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ON THE APPROXIMATE SOLUTION OF NONLINEAR VOLTERRA – FREDHOLM INTEGRAL EQUATIONS ON A COMPLEX DOMAIN BY DZYADYK'S METHOD

ПРО НАБЛИЖЕНЕ РОЗВ'ЯЗУВАННЯ НЕЛІНІЙНИХ ІНТЕГРАЛЬНИХ РІВНЯНЬ ВОЛЬТЕРРА – ФРЕДГОЛЬМА У КОМПЛЕКСНІЙ ОБЛАСТІ МЕТОДОМ ДЗЯДИКА

In 1980–1984, V. K. Dzyadyk suggested and modified an iterative approximation method (IA-method) for numerical solution of the Cauchy problem $y' = f(x, y)$, $y(x_0) = x_0$. Particular cases of nonlinear mixed Volterra–Fredholm integral equations of the second kind arise in the mathematical simulation of the space–time development of an epidemic. This paper is concerned with the approximate solution of integral equations of this type by the Dzyadyk method on complex domains. Finally, we test this method numerically by four different examples.

У 1980–1984 рр. В. К. Дзядик запропонував та розвинув ітеративно-апроксимаційний метод (IA-метод) для числового розв'язування проблеми Коші $y' = f(x, y)$, $y(x_0) = x_0$. Частинні випадки нелінійних мішаних інтегральних рівнянь Вольтерра–Фредгольма другого роду виникають у задачах математичного моделювання часово-просторового розвитку епідемій. У цій статті розглянуто задачу наближеного розв'язування таких інтегральних рівнянь методом В. К. Дзядика на комплексній області. Наприкінці статті наведено результати чисельної перевірки методу на чотирьох різних прикладах.

Introduction. One of advantages of the iterative approximation method suggested by V. K. Dzyadyk is its high accuracy [1 – 3]. Later, the Dzyadyk's method was successfully applied to the numerical solution of many mathematical problems.

In 1985, by using the Dzyadyk's method, Karpenko [4] studied some nonlinear integral equations such as Volterra, Urysohn, and Lyapunov–Lichtenstein equations.

In 1986, Dzyadyk and Romanenko [5] studied the Cauchy, Darboux, and Goursat hyperbolic partial differential equations. In 1989, Bassov [6] studied the boundary-value problem for ordinary differential equations of the second order.

In 1991, Dzyadyk, Bassov, and Rizk [7] carried out a comparative analysis of this method and implicit methods of Runge–Kutta type. They proved that the iterative approximation method is an $(n+1)$-stage implicit method of Runge–Kutta type of order $(n+1)$. Furthermore, by using counterexamples, they showed that this method is better than the four-stage implicit Butcher method.

Also, in 1991, Dzyadyk and Vassilenko [8] studied stiff ordinary differential equations.

In this paper, we use the same Dzyadyk method for studying the approximation and convergence of solutions of nonlinear Volterra–Fredholm integral equations on complex domains.

Finally, we test this method by some counterexamples.

Preliminaries. Let $[0, h_1]$, $[0, h_2]$ be two segments in $\mathbb{R}$, let $r_1, r_2 \geq 1$ and $H > 0$ be real numbers, and let D and $\mathcal{E}_{r_k}$, $k = 1, 2$, be regions in $\mathbb{C}$ defined as follows:

$$\mathcal{E}_{r_1} := \mathcal{E}_{r_1}(h_1) = \left\{ (x, y) \in \mathbb{R}^2 \colon \frac{[x - h_1/2]^2}{a_{r_1}^2} + \frac{y^2}{b_{r_1}^2} \leq 1 \right\},$$

$$\mathcal{E}_{r_2} := \mathcal{E}_{r_2}(h_2) = \left\{ (x, y) \in \mathbb{R}^2 \colon \frac{x^2}{a_{r_2}^2} + \frac{[y - h_2/2]^2}{b_{r_2}^2} \leq 1 \right\},$$

where

$$a_{r_1} = \frac{h_1}{4}\left(r_1 + \frac{1}{r_1}\right), \quad b_{r_1} = \frac{h_1}{4}\left(r_1 - \frac{1}{r_1}\right),$$

$$a_{r_2} = \frac{h_2}{4}\left(r_2 - \frac{1}{r_2}\right), \quad b_{r_2} = \frac{h_2}{4}\left(r_2 + \frac{1}{r_2}\right),$$

and

$$D := D(H) = \{z \in \mathbb{C} : |z| \le H\}.$$

Let Ω be a compact subset of $\mathbb{C}^n$ and let $AC(\Omega)$ be the Banach space of all functions f continuous on Ω and analytic in $\operatorname{int}(\Omega)$ with the norm

$$\|f\|_{AC(\Omega)} = \max_{z \in \Omega} |f(z)|.$$

Consider the following nonlinear Volterra–Fredholm integral equation:

$$u(z, w) = f(z, w) + \int_0^z \int_0^{h_2} K(z, \tau, w, \xi, u(\tau, \xi))\, d\xi\, d\tau, \tag{1}$$

where $f(z, w)$ and $K(z, \tau, w, \xi, u)$ belong to $AC(\Omega_1)$ and $AC(\Omega_2)$, respectively, $\Omega_1 = \mathcal{E}_{r_1} \times \mathcal{E}_{r_2}$, and $\Omega_2 = \mathcal{E}_{r_1} \times \mathcal{E}_{r_1} \times \mathcal{E}_{r_2} \times \mathcal{E}_{r_2} \times D$.

Picard iteration. Let $\mathcal{K}_H := \{u : u \in AC(\Omega_1),\ \|u\|_{AC(\Omega_1)} \le H\}$ and define the mapping (operator) T on $\mathcal{K}_H$ as

$$(Tu)(z, w) = f(z, w) + \int_0^z \int_0^{h_2} K(z, \tau, w, \xi, u(\tau, \xi))\, d\xi\, d\tau. \tag{2}$$

Theorem 1. *If the conditions*

(a) $|K(z_1, z_2, z_3, z_4, z_5) - K(z_1, z_2, z_3, z_4, z_5')| \le A|z_5 - z_5'|$,

$A \equiv \text{const}, \quad \forall\, z_1, z_2 \in \mathcal{E}_{r_1}, \quad \forall\, z_3, z_4 \in \mathcal{E}_{r_2}, \quad \forall\, z_5, z_5' \in D$;

(b) $S := Aq < 1, \quad q = h_2\left(a_{r_1} + \dfrac{h_1}{2}\right)$;

(c) $(\|f\|_{AC(\Omega_1)} + q\|K\|_{AC(\Omega_2)}) \le H$

are satisfied, then there exists a unique solution $u \in \mathcal{K}_H$ *of* (1) *given by*

$$u(z, w) = \sum_{\mu=0}^{\infty} [u^{[\mu+1]}(z, w) - u^{[\mu]}(z, w)],$$

where

$$u^{[0]}(z, w) \equiv 0, \tag{3a}$$

$$u^{[\mu+1]}(z, w) = f(z, w) + \int_0^z \int_0^{h_2} K(z, \tau, w, \xi, u^{[\mu]}(\tau, \xi))\, d\xi\, d\tau, \tag{3b}$$

$$\mu = 0, 1, 2, \ldots,$$

and

$$\|u - u^{[\mu]}\|_{\mathcal{K}_H} \le (\|f\|_{AC(\Omega_1)} + q\|K\|_{AC(\Omega_2)})\, \frac{S^{\mu}}{1 - S}.$$

Proof. (i) By (2) and condition (c), if $u \in \mathcal{K}_H$ and $(z, w) \in \Omega_1$, then

$$|(Tu)(z,w)| \le |f(z,w)| + \int_0^z \int_0^{h_2} |K(z,\tau,w,\xi,u(\tau,\xi))||d\xi||d\tau| \le$$

$$\le \|f\|_{AC(\Omega_1)} + \|K\|_{AC(\Omega_2)} h_2 \left(a_\eta + \frac{h_1}{2}\right) \le H.$$

Thus, $T\mathcal{K}_H \subseteq \mathcal{K}_H$.

(ii) By (2) and conditions (a) and (b), if $u_1, u_2 \in \mathcal{K}_H$, $(z,w) \in \Omega_1$, then

$$|(Tu_1)(z,w) - (Tu_2)(z,w)| \le$$

$$\le \int_0^z \int_0^{h_2} |K(z,\tau,w,\xi,u_1(\tau,\xi)) - K(z,\tau,w,\xi,u_2(\tau,\xi))||d\xi||d\tau| \le$$

$$\le A\,q\,\|u_1 - u_2\|_{\mathcal{K}_H} = S\,\|u_1 - u_2\|_{\mathcal{K}_H}, \quad S < 1.$$

It follows from (i) and (ii) that T is a contraction mapping on $\mathcal{K}_H$ and, according to the fixed-point theorem [9], there exists a unique element $u \in \mathcal{K}_H$ such that $Tu = u$, which is given by the uniform limit of the sequence

$$u^{[0]}(z,w) \equiv 0, \quad u^{[\mu+1]}(z,w) = Tu^{[\mu]}(z,w), \quad \mu = 0, 1, 2, \ldots .$$

(iii) It is clear from (i) and (ii) that

$$\|u^{[\mu+1]} - u^{[\mu]}\|_{\mathcal{K}_H} \le S^\mu \|u^{[1]} - u^{[0]}\|_{\mathcal{K}_H} \le S^\mu (\|f\|_{AC(\Omega_1)} + q\|K\|_{AC(\Omega_2)})$$

and, hence,

$$\|u - u^{[\mu]}\|_{\mathcal{K}_H} \le \|u^{[\mu+1]} - u^{[\mu]}\|_{\mathcal{K}_H} + \|u^{[\mu+2]} - u^{[\mu+1]}\|_{\mathcal{K}_H} + \ldots \le$$

$$\le (\|f\|_{AC(\Omega_1)} + q\|K\|_{AC(\Omega_2)})[S^\mu + S^{\mu+1} + \ldots] =$$

$$= (\|f\|_{AC(\Omega_1)} + q\|K\|_{AC(\Omega_2)}) \frac{S^\mu}{1-S}, \quad S < 1.$$

Theorem 1 is proved.

Algorithm. Let n be an arbitrary natural number and let $-1 \le \xi_0 < \xi_1 < \ldots$ $\ldots < \xi_n \le 1$ be nodes on $[-1, 1]$. The Lagrangian interpolation polynomial for a continuous function $g(t)$ on $[-1, 1]$ constructed by $\{\xi_j\}_{j=0}^n$ is denoted by $L_n(g;t)$ and defined as

$$L_n(g;t) = \sum_{j=0}^n g(\xi_j)\, l_{n,j}(t),$$

where $l_{n,j}(t)$ is called the fundamental Lagrangian polynomial and defined by

$$l_{n,j}(t) = \prod_{\substack{i=0 \\ i \ne j}}^n \frac{t - \xi_i}{\xi_j - \xi_i}. \tag{4}$$

Remarks. **1.** If

$$\xi_j = \xi_j^* = -\cos\frac{j\pi}{n}, \quad j = \overline{0, n}, \tag{5}$$

then $l_{n,j}(t)$ can be written as [3]

$$l_{n,j}(t) = l_{n,j}^*(t) = \frac{\varepsilon_j}{n}\left[1 + 2\sum_{r=1}^{n-1}(-1)^r \cos\frac{rj\pi}{n} T_r(t) - (-1)^{n+j} T_n(t)\right], \quad j = \overline{0, n}, \tag{4*}$$

where $\varepsilon_0 = \varepsilon_n = 1/2$, $\varepsilon_j = 1$, $j = \overline{1, n-1}$, and $T_n(t) = \cos(n \arccos t)$ is the Chebyshev polynomial of degree n.

2. If

$$\xi_j = \xi_j^0 = -\cos\frac{(2j+1)\pi}{2n+2}, \quad j = \overline{0, n}, \tag{6}$$

then $l_{n,j}(t)$ can be written as [3]

$$l_{n,j}(t) = l^{\circ}_{n,j}(t) =$$

$$= \frac{1}{n+1}\left[1 + 2\sum_{r=1}^{n-1}(-1)^r \cos\frac{r(2j+1)\pi}{2n+2} T_r(t)\right], \quad j = \overline{0, n}.$$

Let $n, m \in \mathbb{N}$ and let $-1 \le \xi_0 < \xi_1 < \ldots < \xi_n \le 1$ and $-1 \le \tau_0 < \tau_1 < \ldots < \tau_m \le 1$ be two nodes on $[-1, 1]$.

Consider the $(m+1) \times (m+1)$ matrix $[a_{ij}^{(m)}]_{i,j=0}^{m}$,

$$a_{ij}^{(m)} = a_{ij}(m, r_j) = \int_{-1}^{\tau_j} l_{m,i}(t)\, dt, \tag{7}$$

where $l_{m,i}(t)$ is the fundamental Lagrangian polynomial constructed by $\{\tau_j\}_{j=0}^{m}$, and the vector matrix $\{b_i^{(n)}\}_{i=0}^{n}$,

$$b_i^{(n)} = b_i(n, \xi_i) = \int_{-1}^{1} l_{n,i}(t)\, dt, \tag{8}$$

where $l_{n,i}(t)$ is the fundamental Lagrangian polynomial constructed by $\{\xi_i\}_{i=0}^{n}$.

Note that $[a_{ij}^{(m)}]_{i,j=0}^{m}$ and $[b_i^{(n)}]_{i=0}^{n}$ depend only on the natural numbers m and n and the choice of the nodes $\{\tau_j\}_{j=0}^{m}$ and $\{\xi_i\}_{i=0}^{n}$, but do not depend on the Volterra–Fredholm integral equation (1).

Remarks. **3.** If $\{\tau_j\}_{j=0}^{m}$ is chosen by (5), then the matrix $[a_{ij}^{(m)}]_{i,j=0}^{m}$ has the following explicit form [3]:

$$a_{ij}^{*(m)} = a_{ij}(m, \tau_j^*) = \frac{\varepsilon_i}{m}\left[1 + S_{j,1}^* + \frac{1}{2} S_{i,1}^*(S_{j,2}^* - 1) + \right.$$

$$\left. + \sum_{\nu=1}^{m} \varepsilon_\nu S_{i,\nu}^* \left(\frac{S_{j,\nu+1}^*}{\nu+1} - \frac{S_{j,\nu-1}^*}{\nu-1} - (-1)^{\nu+j}\frac{2}{\nu^2-1}\right)\right], \quad i, j = \overline{0, m}, \tag{7*}$$

where

$$\varepsilon_0 = \varepsilon_m = 1/2, \quad \varepsilon_i = 1, \quad i = \overline{1, m-1}, \quad S_{i,j}^* := \cos\frac{j(m-i)\pi}{m}.$$

4. If $\{\xi_i\}_{i=0}^{n}$ is chosen by (5), then

$$b_i^{*(n)} = b_i(n, \xi_i^*) = a_{in}(n, \xi_i^*), \quad i = \overline{0, n}. \tag{8*}$$

5. If $\{\tau_j\}_{j=0}^{m}$ is chosen by (6), then the matrix $[a_{ij}^{(m)}]_{i,j=0}^{m}$ has the following explicit form [7]:

$$a_{ij}^{\circ(m)} = a_{ij}(m, \tau_j^\circ) = \frac{1}{m+1}\left[1 + S_{j,1}^\circ + \frac{1}{2} S_{i,1}^\circ (S_{j,2}^\circ - 1) + \right.$$

$$\left. + \sum_{\nu=2}^{m} S_{i,\nu}^\circ \left(\frac{S_{j,\nu+1}^\circ}{\nu+1} - \frac{S_{j,\nu-1}^\circ}{\nu-1} + (-1)^{\nu+1} \frac{2}{\nu^2-1} \right)\right], \qquad i, j = \overline{0, m},$$

where $S_{i,j}^\circ := \cos \dfrac{j(2m+1-2i)\pi}{2m+2}$.

6. If $\{\xi_i\}_{i=0}^n$ is chosen by (6), then

$$b_i^{\circ(n)} = \frac{2}{n+1}\left[1 - 2\sum_{r=2}^{n} S_{i,r}^\circ \frac{\delta_r}{r^2-1}\right], \qquad i = \overline{0, n},$$

where $\delta_r = 1$ if r is even, and $\delta_r = 0$ if r is odd.

By using matrices (7) and (8), for every $\mu = 0, 1, 2, \ldots$, we get a system for the approximate solution of (1) at the points (x_r, y_k), $r = \overline{0, m}$, $k = \overline{0, n}$ $(0 \le x_0 < < x_1 < \ldots < x_m \le h_1$, $0 \le y_0 < y_1 < \ldots < y_n \le h_2)$, where $x_r = h_1(1+\tau_r)/2$, $r = \overline{0, m}$, and $y_k = h_2(1+\xi_k)/2$, $k = \overline{0, n}$, by the formulas

$$u_{r,k}^{[0]} = 0, \quad r = \overline{0, m}, \quad k = \overline{0, n}, \tag{9}$$

$$u_{r,k}^{[\mu+1]} = f(x_r, y_k) + \frac{h_1 h_2}{4} \sum_{i=0}^{n} \sum_{j=0}^{m} b_i^{(n)} a_{jr}^{(m)} K(x_r, x_j, y_k, y_i, u_{j,i}^{[\mu]}), \tag{10}$$

$$r = \overline{0, m}, \qquad k = \overline{0, n}.$$

Lemma 1. *The values* $u_{r,k}^{[\mu+1]}$ *given by* (9), (10) *represent the values of the following function* (12) *at the points* $z = x_r \in [0, h_1]$, $r = \overline{0, m}$, $w = y_k \in [0, h_2]$, $k = \overline{0, n}$:

$$\overline{\mathcal{U}}^{[0]}(z, w) \equiv 0, \tag{11}$$

$$\overline{\mathcal{U}}^{[\mu+1]}(z, w) = f(z, w) + \int_0^z \int_0^{h_2} \overline{L}_n[K(z, \sigma, w, \cdot, \overline{\mathcal{U}}^{[\mu]}(\sigma, \cdot)); \rho]\, d\rho\, d\sigma =$$

$$= f(z, w) + \int_0^z \int_0^{h_2} \overline{L}_m\left[\overline{L}_n[K(z, \cdot\cdot, w, \cdot, \overline{\mathcal{U}}^{[\mu]}(\cdot\cdot, \cdot)); \rho]; \sigma\right] d\rho\, d\sigma = f(z, w) +$$

$$+ \sum_{i=0}^{n} \sum_{j=0}^{m} K(z, x_j, w, y_i, \overline{\mathcal{U}}^{[\mu]}(x_j, y_i)) \int_0^{h_2} \bar{l}_{n,i}(\rho)\, d\rho \int_0^z \bar{l}_{m,j}(\sigma)\, d\sigma, \tag{12}$$

where $\overline{L}_n$, $\bar{l}_{n,i}$ *denote the transfer of* L_n, $l_{n,i}$ *from* $[-1, 1]$ *to* $[0, h_2]$, *and* $\overline{L}_m$, $\bar{l}_{m,j}$ *from* $[-1, 1]$ *to* $[0, h_1]$, *respectively; in other words,*

$$u_{r,k}^{[\mu]} = \overline{\mathcal{U}}^{[\mu]}(x_r, y_k), \quad \mu = 0, 1, 2, \ldots, \quad k = \overline{0, n}, \quad r = \overline{0, m}. \tag{13}$$

Proof. For $\mu = 0$, relation (13) obviously follows from (9) and (10). By using the transformations

$$x = \frac{h_1}{2}(1+\tau) : [-1, 1] \to [0, h_1] \;\leftrightarrow\; \tau = -1 + \frac{2}{h_1} x : [0, h_1] \to [-1, 1],$$

$$y = \frac{h_2}{2}(1+\xi) : [-1, 1] \to [0, h_2] \leftrightarrow \xi = -1 + \frac{2}{h_2} x : [0, h_2] \to [-1, 1],$$

we get

$$\int_0^{h_2} \bar{l}_{n,i}(\xi)\, d\xi = \int_0^{h_2} l_{n,i}\left(-1+\frac{2}{h_2}\xi\right) d\xi = \frac{h_2}{2}\int_{-1}^{1} l_{n,i}(\alpha)\, d\alpha = \frac{h_2}{2} b_i^{(n)}, \quad i = \overline{0, n}.$$

Similarly,

$$\int_0^{x_r} \bar{l}_{m,j}(\tau)\, d\tau = \int_0^{x_r} l_{m,j}\left(-1+\frac{2}{h_1}\tau\right) d\tau = \frac{h_1}{2}\int_{-1}^{\tau_r} l_{m,j}(\beta)\, d\beta = \frac{h_1}{2} a_{jr}^{(m)}, \quad j,\ r = \overline{0, m},$$

and, by mathematical induction on μ, we obtain the required result.

Lemma 1 is proved.

By Lemma 1, (9), and (10), we now get a function on Ω_1 given by

$$\overline{\mathcal{U}}^{[0]}(z, w) \equiv 0, \tag{14}$$

$$\overline{\mathcal{U}}^{[\mu]}(z, w) = f(z, w) + $$

$$+ \frac{h_2}{2}\sum_{i=0}^{n}\sum_{j=0}^{m} b_i^{(n)} K(z, x_j, w, y_i, u_{j,i}^{[\mu-1]}) \int_0^z \bar{l}_{m,j}(\sigma)\, d\sigma, \quad \mu = 1, 2, \ldots . \tag{15}$$

Remarks. 7. If τ_j is chosen by (5), then $\int_0^z \bar{l}_{m,j}(\sigma)\, d\sigma$ has the form [10]

$$\int_0^z \bar{l}_{m,j}(\sigma)\, d\sigma = \frac{h_1 \varepsilon_j}{2m}\left[1 + T_1(\tau) + \frac{1}{2} S_{j,1}^*(T_2(\tau) - 1) + \right.$$

$$\left. + \sum_{\nu=2}^{m} \varepsilon_\nu S_{j,\nu}^*\left(\frac{T_{\nu+1}(\tau)}{\nu+1} - \frac{T_{\nu-1}(\tau)}{\nu-1} + (-1)^{\nu+1}\frac{2}{\nu^2-1}\right)\right], \quad j = \overline{0, m},$$

where $\tau = -1 + 2z/h_1$, $S_{i,j}^* = \cos\dfrac{j(m-i)\pi}{m}$, $i, j = \overline{0, m}$, and ε_j and $T_j(\tau)$ are the same as in (4*).

8. If τ_j is chosen by (6), then $\int_0^z \bar{l}_{m,j}(\sigma)\, d\sigma$ has the form [7]

$$\int_0^z \bar{l}_{m,j}(\sigma)\, d\sigma = \frac{h_1}{2(m+1)}\left[(\tau+1) + S_{j,1}^\circ(\tau^2 - 1) + \right.$$

$$\left. + \sum_{\nu=2}^{m} S_{j,\nu}^\circ\left(\frac{T_{\nu+1}(\tau)}{\nu+1} - \frac{T_{\nu-1}(\tau)}{\nu-1} + (-1)^{\nu+1}\frac{2}{\nu^2-1}\right)\right], \quad j = \overline{0, m},$$

where $S_{i,j}^\circ = \cos\dfrac{j(2m+1-2i)\pi}{2m+2}$, $i, j = \overline{0, m}$, and $\tau = -1 + 2z/h_1$.

In the next section, we prove that the function defined by (14) and (15) converges to a solution of (1) as n, m, and μ tend to infinity.

Convergence. If F is a continuous function on $[0, h]$, then it follows from the Lebesgue theorem [3] that

$$\| F(x) - L_n(F; x)\|_{C[0,h]} \le (1 + \lambda_n)\, E_n(F)_{C[0,h]},$$

where $\lambda_n := \left\| \sum_{i=0}^{n} |l_{n,i}(x)| \right\|_{C[0,h]}$ and $E_n(F)_{C[0,h]}$ is the functional of the best approximation of F on $C[0, h]$.

Furthermore, if F is defined and continuous on $\mathcal{E}_{\tilde{r}}$, $\tilde{r} > 1$, and analytic in int $\mathcal{E}_{\tilde{r}}$, then, for any $r \in [1, \tilde{r})$, we have the following bound for $E_n(F)_{AC(\mathcal{E}_r)}$ (by the Bernshtein theorem [12, p. 73–75]; see also [3, p. 60–62]):

$$E_n(F)_{AC(\mathcal{E}_r)} \le \|F\|_{AC(\mathcal{E}_r)}\, \alpha_n \left(\frac{r}{\tilde{r}}\right)^n,$$

where $\alpha_n = \left(\dfrac{r}{\tilde{r}-r}\right)\left(1 + \dfrac{1}{r^{2n+2}}\right)$.

Now define the mapping (operator) $\tilde{T}$ on $\mathcal{K}_H$ as follows:

$$(\tilde{T}u)(z, w) = f(z, w) + \sum_{i=0}^{n} \sum_{j=0}^{m} K(z, x_j, w, y_i, u(x_j, y_i)) \int_0^{h_2} \bar{l}_{n,i}(\rho)\, d\rho \int_0^{z} \bar{l}_{m,j}(\sigma)\, d\sigma. \tag{16}$$

It is clear from (12), (13), (15), and (16) that

$$\overline{\mathcal{U}}^{[\mu+1]} = \tilde{T}\,\overline{\mathcal{U}}^{[\mu]}, \quad \mu = 0, 1, 2, \ldots . \tag{17}$$

Lemma 2. *If $\tilde{T}\mathcal{K}_H \subseteq \mathcal{K}_H$, then*

$$\|u^{[\mu]} - \overline{\mathcal{U}}^{[\mu]}\|_{\mathcal{K}_H} \le \delta \frac{1-S^{\mu}}{1-S}, \quad \mu = 0, 1, 2, \ldots,$$

where $u^{[\mu]}$ is given by (3a), (3b), *$\overline{\mathcal{U}}^{[\mu]}$ by* (14), (15), *S by condition (a) of Theorem* 1, *and*

$$\delta = \sup_{u \in \mathcal{K}_H} \|Tu - \tilde{T}u\|_{\mathcal{K}_H}.$$

Proof. Lemma 2 can be proved by mathematical induction on μ with the use of step (ii) in the proof of Theorem 1, (3a), (3b), and (17).

Theorem 2. *Let $\tilde{r}_1, \tilde{r}_2 > 1$, $H > 0$, $\tilde{\Omega}_1 = \mathcal{E}_{\tilde{r}_1} \times \mathcal{E}_{\tilde{r}_2}$, and let $\tilde{\Omega}_2 = \mathcal{E}_{\tilde{r}_1} \times \mathcal{E}_{\tilde{r}_1} \times \mathcal{E}_{\tilde{r}_2} \times \mathcal{E}_{\tilde{r}_2} \times D$. Suppose that $f(z, w) \in AC(\tilde{\Omega}_1)$ and $K(z_1, z_2, z_3, z_4, z_5) \in AC(\tilde{\Omega}_2)$ satisfy the following conditions:*

(a) $|K(z_1, z_2, z_3, z_4, z_5) - K(z_1, z_2, z_3, z_4, z_5')| \le A|z_5 - z_5'|$,

$$A \equiv \text{const}, \quad \forall\, z_1, z_2 \in \mathcal{E}_{r_1}, \quad \forall\, z_3, z_4 \in \mathcal{E}_{r_2}, \quad \forall\, z_5, z_5' \in D;$$

(b) $S := Aq < 1, \quad q = h_2\left(a_{r_1} + \dfrac{h_1}{2}\right)$;

(c) *for some $\varepsilon > 0$, we have $M(1+\varepsilon) \le H$, where*

$$M = \max\{(\|f\|_{AC(\Omega_1)} + q\|K\|_{AC(\Omega_2)}),\ q\|K\|_{AC(\tilde{\Omega}_2)})\}.$$

Then the Volterra–Fredholm integral equation (1) *has a unique solution $u(z, w)$ (from Theorem* 1*), and the functions $\overline{\mathcal{U}}^{[\mu]}(z, w)$ approximate $u(z, w)$ in the sense that, for any $r_k \in [1, \tilde{r}_k)$, $k = 1, 2$, and all n and m such that*

$$\delta_{nm} = (1+\lambda_m)\,\alpha_m^{(1)}\left(\frac{r_1}{\tilde{r}_1}\right)^m + (1+\lambda_n)\,\lambda_m\,\alpha_n^{(2)}\left(\frac{r_2}{\tilde{r}_2}\right)^n < \varepsilon,$$

where $\alpha_m^{(1)} = \left(\dfrac{r_1}{\tilde{r}_1 - r_1}\right)\left(1 + \dfrac{1}{r_1^{2m+2}}\right)$ and $\alpha_n^{(2)} = \left(\dfrac{r_2}{\tilde{r}_2 - r_2}\right)\left(1 + \dfrac{1}{r_2^{2n+2}}\right)$, the following inequality holds:

$$\|u - u^{[\mu]}\|_{\mathcal{K}_H} \le (\|f\|_{AC(\Omega_1)} + q\|K\|_{AC(\Omega_2)})\,\frac{S^{\mu}}{1-S} + q\|K\|_{AC(\tilde{\Omega}_2)}\,\delta_{nm}\,\frac{1-S^{\mu}}{1-S}.$$

Proof. I. For all $(z, w) \in \Omega_2$ and $u \in \mathcal{K}_H$, it follows from (2) and (16) that

$$|(Tu)(z, w) - (\tilde{T}u)(z, w)| \le \int_0^z \int_0^{h_2} \Bigg| K(z, \tau, w, \xi, u(\tau, \xi)) -$$

$$- \sum_{j=0}^{m} K(z, x_j, w, \xi, u(x_j, \xi))\, \bar{l}_{m,j}(\tau) \Bigg| |d\xi||d\tau| +$$

$$+ \int_0^z \int_0^{h_2} \sum_{j=0}^{m} \Bigg| K(z, x_j, w, \xi, u(x_j, \xi)) -$$

$$- \sum_{i=0}^{n} K(z, x_j, w, y_i, u(x_j, y_i))\, \bar{l}_{n,i}(\xi) \Bigg| |\bar{l}_{m,j}(\tau)||d\xi||d\tau| \le$$

$$\le (1+\lambda_m) E_m(K)_{AC(\Omega_2)}\, q +$$

$$+ (1+\lambda_m)\, q \lambda_m \max_{0 \le j \le m} E_n\big(K(z, x_j, w, \xi, u(x_j, \xi))\big)_{AC(\tilde{\Omega}_2)} \le$$

$$\le (1+\lambda_m) \|K\|_{AC(\tilde{\Omega}_2)}\, \alpha_m^{(1)} \left(\frac{r_1}{\tilde{r}_1}\right)^m q + (1+\lambda_n)\, \lambda_m\, q \|K\|_{AC(\tilde{\Omega}_2)}\, \alpha_m^{(2)} \left(\frac{r_2}{\tilde{r}_2}\right)^n \le$$

$$\le q \|K\|_{AC(\tilde{\Omega}_2)}\, \delta_{nm}. \tag{18}$$

II. If $u \in \mathcal{K}_H$, $(z, w) \in \Omega_1$, then it follows from from step (i) in the proof of Theorem 1 and (18) that

$$|(\tilde{T}u)(z, w)| \le |(Tu)(z, w)| + |(\tilde{T}u)(z, w) - (Tu)(z, w)| \le$$

$$\le \left(\|f\|_{AC(\Omega_1)} + q \|K\|_{AC(\Omega_2)}\right) + q \|K\|_{AC(\tilde{\Omega}_2)}\Bigg) \left\{(1+\lambda_m)\, \alpha_m^{(1)} \left(\frac{r_1}{\tilde{r}_1}\right)^m + \right.$$

$$\left. + \lambda_m (1+\lambda_n)\, \alpha_n^{(2)} \left(\frac{r_2}{\tilde{r}_2}\right)^n \right\} \le (1+\varepsilon) M \le H.$$

Thus, $\tilde{T}\mathcal{K}_H \subseteq \mathcal{K}_H$.

III. By virtue of (18) and Lemma 2, we get

$$\|u^{[\mu]} - \overline{\mathcal{U}}^{[\mu]}\|_{\mathcal{K}_H} \le \sup_{u \in \mathcal{K}_H} \|Tu - \tilde{T}u\|_{\mathcal{K}_H} \frac{1 - S^\mu}{1 - S} \le$$

$$\le q \|K\|_{AC(\tilde{\Omega}_2)}\, \delta_{nm} \frac{1 - S^\mu}{1 - S}.$$

IV. Finally, it follows from step III and Theorem 1 that

$$\|u - \overline{\mathcal{U}}^{[\mu]}\|_{\mathcal{K}_H} \le \|u - u^{[\mu]}\|_{\mathcal{K}_H} + \|u - \overline{\mathcal{U}}^{[\mu]}\|_{\mathcal{K}_H} \le$$

$$\le \left(\|f\|_{AC(\Omega_1)} + q \|K\|_{AC(\Omega_2)}\right) \frac{S^\mu}{1 - S} + q \|K\|_{AC(\tilde{\Omega}_2)}\, \delta_{nm} \frac{1 - S^\mu}{1 - S}.$$

Theorem 2 is proved.

Remarks. **9.** If ξ_j is chosen by (5), then [3]

$$\frac{2}{\pi} \ln(n-1) \le \lambda_n = \lambda_n^* \le \frac{2}{\pi} \ln n + 1.$$

10. If ξ_j is chosen by (6), then [3]

$$\frac{2}{\pi}\ln(n-1) \le \lambda_n = \lambda_n^{\circ} \le \frac{2}{\pi}\ln(n-1)+1.$$

11. If $r_1 = r_2 = 1$ (i.e., $\mathcal{E}_{r_1} = [0, h_1]$ and $\mathcal{E}_{r_1} = [0, h_2]$), then

$$\alpha_m^{(1)} = \frac{2}{\tilde{r}_1 - 1}, \qquad \alpha_n^{(2)} = \frac{2}{\tilde{r}_2 - 1},$$

and

$$\delta_{nm} = (1+\lambda_m)\frac{2}{\tilde{r}_1 - 1}\left(\frac{1}{\tilde{r}_1}\right)^m + (1+\lambda_n)\,\lambda_m \frac{2}{\tilde{r}_2 - 1}\left(\frac{1}{\tilde{r}_2}\right)^n.$$

12. If $\dfrac{\tilde{r}_1}{r_1} \ge 2$ and $\dfrac{\tilde{r}_2}{r_2} = 1$, then $\alpha_m^{(1)} \le 2$, $\alpha_n^{(2)} \le 2$, and

$$\delta_{nm} = \left(\frac{1+\lambda_m}{2^{m-1}} + \frac{\lambda_m(1+\lambda_n)}{2^{n-1}}\right).$$

Examples. The examples presented below, which have known analytic solutions, were solved with a PC 286AT with respect to each of nodes (5) and (6) by using functions (14) and (15).

Table gives the errors $e_n := e_n^*$ (with respect to nodes (5)) and $e_n := e_n^{\circ}$ (with respect to nodes (6)) at the last point $x = h_1$, $y = h_2$, $h_1 = h_2 = 1$, for different values of n $(n = 3, 5, 7, 9, 11)$, i.e.,

$$e_n = e_n(h, \mu) = |\overline{\mathcal{U}}^{[\mu]}(h_1, h_2) - u(h_1, h_2)|,$$

where μ is the number of iterations.

Example 1.

$$u(t,x) = -\ln\left(1 + \frac{xt}{1+t^2}\right) + \frac{xt^2}{8(1+t)(1+t^2)} + \int_0^t\int_0^1 \frac{x(1-\xi^2)}{(1+t)(1+\tau^2)}\left(1 - e^{-u(\tau,\xi)}\right) d\xi\, d\tau.$$

Its exact solution is $u(t,x) = -\ln\left(1 + \dfrac{xt}{1+t^2}\right)$.

Example 2.

$$u(t,x) = \sin(t+x) + (x+t^2)\sin(t+1) - t^2\sin t - x\sin(1) +$$
$$+ (x+t)\left[\cos(t+1) - \cos t - \cos(1) + 1\right] + \int_0^t\int_0^1 (t\tau + x\xi)\, u(\tau,\xi)\, d\xi\, d\tau.$$

Its exact solution is $u(t,x) = \sin(t+x)$.

Example 3.

$$u(t,x) = 1 + \sin(tx) + \sin t - e^{\sin t} + \int_0^t\int_0^1 \tau\cos(\tau\xi)\cos\tau\, e^{u(\tau,\xi)}\, d\xi\, d\tau.$$

Its exact solution is $u(t,x) = \sin(tx)$.

Example 4.

$$u(t,x) = 1 + t(1+x) - e^t + \int_0^t\int_0^1 \tau e^{u(\tau,\xi)}\, d\xi\, d\tau.$$

...ts exact solution is $u(t, x) = tx$.

	Example						
	μ	2	μ	3	μ	4	μ
...-04	09	3,62538E-04	05	1,92854E-04	07	1,06140E-04	06
...-06	03	7,36132E-05	16	6,81741E-06	13	1,64719E-05	06
...E-07	05	1,48619E-07	11	2,08536E-06	12	4,51354E-08	12
...0E-08	05	5,40094E-08	09	4,06230E-07	06	7,33962E-09	09
...65E-08	05	7,36690E-11	12	1,19734E-08	08	2,36468E-11	12
...611E-08	09	1,23691E-10	14	2,64708E-08	11	2,18279E-11	12
...782E-09	09	1,27329E-11	13	1,18234E-10	12	8,18545E-12	13
...956E-10	06	7,27596E-12	13	1,85537E-10	10	1,81899E-12	13
...32121E-12	07	8,18545E-12	14	9,09494E-13	12	2,72848E-12	12
...22915E-11	09	9,09495E-13	13	1,81899E-12	11	9,09495E-13	12

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