UDC 515.12

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ON UNIVERSALITY OF COUNTABLE POWERS OF ABSOLUTE RETRACTS*

ПРО УНІВЕРСАЛЬНІСТЬ ЗЛІЧЕННИХ СТЕПЕНІВ АБСОЛЮТНИХ РЕТРАКТІВ

We construct an absolute retract X of arbitrary high Borel complexity, such that the countable power X^ω is not universal for the Borelian class $\mathcal{A}_1$ of sigma-compact spaces, and the product $X^\omega \times \Sigma$, where Σ is the radial interior of the Hilbert cube, is not universal for the Borelian class $\mathcal{A}_2$ of absolute $G_{\delta\sigma}$-spaces.

Побудовано абсолютний ретракт X як завгодно високої борелівської складності, злічення степінь якого X^ω не універсальна для борелівського класу $\mathcal{A}_1$, що складається з сігма-компактних просторів. Доведено, що добуток $X^\omega \times \Sigma$ не є універсальним для борелівського класу $\mathcal{A}_2$ абсолютних $G_{\delta\sigma}$-просторів (тут Σ – радіальна внутрішність гільбертового куба).

By $\mathcal{A}_1$, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{A}_2$ we denote respectively the class of all sigma-compact spaces, the class of all Polish spaces, the class of all absolute $F_{\delta\sigma}$-spaces, and the collection of all absolute $G_{\delta\sigma}$-spaces; $Q = [-1, 1]^\omega$ is the Hilbert cube, and $\Sigma = \{ (t_i)_{i=1}^\infty \in Q : \sup_{i \in N} |t_i| < 1 \}$ is its radial interior. A closed set $A \subset X$ in an absolute retract X is called a Z-set, provided every map $f: Q \to X$ can be uniformly approximated by maps into $X \setminus A$ [1]. An absolute retract X is called a Z_σ-space, provided X is a countable union of its Z-sets.

All spaces considered are metrizable and separable, all maps are continuous.

Let C be a collection of spaces. We say that a space X is *C-universal*, provided for every space $C \in C$ there is a closed embedding $f: C \to X$.

In [2] (Corollary 2.5) T. Dobrowolski and J. Mogilski proved that, if an absolute retract X is a Z_σ-space, then the countable power X^ω is $\mathcal{M}_2$-universal. In this note we show that in the above result, the condition on X to be a Z_σ-space can not be replaced by the conditions on Borelian complexity of X (for example, $X \notin \mathcal{M}_1$ or $X \notin \mathcal{M}_2$).

By $\overline{D} = \{ z \in \mathbb{C} : |z| \le 1 \}$ and $D = \{ z \in \mathbb{C} : |z| < 1 \}$ we denote respectively the closed and the open disks in the complex plane $\mathbb{C}$, and by $P = \{ z \in \mathbb{C} : |z| = 1, \arg(z)/\pi \text{ is irrational} \}$ the set of irrationals in the circle $S^1 = \overline{D} \setminus D$. It is obvious

* The work is supported by the State Committee of Ukraine for Science and Technologies.

that, for every dense $A \subset P$, both A and $S^1 \backslash D$ are zero-dimensional. Moreover, the set $D \cup A$ is convex, and consequently, is an absolute retract (see Theorem 3.1 [1] (II, § 3)).

Theorem 1. *For every dense set $A \subset P$ the space $(D \cup A)^\omega$ is not $\mathcal{A}_1$-universal.*

Proof. Assume, on the contrary, that the space $(D \cup A)^\omega$ is $\mathcal{A}_1$-universal. Then there exists a closed embedding $f\colon \Sigma \to (D \cup A)^\omega$. We consider the space $(D \cup A)^\omega$ to be a subset in the compactum $\overline{D}^\omega$. According to Lavrentiev Theorem [3] (Theorem 4.3.21), there exists an embedding $\bar{f}: G \to \overline{D}^\omega$ of some G_σ-set G, $\Sigma \subset G \subset Q$, extending the embedding f. Since Σ is dense in G and $f(\Sigma)$ is closed in $(D \cup A)^\omega$, $\bar{f}(G \backslash \Sigma) \subset \overline{D}^\omega \backslash (D \cup A)^\omega$. Now notice that $\overline{D}^\omega \backslash (D \cup A)^\omega = \bigcup_{n=1}^\infty X_n$, where $X_n = \{ (t_i)_{i=1}^\infty \in \overline{D}^\omega \mid t_n \in S^1 \backslash A \}$. Since $G \backslash \Sigma$ is a G_σ-set in Q, by the Baire Theorem [3] (Theorem 3.9.3), there is an open set $U \subset G \backslash \Sigma$ such that the set $\bar{f}(U) \cap X_n$ is dense in $\bar{f}(U)$ for some $n \in \mathbb{N}$. Let $V = Q \backslash \mathrm{cl}_Q((G \backslash \Sigma) \backslash U)$. Obviously, V is an open set in Q such that $V \cap (G \backslash \Sigma) = U$. Let $V' \subset V$ be an open set of the form $V' = \{ (t_i)_{i=1}^\infty \in Q \mid a_i < t_i < b_i,\ 1 \le i \le m \}$, where $m \in \mathbb{N}$, and $a_i < b_i$, $1 \le i \le m$, are reals. Put finally, $W = V' \cap G$. One can verify that $W \cap \Sigma = V' \cap \Sigma$ is a connected (even convex) dense set in W and the set $W \backslash \Sigma$ is dense in W. Since $\bar{f}$ is an embedding, $\bar{f}(W) \cap X_n$ is dense in $\bar{f}(W)$. Denote by $\mathrm{pr}_n\colon \overline{D}^\omega \to \overline{D}$ the projection onto the n–th factor. Note that $\mathrm{pr}_n^{-1}(S^1 \backslash A) = X_n$. Since the set $\bar{f}(W \backslash \Sigma) \cap X_n$ is dense in $\bar{f}(W \backslash \Sigma)$ (remark that $W \backslash \Sigma$ is an open set in U) and $W \backslash \Sigma$ is dense in W, $\mathrm{pr}_n(\bar{f}(W)) \subset S^1$ and $\mathrm{pr}_n(\bar{f}(W)) \cap (S^1 \backslash A) \neq \varnothing$. Recalling that $\bar{f}(\Sigma) \subset (D \cup A)^\omega$ we obtain that $\mathrm{pr}_n(\bar{f}(W \cap \Sigma)) \subset S^1 \cap (D \cup A) = A$. Since the set $W \cap \Sigma$ is connected, and A is zero-dimensional, the image $\mathrm{pr}_n(\bar{f}(W \cap \Sigma))$ consists of only the point $a \in A$. Since $W \cap \Sigma$ is dense in W, we obtain $\mathrm{pr}_n(\bar{f}(W)) = \{a\}$. But this contradicts to $\mathrm{pr}_n(\bar{f}(W)) \cap (S^1 \backslash A) \neq \varnothing$. Theorem is proved.

In connection with [4] (Question 6.3), the following problem seems to be interesting.

Question. Let $A \subset P$ be a dense set. Can the space $(D \cup A)^\omega \times \Sigma^\omega$ be $\mathcal{A}_2$-universal?

Theorem 2. *For every dense set $A \subset P$ the space $(D \cup A)^\omega \times \Sigma$ is not $\mathcal{A}_2$-universal.*

Proof. We will slightly modify the proof of Theorem 1. Let $s = Q \backslash \Sigma$. Assume that the space $(D \cup A)^\omega \times \Sigma$ is $\mathcal{A}_2$-universal. Then, since $Q^\omega \backslash \Sigma^\omega \in \mathcal{A}_2$, there is a closed embedding $f\colon Q^\omega \backslash \Sigma^\omega \to (D \cup A)^\omega \times \Sigma$. We consider the space $(D \cup A)^\omega \times \Sigma$ to be a subset of the compactum $\overline{D}^\omega \times Q$. According to Lavrentiev Theorem, there exists an embedding $\bar{f}: G \to \overline{D}^\omega \times Q$ of some G_δ-set G, $Q^\omega \backslash \Sigma^\omega \subset G \subset Q^\omega$, extending the embedding f. Since $Q^\omega \backslash \Sigma^\omega$ is dense in G and $f(Q^\omega \backslash \Sigma^\omega)$ is closed in $(D \cup A)^\omega \times \Sigma$, we have $\bar{f}(G \backslash (Q^\omega \backslash \Sigma^\omega) \backslash) \subset (\overline{D}^\omega \times Q) \backslash ((D \cup A)^\omega \times \Sigma) = (\overline{D}^\omega \backslash (D \cup A)^\omega) \times Q \cup (\overline{D}^\omega \times (Q \backslash \Sigma))$. Notice that

$\overline{D}^{\omega} \times (Q \backslash \Sigma) = \overline{D}^{\omega} \times s$ is an absolute G_{δ}-set. Consequently, the intersection $\bar{f}(G) \cap (\overline{D}^{\omega} \times s)$ is also an absolute G_{δ}-set. Moreover, since $\bar{f}(Q^{\omega} \backslash \Sigma^{\omega}) \subset \subset \overline{D}^{\omega} \times \Sigma$, we have $\bar{f}(G) \cap (\overline{D}^{\omega} \times s) \subset \bar{f}(G \cap \Sigma^{\omega})$. Let us show that the space $G \cap \Sigma^{\omega}$ is of the first Baire category. Indeed, since the complement $\Sigma^{\omega} \backslash G = Q^{\omega} \backslash G$ is sigma-compact and the space Σ^{ω} is nowhere sigma-compact, the set $\Sigma^{\omega} \cap G = = \Sigma^{\omega} \backslash (Q^{\omega} \backslash G)$ is dense in Σ^{ω}, and consequently, in Q^{ω}. Now, since the space Σ^{ω} is of the first Baire category [5] (§ 10, IV, 2) implies that the intersection $G \cap \Sigma^{\omega}$ is also of the first Baire category. By the Baire Theorem [3] (Theorem 3.9.3) and [5] (§ 10, IV, 3), the absolute G_{δ}-set $\bar{f}(G) \cap (\overline{D}^{\omega} \times s)$ is nowhere dense in $f(G \cap \Sigma^{\omega})$. Then the set $F = \mathrm{cl}(\bar{f}^{-1}(\overline{D} \times s)) Q^{\omega}$ is nowhere dense in Q^{ω}. Using known universal properties of the couple $(Q^{\omega}, \Sigma^{\omega})$ (see e.g. [6]), one can find a compactum $K \subset Q^{\omega} \backslash F$ such that the pair $(K, K \cap \Sigma^{\omega})$ is homeomorphic to (Q, s). Then $K \backslash \Sigma^{\omega}$ is homeomorphic to $Q \backslash s = \Sigma$. Let $(X, Y) = (K \cap G, (K \cap G) \backslash \Sigma^{\omega}) = (K \cap G, K \backslash \Sigma^{\omega})$ (recall that $Q^{\omega} \backslash \Sigma^{\omega} \subset G$). Considering the restriction $g = \bar{f} \mid K \cap G$ we obtain the embedding $g : X \to \overline{D}^{\omega} \times Q$ of absolute G_{δ}-set such that $g(Y) \subset (D \cup \cup A)^{\omega} \times \Sigma$, $g(X \backslash Y) \subset (\overline{D}^{\omega} \backslash (D \cup A)^{\omega}) \times \Sigma$, and the space $Y = K \backslash \Sigma^{\omega}$ is homeomorphic to Σ. Proceeding by analogy with the proof of Theorem 1 we obtain a contradiction.

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Received 30.01.95