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ON A CLASS OF λ-MODULES

ПРО ОДИН КЛАС λ-МОДУЛІВ

Smith in paper [*Mapping between module lattices*, Int. Electron. J. Algebra, **15**, 173 – 195 (2014)] introduced maps between the lattice of ideals of a commutative ring and the lattice of submodules of an R-module M, i.e., μ and λ mappings. The definitions of the maps were motivated by the definition of multiplication modules. Moreover, some sufficient conditions for the maps to be a lattice homomorphisms are studied. In this work we define a class of λ-modules and observe the properties of the class. We give a sufficient conditions for the module and the ring such that the class λ is a hereditary pretorsion class.

У роботі [*Mapping between module lattices*, Int. Electron. J. Algebra, **15**, 173 – 195 (2014)] Сміт увів у розгляд відображення між решіткою ідеалів комутативного кільця та решіткою субмодулів R-модуля M, тобто відображення μ і λ. Ці означення були мотивовані означеннями мультиплікативних модулів. Також було вказано деякі достатні умови, за яких ці відображення є гомоморфізмами решіток. У цій роботі наведено означення класу λ-модулів та зазначено властивості цього класу. Вказано достатні умови на модуль та кільце, за яких клас λ є спадковим преторсійним класом.

1. Preliminaries. By the ring R we mean any commutative ring with unit and the module M means a left R-module, except we state otherwise. An R-module $M$is called a multiplication module if for any submodule N in M, there is an ideal I in R such that $N = IM$. For further explanation of multiplication modules over commutative rings we refer to papers [4, 8, 13]. Moreover, M is a multiplication module if and only if for any submodule N of M we have $N = \mathrm{Ann}_R(M/N)M$ (see [8]).

An R-module M is called a prime module if for any non-zero submodule K in M, $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(M)$. A proper submodule N in M is called a prime submodule of M if M/N is a prime module (see [14]).

Let K, N be submodules of M. The residue of K in N will be denoted by $[N :_R K] = \{r \in R \mid rK \subseteq N\}$. For a special case, that is if $N = 0$, we obtain the annihilator of K as $[0 :_R K] = \mathrm{Ann}_R(K)$.

Let $\mathcal{L}(M)$ be the lattice of submodules of R-module M, where for any submodules N and K in M the 'join' and 'meet' are defined as

$$N \vee K = N + K, \quad N \wedge K = N \cap K,$$

and $N \leq K$ means $N \subseteq K$. Especially, for $M = R$ we have the lattice of ideals in R and it is denoted by $\mathcal{L}(R)$. The definition of μ and λ mappings conducted by Smith in [12] are following:

$$\mu\colon \mathcal{L}(M) \to \mathcal{L}(R), \quad N \mapsto \mathrm{Ann}_R(M/N), \tag{1.1}$$

$$\lambda\colon \mathcal{L}(R) \to \mathcal{L}(M), \quad I \mapsto IM. \tag{1.2}$$

The mappings (1.1) and (1.2) are motivated by the relationship of submodules and ideals in a multiplication module. Then we define a class of modules as following:

$$\lambda = \{M \mid (B \cap C)M = BM \cap CM \ \ \forall B, C \text{ finitely generated ideals of } R\}.$$

Based on Lemma 2.1 of [12], $M \in \lambda$ if and only if M is a λ-module. Note that λ is not necessary a hereditary class.

If R is a ring, then an R-module M is called a chain module if for any submodules N and L in M either $N \subseteq L$ or $L \subseteq N$. The ring R is called a chain ring if the R-module R is a chain module. Smith in Proposition 2.4 of [12] has proved a sufficient condition of a ring such that its modules are in λ as follows:

Proposition 1.1. *If the ring R is a chain ring, then every R-module is in λ.*

Moreover, the class λ is closed under direct summands and direct sums (see Lemma 2.5 of [12]). Theorem 2.3 of [12] gave a necessary and sufficient condition of a module to be in λ as we recall here.

Proposition 1.2. *The following assertions are equivalent:*

a) *R is a Prüfer;*

b) *every R-module is in λ;*

c) *the class λ is closed under the homomorphic image.*

The sufficient conditions in Proposition 2.4 and Theorem 2.3 of [12] have given a motivation for us to study more general situations from category R-modules R-Mod to subcategory $\sigma[M]$ which consits M-subgenerated modules. In this work, we show that with some additional conditions, if the subgenerator M is a Dedekind module or a chain module, then the class λ will be equal to the class $\sigma[M]$.

In the next section, we discuss module Dedekind and the relationship with the class λ. In Section 3, we prove that we can generalize Theorem 2.3 of [12].

2. Dedekind modules and $\boldsymbol{\lambda}$-modules. For intensive study of Dedekind modules, we refer to Alkan et al. [3] and Saraç et al. [11]. For any commutative ring R with identity and a set S consisting of non-zero divisor elements of R, the fraction ring R_S will be naturally formed. By considering the notion of fractional ideals in Larsen and McCarthy [9], a fractional ideal I of R is invertible if there exist a fractional ideal I^{-1} of R such that $I^{-1}I = R$. In a case when I^{-1} exists, then $I^{-1} = [R :_{R_S} I]$. A domain R is called a Prüfer domain providing that each finitely generated ideal of R is invertible. Furthermore, an integral domain R is Dedekind domain iff every non-zero ideal of R is invertible.

Now we generalize the notion of invertibility of fractional ideals in the case of submodules. Many papers have discussed the notion of invertible submodules (see, for example, [3, 11]).

For any R-module M, consider $T = \{t \in S \mid \text{ for some } m \in M, tm = 0 \text{ implies } m = 0\}$. We can see that T is a multiplicatively closed subset of S. For any submodule N of M, we define $N' = [M :_{R_T} N]$. Following the concept of invertible ideal, we call a submodule N of M is invertible if $N'N = M$. Then M is called a Dedekind module if every non-zero submodule of M is invertible and M is called a Prüfer module providing every finitely generated non-zero submodule is invertible. The examples of Dedekind modules are the $\mathbb{Z}$-module $\mathbb{Q}$ and $\mathbb{Z}_p$ for prime p.

An R-module M is called a multiplication module provided for each submodule N of M there exist an ideal I of R such that $N = IM$, i.e., $I = [N :_R M]$. If P is a maximal ideal of R, then we define

$$T_P(M) = \{m \in M \mid (1-p)m = 0 \text{ for some } p \in P\}. \tag{2.1}$$

Next, M is P-cyclic if there exist $p \in P$ and $m \in M$ such that $(1-p)M \subseteq Rm$. In El-Bast (2007) it has been shown that M is multiplication module if and only if for every maximal ideal P of R either $M = T_P(M)$ or M is P-cyclic.

Now we show the property of λ-module dealing with the invertibility property of submodules of multiplication modules. Let us recall an important property in paper [1], that is for any finitely generated faithful multiplication R-module and for any invertible submodule N of M, $[N :_R M]$ is an invertible ideal of R.

Proposition 2.1. *Let M be an R-module. Then we have the following assertions:*

1. *If I is a multiplication ideal of a ring R and M is a multiplication R-module, then $\lambda(I)$ is a multiplication R-module.*

2. *Every invertible submodule N of a faithful multiplication finitely generated module M is a λ-module.*

3. *If M is a faithful multiplication module over an integral domain R, then M is a λ-module and for any ideal I of R, $I^{-1} = (\lambda(I))^{-1}$.*

Proof. 1. Let P be a maximal ideal of R. Consider the set T_P in (2.1). If $T_P(M) = M$ or $T_P(I) = I$, then $T_P(IM) = IM$. Hence $\lambda(I)$ is a multiplication module. Now suppose that $T_P(I) \neq I$ and $T_P(M) \neq M$. Then I and M are P-cyclic. Therefore there exist elements $p_1, p_2 \in \in P$, $a \in I$, $m \in M$ such that $(1-p_1)I \subseteq Ra$ and $(1-p_2)M \subseteq Rm$. It follows that $(1-p)IM \subseteq \subseteq R(am)$ where $p = p_1 + p_2 - p_1p_2 \in P$. Thus $\lambda(I) = IM$ is P-cyclic. This proves that $\lambda(I)$ is a multiplication R-module.

2. According to Proposition 2.1 of [1], for any invertible submodule N of M, $[N :_R M]$ is an invertible ideal of R. By using (2), we can easily obtain that $N = [N :_R M]M$ is a multiplication R-module. If $r \in \mathrm{Ann}_R(N)$, then $rN = 0$ and hence $rM = rN^{-1}N = 0$. This implies $r = 0$. Therefore, N is faihtful multiplication R-module. By using Theorem 2.12 of [12], we conclude that N is a λ-module because every faithful multiplication module is λ-module.

3. It is obvious by Theorem 2.12 of [12] and Lemma 1 of [2].

Now we are ready to display the connection of Dedekind module with λ-module by using the following Corollary 3.8 of [3]. We recall a module M is divisible if for any $0 \neq r \in R$, $M = rM$.

Lemma 2.1. *Let M be a Dedekind divisible R-module. Then R is a field.*

It is easy to understand that any vector space is a λ-module. Moreover, we have the following direct consequences of Lemma 2.1.

Proposition 2.2. *If M is a Dedekind divisible R-module, then:*

1) *M is λ-module;*

2) *for any $N \in \sigma[M]$, $N \in \lambda$;*

3) *the class of λ is closed under submodules and homomorphic images.*

Now we apply a result in paper [1].

Lemma 2.2. *If M is a faithful multiplication module, then M is a Dedekind (Prüfer) module if and only if R is a Dedekind (Prüfer) domain.*

Proposition 2.3. *Let M be a faithful multiplication module and Prüfer. Then $\sigma[M] \subseteq \lambda$.*

Proof. By assumption and according to a result in Lemma 2.2 we obtain that R is a Prüfer domain. Proposition 1.2 shows that λ is equal to the category of R-modules. It is clear that $\sigma[M] \subseteq \Lambda$.

For the converse, we give in the following corollary.

***Corollary* 2.1.** *Let R be a semisimple ring, M a faithful multiplication module and Prüfer. If M is a subgenerator for any semisimple module, then $\sigma[M] = \lambda$.*

Proof. Applying Proposition 2.3, $\sigma[M] \subseteq \lambda$. Now take any $N \in \lambda$. Since R is a semisimple ring, N is also a semisimple module. Moreover, $N \in \sigma[M]$ and we prove $\lambda \subseteq \sigma[M]$ as well.

We recall a sufficient condition of a Dedekind module in Lemma 3.3 of [1].

Lemma 2.3. *Let R be an integral domain and M a faithful multiplication module. If for any non-zero prime submodule P of M is invertible, then M is Dedekind.*

Proposition 2.4. *Let R be an integral domain and M a faithful multiplication module. If for any non-zero prime submodule P of M is invertible, then $\sigma[M] \subseteq \lambda$.*

Proof. It is obvious by applying Lemmas 2.3 and 2.2.

The following propositions are another properties of a Dedekind module.

Proposition 2.5. *Let M be a faithful multiplication Dedekind module over an integral domain R. If I is an ideal of R, then M is a λ-module over I and $\lambda(I)$ is a λ-module over R.*

Proof. Since I is an ideal of R, $\lambda(I) = IM$ is a submodule of M. Hence IM is an invertible submodule. We have $I^{-1} = (\lambda(I))^{-1}$ due to Proposition 2.1, i.e., I is an invertible ideal of R. By using a result in [7], we conclude that I is a λ-module over R. For any ideal B and C of R we have

$$(B \cap C)\lambda(I) = (B \cap C)IM = (BI \cap CI)M = BIM \cap CIM = B\lambda(I) \cap C\lambda(I).$$

This proves our assertion.

Now we give a sufficient condition of λ-module.

Proposition 2.6. *Let M be a multiplication Dedekind R-module. Then every R-module is a λ-module.*

Proof. According to Theorem 3.12 of [3], a multiplication Dedekind R-module implies the ring R is a Dedekind domain, i.e., a Prüfer domain. It means every R-module is a λ-module.

If M is an R/I-module, then under scalar multiplication $am = (a + I)m, M$ becomes an R-modules for every $a \in R$ and $m \in M$. Conversely, if M is an R-module, then M is an A/I-module with respect to $(a + I)m = am$ for every $a + I \in R/I$ and $m \in M$.

Proposition 2.7. *Let R be a ring, M an R-module and I an ideal of R where $I \subseteq [0 :_R M]$. M is a λ-module over R if and only if M is a λ-module over R/I.*

Proof. If M is a λ-module over R, then $(B \cap C)M = BM \cap CM$ for every finitely generated ideals B, C of R. Suppose B/I, C/I be any ideals of R. Then $(B/I \cap C/I)M = ((B \cap C)/I)M$. Since $(B\cap C)M = BM\cap CM$, $((B\cap C)/I)M = (B/I)M\cap(C/I)M$. This gives $(B/I\cap C/I)M = = (B/I)M \cap (C/I)M$. So, M is a λ-module over R/I.

Conversely, let M is a λ-module over R/I. Suppose B, C be any ideals of R with $BM \neq \{0\}$ and $CM \neq \{0\}$. Clearly, $B + I/I, C + I/I$ are an ideals of R/I. Since M is a λ-module over R/I, $((B + I/I) \cap (C + I/I))M = (B + I/I)M \cap (C + I/I)M$. On the other hand, $((B + I/I) \cap (C + +I/I))M = ((B+I\cap C+I)/I)M$, consequently $((B+I\cap C+I)/I)M = (B+I/I)M\cap(C+I/I)M$. Then $(B + I \cap C + I)M = (B + I)M \cap (C + I)M$. Since, $I \subseteq (0 : M)$, $(B + I)M \cap (C + I)M = = BM \cap CM$. So, M is a λ-module over R.

3. Chain modules and $\boldsymbol{\lambda}$-modules. In this section, we consider chain modules and the relationship with λ-modules.

Proposition 3.1. *Let M be a chain R-module and faithful. Then $\sigma[M] \subseteq \lambda$.*

Proof. For any $N \in \sigma[M]$, according to (15.1) of [15], $N = \oplus_{\Lambda} Rm_{\lambda}$, where $m_{\lambda} \in M^{(\mathbb{N})}$. Since M is chain, $N = Rm_0$ for some $m_0 \in M^{(\mathbb{N})}$. Now take any ideals B and C in the ring R. We only have to prove that $BN \cap CN \subseteq (B \cap C)N$. Take any $x \in BN \cap CN$. Then $x = bm_0$ for some $b \in B$ and $x = cm_0$ for some $c \in C$. Hence $x = bm_0 = cm_0$ and moreover $(b - c)m_0 = 0$. Since M is faithful, $b = c$ and we obtain that $x \in (B \cap C)N$.

For the converse of Proposition 3.1 we need an extra condition as we give in the next corollary.

Corollary **3.1.** *Let R be a semisimple ring, M a chain R-module, faithful and a subgenerator for all semisimple R-modules. Then $\sigma[M] = \lambda$.*

Proof. We apply Proposition 3.1. It is known that a module over a semisimple ring is semisimple. Take any R-module N in λ, then N is semisimple. From the assumption, $N \in \sigma[M]$.

According to the properties of $\sigma[M]$ in (15.1) of Wisbauer [15], we obtain the following corollary.

Corollary **3.2.** *Let R be a semisimple ring, M a chain R-module, faithful and a subgenerator for all semisimple R-modules. Then:*

1) *λ is a hereditary pretorsion class, i.e., λ is closed under submodules, homomorphic images and any direct sums;*

2) *for any $N \in \lambda$, $N = \sum Rm$, where $m \in M^{(\mathbb{N})}$;*

3) *pullback and pushout of morphisms in λ belong to λ.*

Corollary **3.3.** *Let R be a semisimple ring, M a chain R-module, faithful and a subgenerator for all semisimple R-modules. If N is M-injective, then N is K-injective for all K λ-module.*

Proof. If N is M-injective, then N is K-injective for any $K \in \sigma[M]$. But according to Corollary 3.1, $\sigma[M] = \lambda$. Hence N is K-injective for any $K \in \lambda$.

Now we recall the following definition from Definition 2.6 of [10].

Definition 3.1. *Let M and N be R-modules. We says that M rises to N, denoted by $M \uparrow N$, if every M-injective module is N-injective.*

Based on this definition and properties of injectivity in $\sigma[M]$ we conclude that if $N \in \sigma[M]$, then $M \uparrow N$, but the converse is not necessary true. Theorem 2.8 of [10] has given a sufficient condition such that the converse is holds.

Corollary **3.4.** *Let R be a semisimple ring, M a chain R-module, faithful and a subgenerator for all semisimple R-modules. For any module N which is $M \uparrow N$, N is M-injective if and only if N is K-injective for all $K \in \lambda$.*

Proof. It is straightforward from Corollary 3.3 and Theorem 2.8 of [10].

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