UDC 515.12

M. E. Aydin (Firat Univ., Elazig, Turkey)

CLASSIFICATIONS OF TRANSLATION SURFACES IN ISOTROPIC GEOMETRY WITH CONSTANT CURVATURE

КЛАСИФІКАЦІЇ ТРАНСЛЯЦІЙНИХ ПОВЕРХОНЬ В ІЗОТРОПНІЙ ГЕОМЕТРІЇ ЗІ СТАЛОЮ КРИВИНОЮ

We classify translation surfaces in isotropic geometry with arbitrary constant isotropic Gaussian and mean curvatures under the condition that at least one of translating curves lies in a plane.

Запропоновано класифікацію трансляційних поверхонь в ізотропній геометрії з довільною сталою ізотропною гауссовою та середньою кривиною за умови, що принаймні одна з трансляційних кривих лежить у площині.

1. Introduction. A *translation surface* in a Euclidean space $\mathbb{R}^3$ that is expressed as the sum of two curves can be locally parameterized by [5]

$$r(x,y) = \alpha(x) + \beta(y), \tag{1.1}$$

where α and β are referred to as *translating curves*. Recent results and progress on translation surfaces in $\mathbb{R}^3$ with constant Gaussian and mean curvatures were well-structured in the papers [14–16, 25].

If α and β lie in orthogonal planes, then, up to a change of coordinates, the surface is locally described in explicit form

$$z(x,y) = f(x) + g(y),$$

where f, g are smooth real-valued functions of one variable. In this case, beside the planes, only minimal translation surface (i.e., mean curvature vanishes identically) is the *Scherk surface*, namely, the graph of [34]

$$z(x,y) = \frac{1}{c}\log\left|\frac{\cos(cy)}{\cos(cx)}\right|, \quad c \in \mathbb{R}-\{0\}.$$

Many generalizations on this result in (semi-) Euclidean and homogeneous spaces were done so far, see, for example, [8–10, 12, 17, 19, 20, 23, 24, 26, 28, 29, 35, 37, 38].

Recently, Liu and Yu [22] introduced a new class of translation surfaces in $\mathbb{R}^3$, so-called *affine translation surfaces*, as the graphs of

$$z(x,y) = f(x) + g(y + ax), \quad a \in \mathbb{R}-\{0\}. \tag{1.2}$$

By the change of coordinates $x = u,\ y = v - au$ in (1.2) one can be locally parametrized as

$$r(u,v) = (u, v - au, f(u) + g(v)),$$

where the translating curves lie in the planes $x = 0$ and $ax + y = 0$. Because $a \neq 0$ these planes are not orthogonal to each other and the obtained surface is a natural generalization of the classical

© M. E. AYDIN, 2020

translation surface. In same paper, the authors conjectured that, besides planes, only minimal graph surface of the form (1.2), usually called *affine Scherk surface*, is given in explicit form

$$z(x,y)=\frac{1}{c}\log\left|\frac{\cos\left(c\sqrt{1+a^2}x\right)}{\cos\left(c[y+ax]\right)}\right|,\qquad c\in\mathbb{R}-\{0\}.$$

We also refer to [18, 21, 39, 40] for more recent results on this kind of surfaces.

Following Liu and Yu [22] we introduce and classify a new type of translation surfaces in isotropic geometry with constant isotropic Gaussian curvature (CIGC) and constant isotropic mean curvature (CIMC). In addition, we obtain the surfaces of CIGC and CIMC whose one translating curve is planar and other one space curve.

2. Preliminaries. For fundamental notions of curves and surfaces in isotropic geometry that is one of the Cayley – Klein geometries, we refer the reader to [3, 4, 6, 7, 11, 30 – 33]. Those can briefed by the arguments from Projective Geometry as in next paragraphs.

Let $\mathbb{P}^3$ denote the projective space and Γ a plane in $\mathbb{P}^3$. Then an affine space can be obtained from $\mathbb{P}^3$ by substracting Γ which we call *absolute plane.* If Γ involves a pair of complex-conjugate straight lines l_1 and l_2, so-called the *absolute lines,* then the obtained affine space becomes an *isotropic space* $\mathbb{I}^3$, where the triple (Γ, l_1, l_2) is referred to as the *absolute figure* of $\mathbb{I}^3$.

Let a quadruple $\left(\tilde{t}:\tilde{x}:\tilde{y}:\tilde{z}\right)$ be the projective coordinates, i.e., $\left(\tilde{t}:\tilde{x}:\tilde{y}:\tilde{z}\right)\neq(0:0:0:0)$. Then Γ and l_1, l_2 are, respectively, parameterized by $\tilde{t}=0$ and $\tilde{t}=\tilde{x}\pm i\tilde{y}=0$. The intersection point of l_1 and l_2 is said to be *absolute*, i.e., $(0:0:0:1)$.

We are interested in an affine model of $\mathbb{I}^3$. Thus, by means of the affine coordinates $x=\dfrac{\tilde{x}}{\tilde{t}}$, $y=\dfrac{\tilde{y}}{\tilde{t}}$, $z=\dfrac{\tilde{z}}{\tilde{t}}$, $\tilde{t}\neq 0$, the group of motions of $\mathbb{I}^3$ is a six-parameter group given by

$$(x,y,z)\longmapsto(x',y',z')\colon\begin{cases}x'=a+x\cos\theta-y\sin\theta,\\ y'=b+x\sin\theta+y\cos\theta,\\ z'=c+dx+ey+z,\end{cases}\tag{2.1}$$

where $a,\ b,\ c,\ d,\ e,\ \theta\in\mathbb{R}$. The metric invariants of $\mathbb{I}^3$ under (2.1), such as *isotropic distance* and *angle*, are Euclidean invariants in the Cartesian plane.

A line in $\mathbb{I}^3$ is said to be *isotropic* provided its point at infinity agrees with the absolute point. In the affine model of $\mathbb{I}^3$, it corresponds to a line parallel to z-axes. Otherwise, it is called *non-isotropic line*.

A plane in $\mathbb{I}^3$ involving an isotropic line is said to be *isotropic* and then its line at infinity involves the absolute point. Otherwise, it is called *non-isotropic plane*. For example, the equation $ax+by+cz=d$, $a,\ b,\ c,\ d\in\mathbb{R}$, determines a non-isotropic (resp., isotropic) plane when $c\neq 0$ (resp., $c=0$).

A unit speed curve has the form

$$\alpha\colon I\subseteq\mathbb{R}\longrightarrow\mathbb{I}^3,\qquad s\longmapsto\left(f(s),g(s),h(s)\right),\qquad (f')^2+(g')^2=1,$$

where the derivative with respect to s is denoted by a prime. Therefore, the *curvature* κ and the *torsion* τ are given by

$$\kappa=\sqrt{(f'')^2+(g'')^2}\qquad\text{or}\qquad\kappa=f'g''-f''g'$$

and

$$\tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}, \quad \kappa \neq 0. \tag{2.2}$$

A curve that lies in an isotropic (resp., non-isotropic) plane is said to be *isotropic* (resp., *non-isotropic*) *planar*. Otherwise, we call it *space curve* and then $\tau \neq 0$.

Let M^2 be an admissible surface in $\mathbb{I}^3$, that is, a surface in which tangent plane at each point is non-isotropic. Then the tangent plane $T_p\left(M^2\right)$ at some point $p \in M^2$ has a Euclidean metric. For such a surface, the components E, F, G of the first fundamental form is obtained by the metric on M^2 induced from $\mathbb{I}^3$.

The unit isotropic direction $U = (0,0,1)$ is assumed to be the normal vector field of M^2 which is indeed orthogonal to all tangent vectors in $T_p\left(M^2\right)$. Hence, the components of the second fundamental form are computed with respect to U, namely,

$$l = \frac{\det(r_{xx}, r_x, r_y)}{\sqrt{EG - F^2}}, \qquad m = \frac{\det(r_{xy}, r_x, r_y)}{\sqrt{EG - F^2}}, \qquad n = \frac{\det(r_{yy}, r_x, r_y)}{\sqrt{EG - F^2}},$$

where $r = r(x, y)$ refers to a local parameterization on M^2 and $r_x = \dfrac{\partial r}{\partial x}$, $r_{xy} = \dfrac{\partial^2 r}{\partial x \partial y}$, etc. Note that the admissibility of M^2 implies $EG - F^2 \neq 0$.

The *isotropic Gaussian* (so-called *relative*) K and the *mean curvatures* H are defined by

$$K = \frac{ln - m^2}{EG - F^2}, \qquad H = \frac{En - 2Fm + Gl}{2\left(EG - F^2\right)}.$$

A surface for which H (resp., K) vanishes identically is said to be *isotropic minimal* (resp., *flat*). Moreover, a surface is said to have *CIMC* (resp., *CIGC*) if H (resp., K) is a constant function on whole surface.

3. Categorization of translation surfaces. The translation surfaces in $\mathbb{I}^3$ that are locally given by (1.1) can be categorized in terms of the translating curves and the absolute figure as below:

Type I α and β are planar:

Type I.1: α and β are isotropic planar,

Type I.2: α is isotropic planar and β non-isotropic planar,

Type I.3: α and β are non-isotropic planar.

Type II: α is isotropic planar and β space curve.

Type III: α is non-isotropic planar and β space curve.

Type IV: α and β are space curves.

A surface which belongs to one type is no equivalent to that of another type up to the absolute figure.

Let us assume for a surface of Type 1 that the translating curves lie orthogonal planes. Denoting f and g smooth functions, after a change of coordinates, such a surface can be locally given in one of the following explicit forms:

Type I.1*: Both translating curves are isotropic planar $z(x, y) = f(x) + g(y)$.

Type I.2*: One translating curve is non-isotropic planar and other one isotropic planar $y(x, z) = f(x) + g(z)$.

Type I.3*: Both translating curves are non-isotropic planar

$$x(y, z) = \frac{1}{2}\left[f\left(\frac{y + z - \pi}{2}\right) + g\left(\frac{-y + z + \pi}{2}\right)\right].$$

These surfaces with CIMC and CIGC were obtained in [27, 36].

Next let assume for a surface of Type 1 that the translating curves lie in arbitrary planes. Let $[a_{ij}]$ be a 2×2 matrix and $|a_{ij}| = a_{11}a_{22} - a_{12}a_{21} \neq 0$. More generally the surfaces of Type I.1 are locally given by

$$r(u,v) = \left(\frac{a_{22}u}{|a_{ij}|} - \frac{a_{12}v}{|a_{ij}|}, -\frac{a_{21}u}{|a_{ij}|} + \frac{a_{11}v}{|a_{ij}|}, f(u) + g(v) \right). \tag{3.1}$$

Up to a change of coordinates (3.1) turns to the graph of the form

$$z = f\left(a_{11}x + a_{12}y\right) + g\left(a_{21}x + a_{22}y\right). \tag{3.2}$$

Such surfaces with CIMC and CIGC, which we call *translation graphs of first kind*, were presented in [2]. In this paper, we are interested in the surfaces of Types I.2 – III.

In the case that one curve is isotropic planar and another one without condition the translation surfaces with CIMC and CIGC were provided in [1].

4. Surfaces of Types I.2 and I.3. Let $[a_{ij}]$ denote a 2×2 matrix and $\omega = |a_{ij}| \neq 0$. We consider the following translation surface generated by planar curves:

$$r(u,v) = \left(\frac{a_{22}u}{\omega} - \frac{a_{12}v}{\omega}, f(u) + g(v), -\frac{a_{21}u}{\omega} + \frac{a_{11}v}{\omega} \right), \tag{4.1}$$

where the translating curves and the planes involving them are given by

$$\alpha(u) = \left(\frac{a_{22}u}{\omega}, f(u), -\frac{a_{21}u}{\omega} \right), \qquad \Gamma_\alpha : a_{21}x + a_{22}z = 0$$

and

$$\beta(v) = \left(-\frac{a_{12}v}{\omega}, g(v), \frac{a_{11}v}{\omega} \right), \qquad \Gamma_\beta : a_{11}x + a_{12}z = 0.$$

***Remark* 4.1.** Since the roles of f and g are symmetric we only discuss the cases depending on f throughout the section.

For a surface given by (4.1) we have:

The planes Γ_α and Γ_β are orthogonal to each other, provided $[a_{ij}]$ is an orthogonal matrix.

If $a_{12} = 0$ then, due to $\omega \neq 0$, Γ_α becomes a non-isotropic plane and Γ_β an isotropic plane. Thereby the obtained surface belongs to Type I.2.

If $a_{12} \neq 0$, then by symmetry $a_{22} \neq 0$ and the planes Γ_α, Γ_β are non-isotropic. Therefore, the obtained surface belongs to Type I.3.

After the change of coordinates, so-called *affine parameter coordinates,*

$$u = a_{11}x + a_{12}z, \qquad v = a_{21}x + a_{22}z,$$

the local surface given by (4.1) turns to the graph of the form

$$y = f(u) + g(v). \tag{4.2}$$

We call the surface of the form (4.2) *translation graph of second kind*. Notice that it is no equivalent to the graph of the form (3.2) up to the absolute figure. The positive side of this notion is to express the surfaces of Types I.2 and I.3 into one format.

Now we purpose to present the translation graphs of second kind in $\mathbb{I}^3$ with CIGC. For this, the admissibility implies that

$$a_{12}f' + a_{22}g' \neq 0, \qquad f' = \frac{df}{du}, \qquad g' = \frac{dg}{dv}.$$

By a calculation, the Gaussian curvature K turns to

$$K = \frac{\omega^2 f'' g''}{(a_{12}f' + a_{22}g')^4}. \tag{4.3}$$

It is seen from (4.3) that K vanishes identically provided $f'' = 0$, namely, the surface is a *generalized cylinder* (see [13, p. 439]) with non-isotropic rulings. So, next result can be stated in order for K to be a non-vanishing constant.

Remark **4.2.** In order to provide convenience in calculations, we denote nonzero constants by c_1, $c_2, \ldots$ and some constants by $d_1,\ d_2, \ldots$ throughout the paper unless otherwise stated.

Theorem 4.1. *For a translation graph of second kind in $\mathbb{I}^3$ with nonzero CIGC the following holds:*

$$f(u) = \frac{c_1}{2}u^2 + d_1 u + d_2, \qquad g(v) = \frac{-c_1 a_{11}^2}{2K_0 a_{22}^2}\left(-\frac{3K_0 a_{22}^2}{c_1 a_{11}^2}v + d_3\right)^{\frac{2}{3}} + d_4. \tag{4.4}$$

Proof. Because $K = K_0 \neq 0,\ K_0 \in \mathbb{R}$, in (4.3), we have $f''g'' \neq 0$. The partial derivative of (4.3) with respect to u gives

$$\frac{4K_0}{\omega^2}\left(a_{12}f' + a_{22}g'\right)^3\left(a_{12}f''\right) = f'''g''. \tag{4.5}$$

We distinguish two cases to proceed (4.5):

Case 1: $a_{12} = 0$. Then $a_{11}a_{22} \neq 0$ due to $\omega \neq 0$. By (4.5) we have $f'' = c_1$ and, thus, it follows from (4.3) that

$$\frac{K_0 a_{22}^2}{c_1 a_{11}^2} = \frac{g''}{(g')^4}. \tag{4.6}$$

Solving the equations $f'' = c_1$ and (4.6) leads to (4.4).

Case 2: $a_{12} \neq 0$. By symmetry, we deduce $a_{22} \neq 0$. Then (4.5) can be arranged as

$$\frac{\left(a_{12}f' + a_{22}g'\right)^3}{g''} = \frac{\omega^2}{4K_0 a_{12}}\left(\frac{f'''}{f''}\right). \tag{4.7}$$

The partial derivative of (4.7) with respect to v yields

$$3a_{22}(g'')^2 - \left(a_{12}f' + a_{22}g'\right)g''' = 0, \tag{4.8}$$

where $g''' \neq 0$ because $a_{22}g'' \neq 0$. After taking partial derivative of (4.8) with respect to u we immediately achieve a contradiction.

Theorem 4.1 is proved.

By a calculation, the isotropic mean curvature H of (4.2) is

$$H = -\frac{\left[a_{12}^2 + (\omega g')^2\right]f'' + \left[a_{22}^2 + (\omega f')^2\right]g''}{2\left(a_{12}f' + a_{22}g'\right)^3}. \tag{4.9}$$

First, we concern minimality case via the following result.

Theorem 4.2. *For a minimal translation graph of second kind in* $\mathbb{I}^3$ *one of the following occurs:*
(1) *it is a non-isotropic plane;*
(2) $f(u) = \dfrac{1}{c_1 a_{11}^2 a_{22}^2} \log\left|\cos\left(c_1 a_{11} a_{22}^2 u + d_1\right)\right| + d_2,$

$$g(v) = \frac{-1}{c_1 a_{11}^2 a_{22}^2} \log\left|c_1 a_{11}^2 a_{22}^2 v + d_3\right| + d_4,$$

(3) $f(u) = \dfrac{1}{c_1 \omega^2} \log\left|\cos\left(\omega c_1 a_{22} u + d_1\right)\right| + d_2,$

$$g(v) = \frac{-1}{c_1 \omega^2} \log\left|\cos\left(\omega c_1 a_{12} v + d_3\right)\right| + d_4.$$

Proof. Since H vanishes identically, (4.9) reduces to

$$\left[a_{12}^2 + (\omega g')^2\right] f'' + \left[a_{22}^2 + (\omega f')^2\right] g'' = 0. \tag{4.10}$$

$f'' = g'' = 0$ is a solution for (4.10), and, in this case, the surface is a non-isotropic plane. Suppose that $f'' g'' \neq 0$. Hence, (4.10) implies

$$-\frac{f''}{a_{22}^2 + (\omega f')^2} = c_1 = -\frac{g''}{a_{12}^2 + (\omega g')^2}. \tag{4.11}$$

We have two cases:

Case 1: $a_{12} = 0$. Then we have $a_{11} a_{22} \neq 0$ because $\omega \neq 0$. By solving (4.11), we obtain the item (2) of the theorem.

Case 2: $a_{12} \neq 0$. By symmetry, we get $a_{22} \neq 0$ and solving (4.11) leads to last item of the theorem.

Theorem 4.3. *For a translation graph of second kind in* $\mathbb{I}^3$ *with nonzero CIMC, we have*

(a) $f(u) = c_1 u + d_1, \qquad g(v) = -H_0 \dfrac{a_{12} c_1}{a_{21}^2} v^2 + d_2 v + d_3,$

or

(b) $f(u) = c_1 u + d_1, \qquad g(v) = \dfrac{a_{22}^2 + (\omega c_1)^2}{2 H_0 a_{22}^2} \left(\dfrac{4 H_0 a_{22}}{a_{22}^2 + (\omega c_1)^2} v + d_2\right)^{\frac{1}{2}} - \dfrac{a_{12} c_1}{a_{22}} v + d_3.$

Proof. Assume $H = H_0 \neq 0$, $H_0 \in \mathbb{R}$, in (4.9). The partial derivatives of (4.9) with respect to u and v yield

$$-6 H_0 \omega^{-2} a_{12} a_{22} \left(a_{12} f' + a_{22} g'\right)\left(f'' g''\right) = g' g'' f''' + f' f'' g'''. \tag{4.12}$$

The situation for which both f'' and g'' vanish is a solution for (4.12), however we omit this one because $H_0 \neq 0$. We distinguish the remaining cases:

Case 1: $f = c_1 u + d_1$ and $g'' \neq 0$. This assumption is a solution for (4.12). Then, from (4.9), we derive

$$\frac{g''}{\left(a_{12} c_1 + a_{22} g'\right)^3} = \frac{-2 H_0}{a_{22}^2 + (\omega c_1)^2}. \tag{4.13}$$

We have two cases:

(1.1) $a_{22} = 0$. Here $a_{12}a_{21} \neq 0$ due to $\omega \neq 0$. Solving (4.13) implies the first item of the theorem.

(1.2) $a_{22} \neq 0$. By symmetry we have $a_{12} \neq 0$. After solving (4.13) we obtain the second item of the theorem.

Case 2: $f''g'' \neq 0$. By dividing (4.12) with $f''g''$, one can be rewritten as

$$-6H_0\omega^{-2}a_{12}a_{22}\left(a_{12}f' + a_{22}g'\right) = g'\frac{f'''}{f''} + f'\frac{g'''}{g''}. \tag{4.14}$$

We have again cases:

(2.1) $a_{12} = 0$. Then $\omega \neq 0$ implies $a_{11}a_{22} \neq 0$. (4.14) turns to

$$\frac{f'''}{f'f''} = d_1 = -\frac{g'''}{g'g''},$$

which gives that $f'' = c_1e^{d_1f}$ and $g'' = c_2e^{-d_1g}$. By substituting those in (4.9), we derive

$$-2H_0a_{22}(g')^3 = c_1a_{11}^2(g')^2e^{d_1f} + \left[c_2 + c_2a_{11}^2(f')^2\right]e^{-d_1g}. \tag{4.15}$$

Put $f' = p$ and $g' = q$ in (4.15). Then taking partial derivative of (4.15) with respect to f yields

$$0 = d_1c_1q^2e^{d_1f} + 2c_2p\dot{p}e^{-d_1g}, \tag{4.16}$$

where $\dot{p} = \dfrac{dp}{df} = \dfrac{f''}{f'}$. If $d_1 = 0$ in (4.16), then we obtain the contradiction $\dot{p} = 0$. Otherwise, we have

$$\frac{d_1c_1e^{d_1f}}{2c_2p\dot{p}} = c_3 = -\frac{e^{-d_1g}}{q^2}. \tag{4.17}$$

By substituting the second equality in (4.17) into (4.15), we conclude

$$-2H_0a_{22}q(g) = c_1a_{11}^2e^{d_1f} - c_2c_3\left[1 + a_{11}^2p(f)^2\right]. \tag{4.18}$$

The left-hand side in (4.18) is a function of g however other side is a function of f. This is not possible.

(2.2) $a_{12} \neq 0$ in (4.14). The symmetry implies $a_{22} \neq 0$. By dividing (4.14) with $f'g'$, we write

$$D\left(\frac{a_{12}}{g'} + \frac{a_{22}}{f'}\right) = \frac{f'''}{f'f''} + \frac{g'''}{g'g''}, \tag{4.19}$$

where $D = -6H_0\omega^{-2}a_{12}a_{22}$.

It follows from (4.19) that

$$f''' = \left(-d_1f' + Da_{22}\right)f'' \quad \text{and} \quad g''' = \left(d_1g' + Da_{12}\right)g''. \tag{4.20}$$

On the other hand, by taking partial derivative of (4.9) with respect to v and considering the second equality in (4.20), we obtain

$$-6Ha_{22}\left(a_{12}f' + a_{22}g'\right)^2 = 2\omega^2g'f'' + \left[a_{22}^2 + (\omega f')^2\right]\left(d_1g' + Da_{12}\right),$$

which is a polynomail equation on g'. The leading coefficient coming from the term $(g')^2$ is $-6Ha_{22}^3$ which cannot vanish. This gives a contradiction.

Theorem 4.3 is proved.

5. Surfaces of Type II. Let α and β denote the isotropic planar and space curves given by, respectively,

$$\alpha(x) = (x, ax, f(x)), \qquad \beta(y) = (y, g(y), h(y)),$$

where $a \in \mathbb{R}$. Because the torsion of β is non-vanishing, we deduce from (2.2) that

$$g''h''' - g'''h'' \neq 0, \tag{5.1}$$

where $g' = \dfrac{dg}{dy}$, $h' = \dfrac{dh}{dy}$ and so on. Then the obtained translation surface belongs to Type II and has the form

$$r(x, y) = (x + y, ax + g(y), f(x) + h(y)). \tag{5.2}$$

The assumption (5.1) ensures the admissibility of (5.2), i.e., $g' - a \neq 0$. Hence, by a calculation, the Gaussian curvature K turns to

$$K = \frac{f''\left[h''(g' - a) - g''(h' - f')\right]}{(g' - a)^3}, \tag{5.3}$$

where $f' = \dfrac{df}{dx}$, etc.

Theorem 5.1. *A translation surface in $\mathbb{I}^3$ of the form* (5.2) *with CIGC* (K_0) *is a generalized cylinder with non-isotropic rulings, i.e.,* $K_0 = 0$.

Proof. If $K_0 \neq 0$, then (5.3) can be rewritten as

$$\frac{K_0}{f''} = \frac{h''}{(g' - a)^2} - \frac{g''}{(g' - a)^3}(h' - f'). \tag{5.4}$$

Taking partial derivative of (5.4) with respect to x yields

$$-K_0 \frac{f'''}{(f'')^3} = \frac{g''}{(g' - a)^3}$$

and solving this one

$$f(x) = \frac{1}{3c_1^2}(-2c_1 x + d_1)^{\frac{3}{2}} + d_2 x + d_3 \tag{5.5}$$

and

$$g(y) = \frac{1}{K_0 c_1}(2K_0 c_1 y + d_4)^{\frac{1}{2}} + ay + d_5. \tag{5.6}$$

Substituting (5.5) and (5.6) into (5.4) gives

$$0 = \frac{h''}{h' - d_2} + \frac{K_0 c_1}{2K_0 c_1 y + d_4}. \tag{5.7}$$

By solving (5.7), we find

$$h(y) = \frac{c_2}{K_0 c_1}(2K_0 c_1 y + d_4)^{\frac{1}{2}} + d_2 y + d_6. \tag{5.8}$$

Comparing (5.6) with (5.8) gives a contradiction due to (5.1). Next assume that $K_0 = 0$. If $f'' \neq 0$, we have $h''(g' - a) = g''(h' - f')$. Taking partial derivative of this one with respect to x yields the contradiction $g'' = 0$ due to (5.1). Henceforth only possibility is that $f'' = 0$, namely, α is a non-isotropic line.

Theorem 5.1 is proved.

By a direct calculation, the mean curvature H is

$$2H = \frac{\left[1 + (g')^2\right](g' - a)f'' + \left(1 + a^2\right)\left[h''(g' - a) - g''(h' - f')\right]}{(g' - a)^3}. \tag{5.9}$$

Theorem 5.2. *A translation surface in $\mathbb{I}^3$ of the form* (5.2) *cannot be isotropic minimal.*

Proof. We prove by contradiction. If $H = 0$, then (5.9) reduces to

$$\left[1 + (g')^2\right](g' - a)f'' + \left(1 + a^2\right)\left[h''(g' - a) - g''(h' - f')\right] = 0. \tag{5.10}$$

The partial derivative of (5.10) with respect to x yields

$$\left[1 + (g')^2\right](g' - a)f''' + \left(1 + a^2\right)g''f'' = 0. \tag{5.11}$$

We have to distinguish two cases:

Case 1: $f'' = 0$, i.e., $f(x) = d_1x + d_2$. By (5.10) we deduce

$$\frac{h''}{h' - d_1} = \frac{g''}{g' - a},$$

which implies

$$h = c_1g + (d_1 - ac_1)\,y - d_3.$$

This is not possible due to (5.1).

Case 2: $f'' \neq 0$. (5.11) implies

$$\frac{f'''}{f''} = -\frac{\left(1 + a^2\right)g''}{\left[1 + (g')^2\right](g' - a)}. \tag{5.12}$$

Hence, it follows from (5.12) that

$$f'' = c_1f' + d_1, \qquad \left[1 + (g')^2\right](g' - a)c_1 = -\left(1 + a^2\right)g''. \tag{5.13}$$

By considering (5.13) into (5.10), we conclude

$$0 = \frac{g''}{g' - a} - \frac{h''}{h' + \dfrac{d_1}{c_1}}. \tag{5.14}$$

Solving (5.14) gives

$$g = c_2h + \left(a + \frac{c_2d_1}{c_1}\right)y + d_2,$$

which is not possible due to (5.1).

Theorem 5.2 is proved.

Theorem 5.3. *For a translation surface in* $\mathbb{I}^3$ *of the form* (5.2) *with nonzero CIMC* (H_0) *one of the following occurs:*

(1) $\alpha(x) = (x, ax, d_1x + d_2)$ *and*

$$\beta(y) = \left(y, g, \frac{H_0}{1+a^2}(g-ay)^2 + d_1y + d_3(g-ay) + d_4\right),$$

(2) $\alpha(x) = (x, ax, c_1\exp(c_2x) + d_1x + d_2)$ *and*

$$\beta(y) = \left(y, g, \frac{H_0}{(1+a^2)}(g-ay)^2 - \frac{d_1}{c_2}y + d_3(g-ay) + d_4\right),$$

where $g = g(y)$ *is a non-linear function and*

$$g - ay \neq \frac{-1}{c_3}\sqrt{-2c_3y + d_5} + d_6. \tag{5.15}$$

Proof. We seperate the proof into two cases:

Case 1: $f'' = 0,\ f(x) = d_1x + d_2$. By substituting it into (5.9), we derive

$$\frac{2H_0}{1+a^2}(g'-a) = \left(\frac{h'-d_1}{g'-a}\right)'. \tag{5.16}$$

Twice integration in (5.16) leads to

$$h = \frac{H_0}{1+a^2}(g-ay)^2 + d_1y + d_3(g-ay) + d_4. \tag{5.17}$$

On the other hand, by (5.1) and (5.17) we deduce (5.15). This completes the proof of the item (1) of the theorem.

Case 2: $f'' \neq 0$. By taking partial derivative of (5.9) with respect to x, we obtain (5.11) again. It means that next steps are similar to those of Theorem 5.2. Thus, we have (5.13), namely,

$$f(x) = c_1\exp(c_2x) + d_1x + d_2 \tag{5.18}$$

and

$$\left[1 + (g')^2\right](g'-a)c_2 = -\left(1+a^2\right)g''. \tag{5.19}$$

Substituting (5.18) and (5.19) into (5.9), we conclude

$$\frac{2H_0}{(1+a^2)}(g'-a) = \frac{-d_1g''}{c_2(g'-a)^2} + \left(\frac{h'}{g'-a}\right)'. \tag{5.20}$$

Twice integration in (5.20) gives

$$h = \frac{H_0}{(1+a^2)}(g-ay)^2 - \frac{d_1}{c_2}y + d_3(g-ay) + d_4. \tag{5.21}$$

By (5.1) and (5.21), we deduce (5.15) again.

Theorem 5.3 is proved.

6. Surfaces of Type III. Let α and β denote the non-isotropic planar and space curves given by, respectively,

$$\alpha(x) = (x, f(x), ax), \qquad \beta(y) = (y, g(y), h(y)),$$

where $a \in \mathbb{R}$. Then the torsion of β is non-vanishing, namely, (2.2) implies

$$g''h''' - g'''h'' \neq 0, \tag{6.1}$$

where $g' = \dfrac{dg}{dy}$, $h' = \dfrac{dh}{dy}$ and so on. Hence, the surface obtained by a sum of α and β belongs to Type III and has the form

$$r(x, y) = (x + y, f(x) + g(y), ax + h(y)). \tag{6.2}$$

It implies from (6.1) that the surface is admissible, i.e., $g' - f' \neq 0$, $f' = \dfrac{df}{dx}$. By a calculation, the isotropic Gaussian curvature K is

$$K = -\frac{f''(h' - a)\,[h''(g' - f') - g''(h' - a)]}{(g' - f')^4}. \tag{6.3}$$

Theorem 6.1. *A translation surface in* $\mathbb{I}^3$ *of the form* (6.2) *with CIGC* (K_0) *is a generalized cylinder with non-isotropic rulings, namely,* $K_0 = 0$.

Proof. Assume that K is a nonzero constant K_0. Then we have $f'' \neq 0$ and the partial derivative of (6.3) with respect to x yields

$$\frac{4(g' - f')^3}{f''} + \frac{(g' - f')^4 f'''}{(f'')^3} = \frac{(h' - a)h''}{K_0}. \tag{6.4}$$

We have two cases:

Case 1: $f''' = 0$, $f'' = c_1$. Then from (6.4), we get

$$4K_0(g' - f')^3 = c_1(h' - a)h''. \tag{6.5}$$

The partial derivative of (6.5) with respect to x gives $f'' = 0$, which is not possible.

Case 2: $f''' \neq 0$. Taking partial derivative of (6.4) with respect to x and after dividing with $(g' - f')^2$ gives

$$-12 - 8(g' - f')\frac{f'''}{(f'')^2} + (g' - f')^2 \left(\frac{f'''}{(f'')^3}\right)' = 0. \tag{6.6}$$

This is a polynomial equation on g' and the leading coefficient $\left(\dfrac{f'''}{(f'')^3}\right)'$ coming from $(g')^2$ has to vanish. Thereby, (6.6) reduces to

$$-12 - 8(g' - f')\frac{f'''}{(f'')^2} = 0. \tag{6.7}$$

Taking partial derivative of (6.7) with respect to y implies $f''' = 0$, which is not our case.

The discussion above yields $K_0 = 0$. In that case, because β is a space curve, only possibility in (6.3) is $f'' = 0$, namely, α is a non-isotropic line.

Theorem 6.1 is proved.

By a direct calculation, the isotropic mean curvature is

$$2H = \frac{\left[1+(f')^2\right]\left[h''(g'-f') - g''(h'-a)\right] - \left[1+(g')^2\right](h'-a)f''}{(g'-f')^3}. \tag{6.8}$$

Theorem 6.2. *A translation surface in* $\mathbb{I}^3$ *of the form* (6.2) *cannot be isotropic minimal.*

Proof. We prove by contradiction. If the surface is isotropic minimal, then (6.8) reduces to

$$\left[1+(f')^2\right]\left[h''(g'-f') - g''(h'-a)\right] - \left[1+(g')^2\right](h'-a)f'' = 0. \tag{6.9}$$

We have two cases:

Case 1: $f'' = 0,\ f = d_1x + d_2$. Then (6.9) reduces to

$$\frac{h''}{h'-a} = \frac{g''}{g'-d_1}.$$

Solving last equation leads to a contradiction due to (6.1).

Case 2: $f'' \neq 0$. By dividing (6.9) with $\left[1+(g')^2\right]\left[1+(f')^2\right](h'-a)$, we derive

$$\frac{h''(g'-f')}{(h'-a)\left[1+(g')^2\right]} + \frac{f''}{1+(f')^2} - \frac{g''}{1+(g')^2} = 0. \tag{6.10}$$

The partial derivatives of (6.10) with respect to x and y yield

$$\frac{h''}{(h'-a)\left[1+(g')^2\right]} = c_1. \tag{6.11}$$

Substituting (6.11) into (6.10) gives

$$\frac{f''}{1+(f')^2} - c_1f' = d_1 \quad \text{and} \quad \frac{g''}{1+(g')^2} - c_1g' = d_1. \tag{6.12}$$

By considering the second equality in (6.12) into (6.11), we obtain

$$\frac{h''}{h'-a} = \frac{c_1g''}{c_1g'+d_1}. \tag{6.13}$$

Solving (6.13) implies a contradiction due to (6.1).

Theorem 6.2 is proved.

Theorem 6.3. *For a translation surface in* $\mathbb{I}^3$ *of the form* (6.2) *with nonzero CIMC* (H_0), *we have* $\alpha(x) = (x, ax, d_1x + d_2)$ *and*

$$\beta(y) = \left(y, g(y), \frac{H_0}{1+d_1^2}(g-d_1y)^2 + ay + d_3(g-d_1y) + d_4\right),$$

where $g = g(y)$ *is a non-linear function and*

$$g - d_1y \neq \frac{-1}{c_1}\sqrt{-2c_1y + d_5} + d_6. \tag{6.14}$$

Proof. We divide the proof into two cases:
Case 1: $f''=0$. Then $f(x)=d_1x+d_2$ and (6.8) reduces to

$$\frac{2H}{1+d_1^2}(g'-d_1)=\left(\frac{h'-a}{g'-d_1}\right)'. \tag{6.15}$$

After twice integration of (6.15), we obtain

$$h=\frac{H_0}{1+d_1^2}(g-d_1y)^2+ay+d_3\left(g-d_1y\right)+d_4. \tag{6.16}$$

By (6.1) and (6.16), we conclude (6.14) and, thus, the hypothesis of the theorem is obtained.

Case 2: $f''\neq 0$. By producting (6.8) with $\dfrac{(g'-f')^3}{1+(f')^2}$ and taking partial derivatives with respect to x and y, we have

$$12H_0\left[\frac{(g'-f')f''g''}{1+(f')^2}+\frac{(g'-f')^2f'f''g''}{[1+(f')^2]^2}\right]=$$

$$=h'''f''+\left\{2g'g''(h'-a)+\left[1+(g')^2\right]h''\right\}\left(\frac{f''}{1+(f')^2}\right)'. \tag{6.17}$$

By dividing (6.17) with $12H_0f''g''$ and putting

$$A(y)=2g'(h'-a)+\frac{1+(g')^2}{g''}h'',\qquad B(x)=\frac{\left\{f''/\left[1+(f')^2\right]\right\}'}{f''},$$

we conclude

$$\frac{g'-f'}{1+(f')^2}+\frac{(g'-f')^2f'}{[1+(f')^2]^2}=\frac{1}{12H_0}\left(\frac{h'''}{g''}+AB\right). \tag{6.18}$$

Taking partial derivative of (6.18) with respect to y gives

$$\frac{1}{1+(f')^2}+\frac{2(g'-f')f'}{[1+(f')^2]^2}=\frac{1}{12H_0}\left[\frac{(h'''/g'')'}{g''}+\frac{A'}{g''}B\right]. \tag{6.19}$$

Again taking partial derivative of (6.19) with respect to y and putting

$$C(y)=\left[\frac{(h'''/g'')'}{g''}\right]', \tag{6.20}$$

we deduce

$$\frac{f'}{[1+(f')^2]^2}=\frac{1}{24H_0}\left[\frac{C}{g''}+\frac{(A'/g'')'}{g''}B\right]. \tag{6.21}$$

The partial derivatives of (6.21) with respect to x and y yield

$$0=\left[\frac{(A'/g'')'}{g''}\right]'B'. \tag{6.22}$$

From (6.22), we have two possibilities:

(1) $B = \text{const}$. Taking partial derivative of (6.19) with respect to x and producting with $\dfrac{\left[1+(f')^2\right]^3}{f''}$ implies the following polynomial equation on f':

$$(f')^3 - 3g'(f')^2 - 3f' + g' = 0$$

which yields a contradiction.

(2) $B \neq \text{const}$. Then, from (6.22), we derive

$$A = d_1(g')^2 + d_2g' + d_3 \tag{6.23}$$

and, from (6.21), $C = d_4g''$. Hence, by (6.20), we deduce

$$\frac{h'''}{g''} = d_4(g')^2 + d_5g' + d_6. \tag{6.24}$$

Substituting (6.23) and (6.24) into (6.18) yields the polynomial equation on g':

$$\left\{\frac{f'}{[1+(f')^2]^2} - \frac{d_1B + d_4}{12H_0}\right\}(g')^2 + \left\{\frac{1-(f')^2}{[1+(f')^2]^2} - \frac{d_2B + d_5}{12H_0}\right\}g' +$$
$$+\frac{f'}{[1+(f')^2]^2} - \frac{d_3B + d_6}{12H_0} = 0,$$

in which the coefficients must be zero, namely,

$$\begin{aligned} \frac{f'}{[1+(f')^2]^2} &= \frac{d_4 + d_1B}{12H_0}, \\ \frac{1-(f')^2}{[1+(f')^2]^2} &= \frac{d_5 + d_2B}{12H_0}, \\ \frac{-f'}{[1+(f')^2]^2} &= \frac{d_6 + d_3B}{12H_0}. \end{aligned} \tag{6.25}$$

Because $f'' \neq 0$ none of $d_1,\ d_2,\ d_3$ can vanish. By using the first and the second equations in (6.25), we obtain the polynomial equation on f':

$$d_1 - d_1(f')^2 - d_2f' = \frac{d_1d_5 - d_2d_4}{12H_0}\left[1+(f')^2\right]^2,$$

which yields a contradiction.

Theorem 6.3 is proved.

7. Several remarks. 1. By considering the obtained results above it can be stated that there do not exist:

surfaces of Types I.3, II and III with non-zero CIGC;

isotropic minimal surfaces of Types II and III.

2. Isotropic minimal translation surfaces belong to the *family of isotropic Scherk surfaces*. When the translating curves lie in orthogonal planes, the members of this family are locally given by [36]

$$r(x,y) = \left(x, y, c\left[x^2 - y^2\right]\right),$$

$$r(x,z) = \left(x, \frac{1}{c}\log\left|\frac{cz}{\cos(cx)}\right|, z\right),$$

$$r(y,z) = \frac{1}{2}\left(\frac{1}{c}\log\left|\frac{\cos(cz)}{\cos(cy)}\right|, y - z + \pi, y + z\right), \quad c \in \mathbb{R} - 0.$$

When the translating curves are in arbitrary planes, the isotropic Scherk surfaces can be described in the following explicit forms:

$$z(x,y) = c\left[(a_{11}x + a_{12}y)^2 - \frac{a_{11}^2 + a_{12}^2}{a_{21}^2 + a_{22}^2}(a_{21}x + a_{22}z)^2\right] \quad \text{(see [2])},$$

$$y(x,z) = \frac{1}{c}\log\left|\frac{\cos\left(\dfrac{cx}{a_{11}}\right)}{c(a_{21}x + a_{22}z)}\right|,$$

$$y(x,z) = \frac{1}{c}\log\left|\frac{\cos\left(\dfrac{ca_{22}}{|a_{ij}|}[a_{11}x + a_{12}z]\right)}{\cos\left(\dfrac{ca_{12}}{|a_{ij}|}[a_{21}x + a_{22}z]\right)}\right|, \quad c \in \mathbb{R} - 0.$$

3. To classify surfaces of Type IV with arbitrary CIGC and CIMC is somewhat complicated, but still it could be a challenging open problem.

References

1. M. E. Aydin, *A generalization of translation surfaces with constant curvature in the isotropic space,* J. Geom., **107**, 603 – 615 (2016).
2. M. E. Aydin, M. Ergut, *Affine translation surfaces in the isotropic space,* Int. Electron. J. Geom., **10**, № 1, 21 – 30 (2017).
3. M. E. Aydin, I. Mihai, *On certain surfaces in the isotropic 4-space,* Math. Commun., **22**, 41 – 51 (2017).
4. B. Y. Chen, S. Decu, L. Verstraelen, *Notes on isotropic geometry of production models,* Kragujevac J. Math., **8**, № 1, 23 – 33 (2014).
5. J. G. Darboux, *Theorie generale des surfaces,* Livre I, Gauthier-Villars, Paris (1914).
6. L. C. B. Da Silva, *The geometry of Gauss map and shape operator in simply isotropic and pseudo-isotropic spaces,* J. Geom., **110**, № 31 (2019).
7. L. C. B. Da Silva, *Rotation minimizing frames and spherical curves in simply isotropic and pseudo-isotropic 3-spaces,* Tamkang J. Math., **51**, № 1, 31 – 52 (2020).
8. F. Dillen, W. Goemans, I. Van de Woestyne, *Translation surfaces of Weigarten type in 3-space,* Bull. Transilv. Univ. Brasov Ser. III, **1(50)**, 109 – 122 (2008).
9. F. Dillen, L. Verstraelen, G. Zafindratafa, *A generalization of the translation surfaces of Scherk,* Different. Geom. in Honor of Radu Rosca: Meeting on Pure and Appl. Different. Geom. (Leuven, Belgium, 1989), KU Leuven, Department Wiskunde (1991), p. 107 – 109.
10. F. Dillen, I. Van de Woestyne, L. Verstraelen, J. T. Walrave, *The surface of Scherk in E^3: A special case in the class of minimal surfaces defined as the sum of two curves,* Bull. Inst. Math. Acad. Sin., **26**, № 4, 257 – 267 (1998).
11. Z. Erjavec, B. Divjak, D. Horvat, *The general solutions of Frenet's system in the equiform geometry of the Galilean, pseudo-Galilean, simple isotropic and double isotropic space,* Int. Math. Forum., **6**, № 1, 837 – 856 (2011).
12. W. Goemans, I. Van de Woestyne, *Translation and homothetical lightlike hypersurfaces of semi-Euclidean space,* Kuwait J. Sci. Eng., **38**, № 2A, 35 – 42 (2011).

13. A. Gray, *Modern differential geometry of curves and surfaces with mathematica,* second ed., CRC Press (1998).
14. T. Hasanis, R. Lopez, *Translation surfaces in Euclidean space with constant Gaussian curvature,* Arxiv 8 Sept 2018: https://arxiv.org/abs/1809.02758v1.
15. T. Hasanis, R. Lopez, *Classification and construction of minimal translation surfaces in Euclidean space,* Arxiv 8 Sept 2018: https://arxiv.org/abs/1809.02759v1.
16. T. Hasanis, *Translation surfaces with non-zero constant mean curvature in Euclidean space,* J. Geom., **110**, № 20 (2019).
17. J. Inoguchi, R. Lopez, M. I. Munteanu, *Minimal translation surfaces in the Heisenberg group* Nil_3, Geom. Dedicata, **161**, 221 – 231 (2012).
18. S. D. Jung, H. Liu, Y. Liu, *Weingarten affine translation surfaces in Euclidean 3-space,* Results Math., **72**, № 4, 1839 – 1848 (2017).
19. B. P. Lima, N. L. Santos, P. A. Sousa, *Generalized translation hypersurfaces in Euclidean space,* J. Math. Anal. and Appl., **470**, № 2, 1129 – 1135 (2019).
20. H. Liu, *Translation surfaces with constant mean curvature in 3-dimensional spaces,* J. Geom., **64**, № 1-2, 141 – 149 (1999).
21. H. Liu, S. D. Jung, *Affine translation surfaces with constant mean curvature in Euclidean 3-space,* J. Geom., **108**, 423 – 428 (2017).
22. H. Liu, Y. Yu, *Affine translation surfaces in Euclidean 3-space,* Proc. Japan Acad. Ser. A, Math. Sci., **89**, 111 – 113 (2013).
23. R. Lopez, *Minimal translation surfaces in hyperbolic space,* Beitr. Algebra und Geom., **52**, № 1, 105 – 112 (2011).
24. R. Lopez, M. I. Munteanu, *Minimal translation surfaces in* Sol_3, J. Math. Soc. Japan, **64**, № 3, 985 – 1003 (2012).
25. R. Lopez, M. Moruz, *Translation and homothetical surfaces in Euclidean space with constant curvature,* J. Korean Math. Soc., **52**, № 3, 523 – 535 (2015).
26. R. Lopez, O. Perdomo, *Minimal translation surfaces in Euclidean space,* J. Geom. Anal., **27**, № 4, 2926 – 2937 (2017).
27. Z. Milin-Sipus, *Translation surfaces of constant curvatures in a simply isotropic space,* Period. Math. Hung., **68**, 160 – 175 (2014).
28. M. Moruz, M. I. Munteanu, *Minimal translation hypersurfaces in* E^4, J. Math. Anal. and Appl., **439**, № 2, 798 – 812 (2016).
29. M. I. Munteanu, O. Palmas, G. Ruiz-Hernandez, *Minimal translation hypersurfaces in Euclidean spaces,* Mediterr. J. Math., **13**, 2659 – 2676 (2016).
30. A. O. Ogrenmis, *Rotational surfaces in isotropic spaces satisfying Weingarten conditions,* Open Phys., **14**, № 9, 221 – 225 (2016).
31. H. Pottmann, P. Grohs, N. J. Mitra, *Laguerre minimal surfaces, isotropic geometry and linear elasticity,* Adv. Comput. Math., **31**, 391 – 419 (2009).
32. H. Pottmann, K. Opitz, *Curvature analysis and visualization for functions defined on Euclidean spaces or surfaces,* Comput. Aided Geom. Design, **11**, 655 – 674 (1994).
33. H. Sachs, *Isotrope Geometrie des Raumes,* Vieweg Verlag, Braunschweig (1990).
34. H. F. Scherk, *Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen,* J. reine und angew. Math., **13**, 185 – 208 (1835).
35. K. Seo, *Translation hypersurfaces with constant curvature in space forms,* Osaka J. Math., **50**, 631 – 641 (2013).
36. K. Strubecker, *Über die isotropen Gegenstucke der Minimalflache von Scherk,* J. reine und angew. Math., **293**, 22 – 51 (1977).
37. H. Sun, *On affine translation surfaces of constant mean curvature,* Kumamoto J. Math., **13**, 49 – 57 (2000).
38. L. Verstraelen, J. Walrave, S. Yaprak, *The minimal translation surfaces in Euclidean space,* Soochow J. Math., **20**, 77 – 82 (1994).
39. D. Yang, Y. Fu, *On affine translation surfaces in affine space,* J. Math. Anal. and Appl., **440**, № 2, 437 – 450 (2016).
40. D. Yang, J. Zhang, Y. Fu, *A note on minimal translation graphs in Euclidean space,* Mathematics, **7**, № 10 (2019).

Received 02.02.17