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THE BOLTZMANN–ENSKOG LIMIT FOR EQUILIBRIUM STATES OF SYSTEMS OF HARD SPHERES IN FRAMEWORK OF CANONICAL ENSEMBLE *

ГРАНИЦЯ БОЛЬЦМАНА–ЕНСКОГА ДЛЯ РІВНОВАЖНИХ СТАНІВ СИСТЕМ ПРУЖНИХ КУЛЬ В РАМКАХ КАНОНІЧНОГО АНСАМБЛЮ

We prove the existence of the Boltzmann–Enskog limit for an equilibrium system of hard spheres. On the basis of analysis of the Kirkwood–Salsburg equations, we show that the limit distribution functions are constants, which can be represented as series in density.

Доведено існування границі Больцмана–Енскога для рівноважної системи пружних куль. На основі аналізу співвідошень Кірквуда–Зальцбурга показано, що граничні функції розподілу є константами, що виражаються рядами за густиною.

Introduction. The problem of derivation and mathematical justification of Boltzmann – Enskog equation is very important and permanently attracts attention of mathematical physicists. In the Boltzmann – Enskog equation, the size of particles is taken into account, while in the Boltzmann equations, point-wise particles are considered.

The problem is as follows: to derive the Boltzmann – Enskog equation from the BBGKY (Bogolubov – Born – Green – Kirkwood – Yvon) hierarchy — the fundamental equations of classical statistical mechanics. This problem is very difficult and it is natural to try to solve it in a more simple situation for equilibrium states of hard spheres. First this problem was considered in framework of grand canonical ensemble in papers [1, 2]. It was shown that the equilibrium distribution function exists in the Boltzmann – Enskog limit when activity z (or density $1/v$) tends to infinity, and the diameter of spheres a tends to zero in such a way that za^3 (a^3/v) is constant.

In this paper, we consider an analogous problem in the framework of canonical ensemble.

It is proved that the equilibrium distribution functions normalized on unity tend to zero in the Boltzmann – Enskog limit, i.e., when the number of particles N tends to infinity, the volume $V(\Lambda)$ of the region Λ tends to infinity, and the diameter of spheres tends to zero in such a way that $Na^3 = \Lambda = \text{const}$, $N/V(\Lambda) = 1/v = \text{const}$. The nonzero limit distribution functions with this normalization can be obtained only for the systems of particles located in a bounded domain.

The system of particles located in the whole space was also investigated for the case of the distribution functions normalized to the number of particles. In this case the limit distribution functions are nonzero only if the diameter of spheres tends to zero and their density tends to infinity so that $a^3/v = \text{const}$.

In this paper, we extensively used the results of papers [1, 2], where the Boltzmann – Enskog limit was investigated in the framework of grand canonical ensemble and the results of paper [3], where the Boltzmann – Grad limit was investigated in the framework of canonical ensemble.

1. The existence of the Boltzmann – Enskog limit for distribution functions normalized on unity.

1. Consider N hard spheres in a region $\Lambda \subset \mathbb{R}^3$ with volume $V(\Lambda) = V$. Denote by $D^{(N)}(q_1, \ldots, q_N)$ an equilibrium distribution function (an equilibrium state) in configurational space, with inverse temperature β.

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The function $D^{(N)}(q_1, \ldots, q_N)$ is defined as follows:

$$D^{(N)}(q_1, \ldots, q_N) = \frac{1}{Q(N, \Lambda)} \exp\left\{-\beta \sum_{i<j=1}^{N} \Phi(q_i - q_j)\right\},$$

$$Q(N, \Lambda) = \int_{\Lambda^n} \exp\left\{-\beta \sum_{i<j=1}^{N} \Phi(q_i - q_j)\right\} dq_1 \ldots dq_N, \qquad (1)$$

where Φ is the interaction potential of hard spheres with diameter a,

$$\Phi(q) = \begin{cases} \infty, & |q| \leq a, \\ 0, & |q| > a. \end{cases} \qquad (2)$$

The function $D^{(N)}(q_1, \ldots, q_N)$ is equal to zero if $q \notin \Lambda$ for at least one number $i \in$ $\in (1, \ldots, N)$. For the sake of simplicity, we suppose that Λ is a sphere centered at the origin of the coordinate system.

Let us define a sequence of reduced distribution functions

$$F_s^{(N)}(q_1, \ldots, q_s) =$$
$$= \int_{\Lambda^{N-s}} D^{(N)}(q_1, \ldots, q_s, q_{s+1}, \ldots, q_N)\, dq_{s+1} \ldots dq_N, \quad 1 \leq s \leq N,$$
$$F^{(N)} = \left(F_1^{(N)}(q_1), \ldots, F_s^{(N)}(q_1, \ldots, q_s), \ldots, F_N^{(N)}(q_1, \ldots, q_N), 0, \ldots\right). \qquad (3)$$

Sequence (3) satisfies the following Kirkwood – Salsburg relations [3, 4]:

$$F_s^{(N)}(q_1, \ldots, q_s) = \frac{Q(N-1, \Lambda)}{Q(N, \Lambda)} \exp\left\{-\beta \sum_{i=2}^{s} \Phi(q_1 - q_i)\right\}\Bigg[F_{s-1}^{(N-1)}(q_2, \ldots, q_s) +$$
$$+ \sum_{k=1}^{N-s} \frac{1}{k!} \prod_{j=1}^{k-1} (N-s-j) \int_{\Lambda^k} \prod_{i=1}^{k} \varphi_{q_1}(y_i) \times$$
$$\times F_{s-1+k}^{(N-1)}(q_2, \ldots, q_s, y_1, \ldots, y_k)\, dy_1 \ldots dy_k\Bigg], \quad 1 < s < N,$$
$$\varphi_{q_1}(y) = \exp\{-\beta\Phi(q_1 - y)\} - 1,$$
$$F_0^{(N-1)} = 1, \qquad (4)$$
$$F_1^{(N)}(q_1) = \frac{Q(N-1, \Lambda)}{Q(N, \Lambda)}\Bigg[1 + \sum_{k=1}^{N-s} \frac{1}{k!} \prod_{j=0}^{k-1} (N-1-j) \times$$
$$\times \int_{\Lambda^k} \prod_{i=1}^{k} \varphi_{q_1}(y_i)\, F_k^{(N-1)}(y_1, \ldots, y_k)\, dy_1 \ldots dy_k\Bigg],$$
$$F_N^{(N)}(q_1, \ldots, q_N) =$$
$$= \frac{Q(N-1, \Lambda)}{Q(N, \Lambda)} \exp\left\{-\beta \sum_{i=2}^{N} \Phi(q_1 - q_i)\right\} F_{N-1}^{(N-1)}(q_2, \ldots, q_N).$$

Relations (4) hold for $N > 2$. For $N = 2$, one has

$$F_1^{(2)}(q_1) = \frac{\int_\Lambda \exp\{-\beta\,\Phi(q_1 - q_2)\}\,dq_2}{\int_{\Lambda^2} \exp\{-\beta\,\Phi(q_1 - q_2)\}\,dq_1\,dq_2},$$

$$F_2^{(2)}(q_1, q_2) = \frac{\exp\{-\beta\,\Phi(q_1 - q_2)\}}{\int_{\Lambda^2} \exp\{-\beta\,\Phi(q_1 - q_2)\}\,dq_1\,dq_2}. \tag{5}$$

Let $\Lambda \nearrow \mathbb{R}^3$, $V(\Lambda) \to \infty$, $N \to \infty$, in such a way that $N/V(\Lambda) = 1/v = \text{const}$. This procedure is known as the thermodynamic limit transition. We suppose that the diameter a also tends to zero in such a way that $Na^3 = \text{const}$. These two limiting procedures are known as the Boltzmann–Enskog limit.

The goal of this article is to prove the existence of the sequence of distribution functions in the Boltzmann–Enskog limit. For this purpose, we use the Kirkwood–Salsburg relations. First, we define the numbers $\lambda = \lambda_0 = Na^3$, $\lambda_1 = (N-1)a^3, \ldots, \lambda_i = (N-i)a^3, \ldots$, and represent relations (4) as follows:

$$F_s^{(N)}((q)_s) = a(N,\Lambda)\exp\left\{-\beta\sum_{i=2}^{s}\Phi(q_1 - q_i)\right\}\Bigg[F_{s-1}^{(N-1)}((q)_s^1) + \\ + \sum_{k=1}^{N-s}\frac{1}{k!}\prod_{j=0}^{k-1}\frac{\lambda_{s+j}}{a^3}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)\times \\ \times F_{s-1+k}^{(N-1)}((q)_s^1,(y)_k)\,d(y)_k\Bigg], \quad 1 < s < N,$$

$$F_1^{(N)}(q_1) = a(N,\Lambda)\left[1 + \sum_{k=1}^{N-1}\frac{1}{k!}\prod_{j=0}^{k-1}\frac{\lambda_{1+j}}{a^3}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)\,F_k^{(N-1)}((y)_k)\,d(y)_k\right], \tag{6}$$

$$F_N^{(N)}((q)_N) = a(N,\Lambda)\exp\left\{-\beta\sum_{i=2}^{N}\Phi(q_1 - q_i)\right\}F_{N-1}^{(N-1)}((q)_N^1),$$

$$(q)_s = (q_1, \ldots, q_s), \quad (q)_s^1 = (q_2, \ldots, q_s),$$

$$(y)_k = (y_1, \ldots, y_k), \quad d(y)_k = dy_1 \ldots dy_k,$$

$$a(N,\Lambda) = \frac{Q(N-1,\Lambda)}{Q(N,\Lambda)}.$$

2. Consider the Banach space E_ξ of sequences of bounded functions $f_s((q)_s)$,

$$f = (f_1(q_1), \ldots, f_s((q)_s), \ldots) \tag{7}$$

with the norm

$$\|f\| = \sup_{s\ge 1}\frac{1}{\xi^s}\sup_{(q)_s}|f_s((q)_s)|, \quad \xi > 0.$$

Define an operator $K^{(N)}$ in E_ξ in the following way:

$$(K^{(N)}f)_s((q)_s) = \exp\left\{-\beta\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\chi_\Lambda((q)_s)\Bigg[f_{s-1}((q)_s^1) + \\ + \sum_{k=1}^{N-s}\frac{1}{k!}\prod_{j=0}^{k-1}\frac{\lambda_{s+j}}{a^3}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)\times \\ \times f_{s-1+k}((q)_s^1,(y)_k)\,d(y)_k\Bigg], \quad 1<s<N,$$

$$(K^{(N)}f)_1((q)_1) = \chi_\Lambda((q)_1)\sum_{k=1}^{N-s}\frac{1}{k!}\prod_{j=0}^{k-1}\frac{\lambda_{1+j}}{a^3}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)\,f_k((y)_k)\,d(y)_k, \tag{8}$$

$$(K^{(N)}f)_N((q)_N) = \chi_\Lambda((q)_N)\exp\left\{-\beta\sum_{i=2}^{N}\Phi(q_1-q_i)\right\}f_{N-1}((q)_N^1),$$

$$(K^{(N)}f)_s((q)_s) = 0, \quad s>N,$$

where $\chi_\Lambda((q)_s)$ is the characteristic function of Λ^s.

We have the following estimate for the norm of the operator $K^{(N)}$:

$$\|K^{(N)}\| \le \sup_{s\ge 1}\xi^{-1}\sum_{k=0}^{N-s}\frac{1}{k!}\left(\frac{4}{3}\pi\lambda\xi\right)^k \le \xi^{-1}\exp\left(\frac{4}{3}\pi\xi\lambda\right). \tag{9}$$

The minimum of the right-hand side of (9) is reached for $\xi = 3/4\pi\lambda$ and it implies

$$\|K^{(N)}\| \le \frac{4}{3}\pi\lambda e. \tag{9$'$}$$

It follows from (9′) that the operator $K^{(N)}$ is defined and bounded in E_ξ.

Estimate the value $a(N,\Lambda)$: By using the inequality [3]

$$Q(N,\Lambda) \ge Q(N-1,\Lambda)\left[V(\Lambda) - (N-1)\frac{4}{3}\pi a^3\right] \ge \\ \ge Q(N-1,\Lambda)\left[V(\Lambda) - \frac{4}{3}\lambda\right], \tag{10}$$

we obtain

$$a(N,\Lambda) = \frac{1}{V(\Lambda) - 4\lambda/3}, \tag{11}$$

which implies that $a(N,\Lambda)\to 0$ in the Boltzmann – Enskog limit.

With the use of the operator $K^{(N)}$, relations (6) can be represented as a single operator relation

$$F^{(N)} = a(N,\Lambda)(K^{(N)}F^{(N-1)} + F_0), \tag{12}$$

where $F_0 = (1, 0, \dots)$.

Consider the problem of existence of the Boltzmann – Enskog limit for the sequence $F^{(N)}$.

Theorem 1. *The sequence* $F^{(N)}$ *tends to zero in the norm of the space* E_ξ *in the Boltzmann–Enskog limit.*

Proof. Consider relations (12) for $N, N-1, \dots, 3$. By using them, one gets

$$F^{(N)} = \sum_{i=1}^{N-3} a(N,\Lambda) K^{(N)} a(N-1,\Lambda) K^{(N-1)} \dots$$

$$\dots a(N-i+1,\Lambda) K^{(N-i+1)} a(N-i,\Lambda) F_0 +$$

$$+ a(N,\Lambda) K^{(N)} a(N-1,\Lambda) K^{(N-1)} \dots a(3,\Lambda) K^{(3)} F^{(2)} + a(N,\Lambda) F_0 . \qquad (13)$$

It follows for estimates (9′) and (11) that

$$\| K^{(N-i+1)} \| \, a(N-i,\Lambda) \le k < 1, \quad 1 \le i \le N-3, \qquad (14)$$

for sufficiently large $V(\Lambda)$ and N, and small a. It is easy to see from (5) that

$$F_1^{(2)}(q_1) < \frac{V(\Lambda)}{V(\Lambda)^2 - 4\pi a^3 V(\Lambda)/3},$$

$$F_2^{(2)}(q_1, q_2) < \frac{1}{V(\Lambda)^2 - 4\pi a^3 V(\Lambda)/3} \qquad (15)$$

and, thus, $\| K^{(3)} \| \, \| F^{(2)} \| \le k < 1$ for sufficiently large $V(\Lambda)$. Then it follows form the estimate

$$\| F^{(N)} \| \le a(N,\Lambda) \sum_{i=0}^{N-2} k^i \le a(N,\Lambda) \frac{1}{1-k} \qquad (16)$$

that $\| F^{(N)} \| \to 0$ in the Boltzmann–Enskog limit, i.e.,

$$N \to \infty, \quad V(\Lambda) \to \infty, \quad \frac{N}{V(\Lambda)} = \frac{1}{v}, \quad Na^3 = \lambda.$$

In order to avoid the phenomenon of tending of the distribution functions to zero, we fix the bounded domain Λ and consider the Boltzmann–Enskog limit when only $N \to \infty$, $a \to 0$, $Na^3 = \lambda$, but Λ is fixed.

Denote $a(N-i,\Lambda)$ by $a_i(N)$ because Λ is fixed, $0 \le i \le N-3$, $a_0(N) \equiv a(N)$.

The sequence $a_i(N)$ is bounded according to (11) (for fixed i). Thus, one can select a convergent subsequence $a_i(N_j)$

$$\lim_{N_j \to \infty} a_i(N_j) = A_i, \quad A_0 \equiv A.$$

By using the diagonal procedure, one can do this for all i. Also the following limits exist for all $i = 0, 1, 2, \dots$:

$$\lim_{N_j \to \infty} a_i(N_j) = A_i. \qquad (17)$$

This does not mean that there are no other convergent subsequences with limits $A_i^{(1)}, \dots, A_i^{(k)}, \dots$, $i = 0, 1, 2, \dots$. Restrict ourselves to the subsequence $a_i(N_j)$ (17). In this notation, relations (12) take the form

$$F^{(N-i)} = a_i(N) \left(K^{(N-i)} F^{(N-i-1)} + F_0 \right). \qquad (18)$$

Represent the operator $K^{(N)}$ in the following form:

$$K^{(N)} = K_1^{(N)} + K_2^{(N)},$$

where

$$(K_1^{(N)} f)_s((q)_s) = \chi_\Lambda((q)_s) \exp\left\{-\beta \sum_{i=2}^{s} \Phi(q_1 - q_i)\right\} f_{s-1}((q)_s^1), \quad 1 < s \le N,$$

$$(K_2^{(N)} f)_s((q)_s) = \chi_\Lambda((q)_s) \exp\left\{-\beta \sum_{i=2}^{s} \Phi(q_1 - q_i)\right\} \times$$

$$\times \sum_{k=1}^{N-s} \frac{1}{k!} \prod_{j=1}^{k-1} \frac{\lambda_{s+j}}{a^3} \int_{\Lambda^k} \prod_{i=1}^{k} \varphi_{q_1}(y_i) f_{s-1+k}((q)_s^1, (y)_k)\, d(y)_k, \tag{19}$$

$$(K_1^{(N)} f)_s((q)_s) = 0, \quad s > N,$$

$$(K_2^{(N)} f)_s((q)_s) = 0, \quad s > N-1.$$

The operators $K_1^{(N)}$, $K_2^{(N)}$ are bounded in E_ξ and the following estimates hold:

$$\|K_1^{(N)}\| \le \xi^{-1} = \frac{4}{3}\pi\lambda,$$

$$\|K_2^{(N)}\| \le \xi^{-1} \exp\left(\frac{4}{3}\pi\lambda\xi\right) = \frac{4}{3}\pi\lambda e. \tag{20}$$

By using relations (18), we get

$$F^{(N)} = \sum_{i=1}^{N-3} a(N) K^{(N)} a(N-1) K^{(N-1)} \ldots$$

$$\ldots a(N-i+1) K^{(N-i+1)} a(N-i) F_0 +$$

$$+ a(N) K^{(N)} a(N-1) K^{(N-1)} \ldots a(3) K^{(3)} F^{(2)} + a(N) F_0. \tag{21}$$

Consider the function

$$(K^{(N)} K^{(N-1)} \ldots K^{(N-i+1)} F_0)_s((q)_s), \tag{22}$$

which is the sum of products of operators $K_1^{(N-i)}$ and $K_2^{(N-i)}$ acting on F_0. This function is expressed via sums of integrals with integrands equal to the product of the kernels

$$K_a(q_j, (y)_m) = \prod_{i=1}^{m} (\exp(-\beta\Phi(q_j - y_i)) - 1), \quad 1 \le j \le s,$$

and the factors

$$\exp\left[-\beta \sum_{i \ne j} \Phi(q_j - q_i)\right],$$

where some q_i are equal to the variables $(y)_m$. The integration is carried out over the domain Λ with respect to the variables $(y)_m$. The kernel $K_a(q_j, (y)_m)$ is different from zero if $|q_j - y_i| \le a$ for all $i = 1, 2, \dots, m$; the factor is different from zero if $|q_j - q_i| > a$, $i \ne j$. Consider a compact domain $D_{a_0} \in \Lambda$ which contains the points $(q)_s$ satisfying the conditions $|q_j - q_i| > a_0$, $(i, j) \in (1, \dots, s)$, where a_0 is an arbitrary fixed number, $a_0 > a$.

Lemma. *For arbitrary small fixed $a_0 > a$ and $(q)_s \in D_{a_0}$, the distribution functions in an arbitrary n th order of the perturbation theory*

$$F_{s,i}((q)_s) = (a(N)K^{(N)}a(N-1)K^{(N-1)}\dots$$
$$\dots a(N-i+1)K^{(N-i+1)}a(N-i)F_0)((q)_s)$$

are independent of a and constants with respect to $(q)_s$ for sufficiently small a.

Proof. An analogous lemma was proved in the paper [2], the only difference is that, instead of a bounded domain Λ, the whole space $\mathbb{R}^3$ was used in it. In our case, functions (22) do not depend on the size of domain Λ because, in virtue of the definition of the support of the kernels K_a, the integrands in (22) are different from zero in spheres with diameter na centered at the points $(q_1, \dots, q_s)$. Here, n is some number less than i. It is obvious that, for sufficiently large Λ, the support of the integrand is completely situated in Λ and function (22) does not depend on the size of Λ. One can put $\Lambda = \mathbb{R}^3$. After these remarks, the proof of our lemma follows directly from the corresponding lemma (Lemma 1) of the paper [2].

Theorem 2. *The distribution functions $F^{(N)}((q)_s)$ tend to constants in the Boltzmann–Enskog limit when $N \to \infty$, $a \to 0$, $Na^3 = \lambda = \text{const}$, and domain Λ is fixed.*

Proof. In Lemma 1, we have established that the function (22) tends to a constant uniformly with respect to $(q)_s \in D_{a_0}$. According to (16), the sequences $a_i(N_j)$ tend to A_i, $i = 1, 2, \dots$. Hence, an arbitrary term in (21)

$$(a(N)K^{(N)}a(N-1)K^{(N-1)}\dots a(N-i+1)K^{(N-i+1)}a(N-i)F_0)_s((q)_s)$$

tends to the constant $F_{s,i}$ uniformly with respect $(q)_s \in D_{a_0}$.

Consider two series

$$F_s^{(N)}((q)_s) =$$

$$= \sum_{i=1}^{N-3}(a(N)K^{(N)}a(N-1)K^{(N-1)}\dots a(N-i+1)K^{(N-i+1)}a(N-i)F_0)_s((q)_s) +$$

$$+ (a(N)K^{(N)}a(N-1)K^{(N-1)}\dots a(3)K^{(3)}F^{(2)})_s((q)_s) + (a(N)F_0)_s((q)_s), \quad (23)$$

$$F_s = \sum_{i=0}^{\infty} F_{s,i}. \quad (24)$$

For sufficiently large Λ and chosen ξ, both series are convergent (the first one uniformly with respect to $(q)_s$) because the estimates of the norms of the operators K^{N-i} are independent of N, Λ, n. Hence, for arbitrary small $\varepsilon > 0$, there exists a number i_0 such that

$$\left| \sum_{i \geq i_0} (a(N) K^{(N)} a(N-1) K^{(N-1)} \ldots a(N-i+1) K^{(N-i+1)} a(N-i) F_0)_s((q)_s) + \right.$$

$$\left. + (a(N) K^{(N)} a(N-1) K^{(N-1)} \ldots a(3) K^{(3)} F^{(2)})_s((q)_s) \right| < \varepsilon, \qquad (25)$$

$$\left| \sum_{i \geq i_0} F_{s,i} \right| < \varepsilon.$$

It follows from the lemma that

$$\sum_{i=1}^{i_0} (a(N) K^{(N)} a(N-1) K^{(N-1)} \ldots a(N-i+1) K^{(N-i+1)} a(N-i) F_0)_s((q)_s) +$$

$$+ (a(N) F_0)_s((q)_s) \to \sum_{i=0}^{i_0} F_{s,i}$$

in the Boltzmann – Enskog limit uniformly with respect to for $(q)_s \in D_{a_0}$. This means that, for arbitrary small $\varepsilon > 0$, there exist sufficiently large N and sufficiently small a such that, for $(q)_s \in D_{a_0}$,

$$|F_s^{(N)}((q)_s) - F_s| < \varepsilon,$$

i.e.,

$$\lim_{\substack{N \to \infty, \ a \to 0, \\ Na^3 = \lambda}} F_s^{(N)}((q)_s) = F_s$$

uniformly with respect to $(q)_s \in D_{a_0}$.

Show that constant F_s does not depend on the subsequences $a_i(N_j)$, i.e., that the limits F_s are unique. Indeed, the functions $F_s^{(N)}((q)_s)$ satisfy the following normalization conditions:

$$\int_{\Lambda^s} F_s^{(N)}((q)_s) \, d(q)_s = 1.$$

By using the Lebesgue theorem and performing the Boltzmann – Enskog limit under the integral sign, we get

$$F_s = \frac{1}{V(\Lambda)^s}, \qquad s = 1, 2, \ldots .$$

Thus, the constant F_s does not depend on the subsequence N_j and is unique.

2. The existence of the Boltzmann – Enskog limit and the thermodynamic limit for distribution functions standardly normalized on numbers of particles. Consider the equilibrium distribution functions normalized as follows

$$F_s^{(N)}(q_1, \ldots, q_s) =$$

$$= \prod_{j=0}^{s-1} (N-j) \int_{\Lambda^{N-s}} D^{(N)}(q_1, \ldots, q_s, q_{s+1}, \ldots, q_N) \, dq_{s+1} \ldots dq_N, \qquad s = 1, 2, \ldots, N. \qquad (26)$$

They satisfy the following Kirkwood – Salsburg relations:

$$F_s^{(N)}((q)_s) = \frac{NQ(N-1,\Lambda)}{Q(N,\Lambda)} \exp\left\{-\beta \sum_{i=2}^{s} \Phi(q_1 - q_i)\right\}\left[F_{s-1}^{(N-1)}((q)_s^1) + \right.$$

$$\left. + \sum_{k=1}^{N-s} \frac{1}{k!} \int_{\Lambda^k} \prod_{i=1}^{k} \varphi_{q_1}(y_i) F_{s-1+k}^{(N-1)}((q)_s^1, (y)_k)\, d(y)_k \right], \quad 1 < s < N,$$

$$F_0^{(N-1)} = 1, \tag{27}$$

$$F_1^{(N)}(q_1) = \frac{NQ(N-1,\Lambda)}{Q(N,\Lambda)} \left[1 + \sum_{k=1}^{N-1} \frac{1}{k!} \int_{\Lambda^k} \prod_{i=1}^{k} \varphi_{q_1}(y_i)\, F_k^{(N-1)}((y)_k)\, d(y)_k \right],$$

$$F_N^{(N)}((q)_N) = \frac{NQ(N-1,\Lambda)}{Q(N,\Lambda)} \exp\left\{-\beta \sum_{i=2}^{N} \Phi(q_1 - q_i)\right\} F_{N-1}^{(N-1)}((q)_N^1).$$

Define $a(N,\Lambda)$ by the ratio

$$a(N,\Lambda) = \frac{NQ(N-1,\Lambda)}{Q(N,\Lambda)} \tag{28}$$

(the value $a(N,\Lambda)$ in (28) differs from the value $a(N,\Lambda)$ in (6)) by the multiplier N and introduce the renormalized distribution functions

$$\tilde{F}_s^{(N)}((q)_s) = a^{3s} F_s^{(N)}((q)_s). \tag{29}$$

They satisfy the following Kirkwood – Salsburg relations (in what follows, for the sake of simplicity, we preserve the previous notation for the renormalized distribution functions):

$$F_s^{(N)}((q)_s) = a(N,\Lambda) a^3 \exp\left\{-\beta \sum_{i=2}^{s} \Phi(q_1 - q_i)\right\}\left[F_{s-1}^{(N-1)}((q)_s^1) + \right.$$

$$\left. + \sum_{k=1}^{N-s} \frac{1}{k!} \frac{1}{a^{3k}} \int_{\Lambda^k} \prod_{i=1}^{k} \varphi_{q_1}(y_i)\, F_{s-1+k}^{(N-1)}((q)_s^1, (y)_k)\, d(y)_k \right], \quad 1 < s < N,$$

$$F_1^{(N)}(q_1) = \tag{30}$$

$$= a(N,\Lambda) a^3 \left[1 + \sum_{k=1}^{N-1} \frac{1}{k!} \frac{1}{a^{3k}} \int_{\Lambda^k} \prod_{i=1}^{k} \varphi_{q_1}(y_i)\, F_k^{(N-1)}((y)_k)\, d(y)_k \right],$$

$$F_N^{(N)}((q)_N) = a(N,\Lambda) a^3 \exp\left\{-\beta \sum_{i=2}^{N} \Phi(q_1 - q_i)\right\} F_{N-1}^{(N-1)}((q)_N^1),$$

$$F_0^{(N-1)} = 1.$$

Relations (30) can be represented as a single operator relation

$$F^{(N)} = a_i(N,\Lambda) a^3 [K^{(N)} F^{(N-1)} + F^0], \tag{31}$$

where the operator $K^{(N)}$ is defined in the space E_ξ as follows:

$$(K^{(N)}f)_s((q)_s) = \chi_\Lambda((q)_s)\exp\left\{-\beta\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\left[f_{s-1}((q)_s^1) + \right.$$

$$\left. + \sum_{k=1}^{N-s}\frac{1}{k!}\frac{1}{a^{3k}}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)\, f_{s-1+k}((q)_s^1,(y)_k)\,d(y)_k\right], \quad 1<s<N,$$

$$(K^{(N)}f)_1((q)_1) = \chi_\Lambda((q)_1)\sum_{k=1}^{N-1}\frac{1}{k!}\frac{1}{a^{3k}}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)\, f_k((y)_k)\,d(y)_k, \tag{32}$$

$$(K^{(N)}f)_N((q)_N) = \chi_\Lambda((q)_N)\exp\left\{-\beta\sum_{i=2}^{N}\Phi(q_1-q_i)\right\} f_{N-1}((q)_N^1),$$

$$(K^{(N)}f)_s((q)_s) = 0, \quad s>N, \quad f_0=0.$$

The operator $K^{(N)}$ is defined and bounded in E_ξ

$$\| K^{(N)}\| \le \xi^{-1}\exp\left(\frac{4}{3}\pi\xi\right), \quad \xi^{-1} = \frac{4}{3}\pi. \tag{33}$$

By analogy with (10), (11), the following estimate can be proved for $a(N,\Lambda)a^3$:

$$a(N,\Lambda)a^3 < \frac{Na^3}{V(\Lambda)-(N-1)4\pi a^3/3} \le \frac{1}{V(\Lambda)-4\lambda/3}. \tag{34}$$

It implies that $a(N,\Lambda)a^3 \to 0$ as $V(A)\to\infty$.

Theorem 3. *The sequence $F^{(N)}$ tends to zero in the norm of the space E_ξ in the Boltzmann–Enskog limit.*

Proof. As in Theorem 1, the proof follows from estimates (33) and (34).

If the domain Λ is fixed, then an analog of Theorem 2 can easily be proved. In order to obtain a nonzero limit $F^{(N)}$ when $N\to\infty$, $V(\Lambda)\to\infty$, $N/V(\Lambda) = 1/v$, $a\to 0$, we consider the case where $1/v\to\infty$ in such a way that $a^3/v = \text{const}$ instead of $Na^3 = \lambda = \text{const}$.

Theorem 4. *The renormalized distribution functions $F_s^{(N)((q)_s)}$ tend to constants in the Boltzmann–Enskog limit*

$$N\to\infty, \quad V(\Lambda)\to\infty, \quad \frac{N}{V(\Lambda)} = \frac{1}{v}\to\infty,$$

$$a\to 0, \quad \frac{1}{v}a^3 = \lambda < \frac{3}{4\pi}.$$

Proof. The value $a(N,\Lambda)a^3$ satisfies the inequality

$$a(N,\Lambda)a^3 < \frac{Na^3}{V(\Lambda)-(N-1)4\pi a^3/3} <$$

$$< \frac{Na^3}{V(\Lambda)-N4\pi a^3/3} = \frac{1}{v}a^3\frac{1}{1-4\pi a^3/3v}. \tag{35}$$

The sequence $a(N-i,\Lambda)a^3$, $i = 1, 2, \ldots$, is also bounded and satisfies inequality (35). Consider the expression

$$F^{(N)} = \sum_{i=1}^{N-3} a(N,\Lambda)a^3 K^{(N)} a(N-1,\Lambda)a^3 K^{(N-1)} \ldots$$

$$\ldots a(N-i+1,\Lambda)a^3 K^{(N-i+1)} a(N-i,\Lambda)a^3 F_0 +$$

$$+ a(N,\Lambda)a^3 K^{(N)} a(N-1,\Lambda)a^3 K^{(N-1)} \ldots$$

$$\ldots a(N-i+1,\Lambda)a^3 K^{(N-i+1)} \ldots a(3,\Lambda)a^3 K^{(3)} F^{(2)} +$$

$$+ a(N,\Lambda)a^3 F_0. \tag{36}$$

The sequences $a(N,\Lambda)a^3$ are bounded; the operators $K^{(N-1)}$ are bounded uniformly with respect to N, Λ, a according to (33). Hence, we can repeat the proof of Theorem 2 word by word. The only difference is that, in this case, $\Lambda \to \mathbb{R}^3$ instead of being fixed as in the case of Theorem 2. But it was shown in the proof of Theorem 2 that $F_s^{(N)}((q)_s)$ does not depend on the size of the domain Λ if $(q)_s \in$ $\in D_{a_0}$. The uniqueness of the limit F_s follows from the normalization condition

$$\frac{1}{N(N-1)\ldots(N-s-1)} \int_{\Lambda^s} F_s^{(N)}((q)_s)\, d(y)_s = a^{3s}.$$

The details of the proof are the same as in [2].

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