{A}} \| d_k U_A^n(x) \| \, |A| < \infty, \tag{50}$$

$$\sup_{k \in \mathbf{Z}^d} \sum_{j \in \mathbf{Z}^d} \sup_{x \in M^{\mathbf{Z}^d}} \sum_{A \in \mathcal{A}} \| d_j d_k U_A^n(x) \| < \infty \tag{51}$$

uniformly in n.

We set

$$V_k^n(x) = \sum_{A \in \mathcal{A}(k),\, d(A) \le n} U_A^n(x) \tag{52}$$

and

$$\Lambda_k^n(x) = d_k V_k^n(x). \tag{53}$$

Let us remark that $U_k^n \in C^\infty(M^{A_n})$, where $A_n = \{ k \in \mathbf{Z}^d : |k| \le n \}$, and, therefore, $\Lambda_k^n = 0$ for $|k| > n$.

For any $n \in \mathbf{N}$, we define a differential operator H_n on the domain $\mathcal{F}C^2(M^{\mathbf{Z}^d}) \subset L_2(M^{\mathbf{Z}^d}, \mu)$ by the formula

$$H_n u(x) = -\frac{1}{2} \sum_{k \in \mathbf{Z}^d} \Delta_k u(x) - \frac{1}{2} \sum_{k \in \mathbf{Z}^d} \left(\Lambda_k^n(x), d_k u(x) \right)_k. \tag{54}$$

We will use the parabolic criterion of essential self-adjointness [15]. Let us consider the following Cauchy problems:

$$\begin{aligned} &\frac{d}{dt} u_n(t) + H_n u_n(t) = 0, \\ &u_n(0) = f, \qquad t \in [0, 1], \end{aligned} \tag{55}$$

where $f \in \mathcal{F}C^2(M^{\mathbf{Z}^d})$ is arbitrary. If we prove the existence of strong solutions

$$u_n : [0, 1] \to L_2(M^{\mathbf{Z}^d}, \mu) \tag{56}$$

of (55) such that

$$u_n(t) \in \mathcal{D}(H_\mu) \tag{57}$$

for any $n \in \mathbf{N}$ and $t \in [0, 1]$, where $\mathcal{D}(H_\mu)$ is the domain of H_μ, and

$$\int_0^1 \| (H_\mu - H_n) u_n(t) \|_{L_2(M^{\mathbf{Z}^d}, \mu)} dt \to 0, \qquad n \to \infty, \tag{58}$$

then the operator H_μ is essentially self-adjoint in $L_2(M^{\mathbf{Z}^d}, \mu)$.

Let us remark first that the Cauchy problem (55) with fixed $f \in \mathcal{F}C^2(M^{\mathbf{Z}^d})$ is finite-dimensional. Hence, there exists the classical solution u_n of (55) and, moreover, $u_n(t) \in \mathcal{F}C^2(M^{\mathbf{Z}^d})$ and is a C^1-function in t. Therefore, conditions (56) and (57) are satisfied.

In order to prove the third condition, let us consider the system of SDE

$$d\xi_k^n = a_k^n(\xi(t))\,dt + B_k(\xi_k(t))\cdot dw_k(t),$$
$$\xi_k^n(0) = x_k \in M, \qquad k \in \mathbf{Z}^d, \tag{59}$$

with $B_k = B$ as above, $a_k^n = \sum_{A\in A_n} V_{A,k}^n(x)$, where $V_{A,k}^n$ is constructed similarly to the proof of Theorem 3, i.e., $V_{A,k}^n = d_k U_A^n$ for $A \neq \{k\}$ and $V_{A,k}^n(x) = d_k U_A^n(x) - b(x_k)$ for $A = \{k\}$.

Obviously, the coefficients of this system satisfy conditions (20) and (21) uniformly in n. Then, by Theorem 2, it is solvable in $M_p^{\mathbf{Z}^d}$ with some weight sequence p for any n. Let $T^n(t)$ be the corresponding semigroup. Then $u_n(t) = T^n(t)f$. Let us remark that the coefficient $\tilde{a}_n$ in the framework of Sec. 2 belongs to the class $\mathcal{F}C_b^\infty\big((R^n)^{\mathbf{Z}^d}\big)$ and $\|(\tilde{a}_n)'(y)\|_{\mathcal{L}(\mathcal{H}_p)} \le C$ for some constant C uniformly in $y \in \mathcal{H}_p$ and n.

It follows from the general theory of SDE in Hilbert spaces that

$$|\tilde{u}_n'(t,y)h| \le \text{const}\, e^{Ct/2}\|h\|_{\mathcal{H}_p}\|f\|_{C_b^1}. \tag{60}$$

We can now check condition (58). We have

$$(H_\mu - H_n)u_n = \sum_k (\Lambda_k - \Lambda_k^n, d_k u_n)_k, \tag{61}$$

and, by virtue of (60) and (17),

$$|(H_\mu - H_n)u_n(x)| \le \text{const}\cdot e^{Ct/2}\|\Lambda(x) - \Lambda^n(x)\|_{\mathcal{H}_p}\|f\|_{C_b^1} \tag{62}$$

uniformly in n and $x \in M_p^{\mathbf{Z}^d}$. Hence,

$$\int_0^1 \|(H_\mu - H_n)u_n(t)\|_{L_2(M^{\mathbf{Z}^d},\mu)}\,dt \le \tag{63}$$

$$\text{const}\cdot \sup_{x\in M^{\mathbf{Z}^d}} \|\Lambda(x) - \Lambda^n(x)\|_{\mathcal{H}_p}, \tag{64}$$

By definition,

$$|\Lambda(x) - \Lambda_k^n(x)| = \left|\sum_{A\in\mathcal{A}} d_k U_A(x) - \sum_{A\in\mathcal{A},\, d(A)\le n} d_k U_A^n(x) + b(x_k)\delta(n,|k|)\right| \le$$

$$\le \sum_{A\in\mathcal{A},\, d(A)\le n} |d_k U_A(x) - d_k U_A^n(x)| +$$

$$+ \sum_{A\in\mathcal{A},\, d(A)>n} |d_k U_A(x)| + |b(x_k)\delta(n,|k|)|, \tag{65}$$

where $\delta(n,m) = 0$ if $n > m$, and $\delta = 1$ if $n \le m$. Then, according to (49),

$$\sup_{x\in M^{\mathbf{Z}^d}} |(\Lambda_k(x) - \Lambda_k^n(x))| \le$$

$$\leq e^{-n} \sum_{A \in \mathcal{A}} e^{-d(A)} + c_1 \cdot \delta + \sup_{x \in M^{\mathbb{Z}^d}} \sum_{A \in \mathcal{A},\, d(A) > n} |d_k U_A(x)| \tag{66}$$

and

$$\sup_{x \in M^{\mathbb{Z}^d}} \| \Lambda(x) - \Lambda^n(x) \|_{\mathcal{H}_p} \leq$$

$$\leq c_2 e^{-n} \sum_{k \in \mathbb{Z}^d} p_{|k|} + c_1 \sum_{k \in \mathbb{Z}^d,\, |k| > n} p_{|k|} +$$

$$+ \sup_{x \in M^{\mathbb{Z}^d}} \sum_{A \in \mathcal{A},\, d(A) > n} |d_k U_A(x)| \to 0, \quad n \to \infty. \tag{67}$$

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