UDC 519.45

M. N. Ivko,
V. I. Senashov, candidates phys.-math. sci. (Computing Center, RAN, Krasnoyarsk)

ON A NEW CLASS OF INFINITE GROUPS*

ПРО НОВИЙ КЛАС НЕСКІНЧЕННИХ ГРУП

In this paper, properties of a new class of Φ-groups are studied. To within this class of accuracy, we characterize the class of layer-finite groups.

Вивчаються властивості нового класу Φ-груп і з точністю до вказаного класу охарактеризовані шарово скінченні групи.

Groups with conditions of finiteness were traditionally studied by V. P. Shunkov's school. Weak conditions imposed on subgroups, normalizers of finite subgroups, suddenly yield unexpected effect and can be extended over the whole group or give it some interesting properties.

This article investigates properties of a new class of Φ-groups. This class of groups is rather broad: among them there are groups of Burnside type [1], Ol'shanskii monsters [2]. It is very closely connected with the groups of Burnside type of odd period $n \geq 665$.

V. P. Shunkov posed the problem of studying groups with some additional limitations provided that, for the given finite subgroup B, the following condition –

(*) normalizer of any non-trivial B-invariant
finite subgroup has a layer-finite periodic part

– is valid.

This work solves this problem partly in the class of locally soluble groups and, for the case $|B| = 2$ and more general limitations, we solve it with Φ-groups accuracy.

The main result of the article is the theorem from the second paragraph proved by V. I. Senashov.

Theorem. *Let G be a group and let a be an involution of G satisfying the following conditions:*

1. *All the subgroups of the form* $\mathrm{gr}(a, a^g)$, $g \in G$, *are finite;*
2. *Normalizer of every non-trivial* (a)*-invariant finite subgroup has a layer-finite periodic part.*

Then, either the set of all elements of finite order forms a layer-finite group or G is an Φ-group.

The first section dwells upon auxiliary results. The authors prove here layer-finiteness of periodic locally soluble groups with condition (*) imposed upon one of its finite subgroups. In Theorem 1, V. I. Senashov established the properties of Φ-groups: conjugateness of involutions, construction of Sylow 2-subgroups and existence of infinite two-generated subgroups.

1. Class of Φ-groups.

Definition. *Let G be a group and let i be an involution of G, satisfying the following conditions:*

1) *all subgroups of the form* $\mathrm{gr}(i, i^g)$, $g \in G$, *are finite;*
2) $C_G(i)$ *is infinite and has a layer-finite periodic part;*
3) $C_G(i) \neq G$ *and* $C_G(i)$ *is not contained in other subgroup from G with a periodic part;*
4) *if K is a finite subgroup from G, which is not inside* $C_G(i)$, *and* $V =$

* This work has been supported in part by the Russian Foundation for fundamental studies (grant 93 – 01 – 16003).

$= K \cap C_G(i) \neq 1$, *then* K *is a Frobenius group with complement* V.

The group G *with some involution* i *satisfying conditions* 1 – 4 *is called a* Φ-*group*.

This class of groups was introduced by V. P. Shunkov.

Example of* Φ*-group. Let $A = \mathrm{gr}(b, c)$ (where $b^n = c^n = d$, n is a positive integer) be a torsion free group and let $A/(d)$ be a free Burnside group with period n [1]. Consider the group $B = A \wr (x) = (A \times A) \leftthreetimes (x)$, where x is an involution. Let us take from $A \times A$ the element $v = (d, d^{-1})$. Obviously, $v \in Z(A \times A)$ and $v^x = v^{-1}$. Further, the group $G = B/(v)$ and its involution $i = x(v)$ (which is easy to see from the abstract properties of the group $A = \mathrm{gr}(b, c)$ [1]) satisfy all the conditions from the definition of Φ-group. Hence, $G = B/(v)$ is an Φ-group.

Recall that the layer-finite group is a group in which the set of elements of each order is finite.

We shall use the term (a, b)-condition of finiteness. It means that, in a group, the element a generates a finite subgroup with almost every (without, possibly, a finite number) element conjugated with b.

An element g of finite order of the group G is called a point if, first, for every non-trivial (g)-invariant finite subgroup K from G, the set of finite subgroups from $N_G(K)$, containing the element g is finite, and, second, in the case where $g = 1$, the set of elements of finite orders from G is finite.

In the sequel, we shall need the following well-known results which, when referred to, are called assertions with corresponding numbers.

1. An infinite layer-finite p-group has such a subgroup of finite index, which is contained in center and is decomposed into a direct product of a finite set of quasi-cyclic subgroups [3].

2. Let G be a group, let A, B be some of its locally finite subgroups with Chernikov primary subgroups. If $A \cap B = D$ has finite indices in A and B, then D contains a finite index subgroup, the normalizer of which contains A and B [4].

3. If, in a periodic locally soluble group G, Sylow p-subgroups, for some prime p, are Chernikov groups, then the quotient-group $G/O_{p'}(G)$ is a Chernikov group [5].

4. Every locally soluble group G of finite rank contains a subgroup of finite index, the commutant of which has an increasing central series [6].

5. Let G be a group, let H be its subgroup strongly embedded in G, and let i be some involution from H satisfying the condition: for almost all elements of the form $g^{-1}ig\,(g \in G \setminus H)$, the subgroup $\mathrm{gr}(i, i^g)$ is finite. Then

1) all involutions from G are conjugate in G;

2) for any involution j from $G \setminus H$, the set of elements from H which are strictly real with respect to j has the same power as the set of involutions from H (assertion 4.3 from [7]).

6. Let G be a finite group and let H be its own subgroup. $H \cap H^g = 1\,(g \in G \setminus H)$. Then $G = F \leftthreetimes H$, where $F \setminus \{1\} = G \setminus \bigcup_{g \in G} H^g$ (see, for example, [8]).

7. Every Chernikov Φ-group has a non-trivial center [5].

8. If a Sylow p-subgroup P of the group G has finitely many conjugate subgroups, then these subgroups conjugate with P are all Sylow p-subgroups of the group G [5].

9. A locally finite p-group is a Chernikov p-group if it has a finite maximal elementary Abelian subgroup [9].

10. Let G be a locally finite group, which is not almost locally soluble, and let H be its strongly embedded subgroup. Then $G/O_{2'}(G) = T$ has either a single involu-

tion or a normal subgroup of odd index in T, which is isomorphic to one of the groups of the types $SL(2,K)$, $Sz(K)$, $PSU(3,K)$, where K is an infinite locally finite field of characteristic 2 [10].

11. If a locally finite group has a Sylow subgroup, which is isomorphic to a cyclic group or to a generalized quaternions group, then $G = O_{2'}(G) \cdot C_G(i)$, where i is an arbitrary involution from G [10, 11].

12. 2-group, which has a single involution, is either locally cyclic or a generalized quaternions group (finite or infinite) [12, 13].

13. Let G be a group and let i be an involution-point of G satisfying (i,i)-condition of finiteness. Then one of the following statements takes place:

A) G has a finite periodic part;

B) the following conditions are valid in G:

Sylow 2-subgroups are cyclic or finite generalized quanternions groups;

$H = C_G(i)$ has a finite periodic part and H is embedded in no other subgroup with this property;

if L is a finite subgroup from G, which is not inside H, and $L \cap H \neq 1$, then L is a Frobenius group with the complement $L \cap H$ [7].

14. Let G be a finite group of the form $G = O_{p'}(G) \leftthreetimes R$, where R is an elementary Abelian subgroup of the order p^2. Then $O_{p'}(G)$ is in the subgroup which is generated by centralizers of non-identity elements from R [14].

15. If g is a point of the group G and $|G : C_G(g)| < \infty$, then G has a finite periodic part [7].

16. If, for some involution $i \in G$, the condition of (i,i)-finiteness is satisfied, then, for any involution $k \in G$, the condition of (k,i)-finiteness is satisfied [7].

17. If a p-group has a finite non-trivial class of conjugate elements, then it has a non-trivial center [15].

18. The extension of a Chernikov group by a Chernikov group is a Chernikov group [15].

19. If the centralizer of some finite p-subgroup of a locally finite group G satisfies the $\min - p$ condition, then all p-subgroups of G are Chernikov [13].

20. Let G be a periodic locally soluble group, let B be a finite subgroup of G. If every B-invariant Abelian subgroup of G has a finite rank, then the rank of G is finite [16, 17].

21. If G is a Frobenius group with a kernel F and a complement H containing an involution i, then $H = C_G(i)$, F is an Abelian group and if $i = f^{-1}$, $f \in F$ [8].

Now we present our own results. Theorem 1 gives an idea about properties of Φ-groups.

Theorem 1 (V. I. Senashov). *An Φ-group G possesses the properties:*

1) *all involutions are conjugate;*

2) *Sylow 2-subgroups are conjugate cyclic or finite generalized quaternions groups;*

3) *there are infinitely many elements of finite orders in G, which are strictly real with respect to the involution i and for every such element c of this set there exists an element s_c from the centralizer of i such that $\mathrm{gr}(c, c^{s_c})$ is an infinite group.*

Proof. Let us prove 1). Let j be an arbitrary involution from the difference $G \setminus C_G(i)$ and let i be an involution of G from the definition of Φ-group. By condition 4 of the definition of Φ-group, $\mathrm{gr}(i,j)$ is a Frobenius group, that makes the involutions i and j conjugate. Hence, i conjugates with every involution from $G \setminus C_G(i)$.

Suppose that the closure $B = \mathrm{gr}(i^G)$ of the involution i is included in $C_G(i)$. For every element c of finite order from $G \setminus C_G(i)$, the group $\mathrm{gr}(c, B)$, which is finite by condition 2 in the definition of Φ-group, is a Frobenius group with a complement containing B by condition 4. By the Frobenius groups properties, in this case, $i^G = i$. This contradicts to condition 3 of the definition of Φ-group.

Let now the group B be finite and not inside $C_G(i)$. Then $B \cdot C_G(i)$ is a group with a periodic part strictly containing $C_G(i)$. Contradiction to condition 3 of the definition of Φ-group means that the class of involutions, conjugate with i, is infinite.

Denote by k some involution from $C_G(i)$ which are not conjugate with i. As we showed above, there are infinitely many involutions outside of $C_G(i)$

$$i_1, i_2, \ldots, i_n, \ldots,$$

which are conjugate with i. Consider subgroups of the form $\mathrm{gr}(k, i_n)$, $n = 1, 2, \ldots$. By Assertion 16 there are infinitely many finite subgroups among them. Let $\mathrm{gr}(k, t)$ be one of them. By the definition of Φ-group, it is a Frobenius group with involution k in its complement. By the Frobenius group properties, the involutions k and t are conjugate. This contradiction means that all involutions from G are conjugate.

Let us prove 2). Let j be an involution of $C_G(i)$ different from i conjugate with i by some element $h \in G$, that is $j = i^h$. Denote the periodic part of $C_G(i)$ by U. Obviously, $j \in U^h \cap U$.

If $N_G(U)$ contains some element b of finite order which does not belong to the centralizer of i, then consider the group $\mathrm{gr}(i, b)$. In view of layer-finiteness of U, it is a finite group which is a Frobenius group with a complement containing the involution i and $i^b \neq i$ by condition 4 of the definition of Φ-group. Contradiction to the construction of a Frobenius group means that $N_G(U)$ has a periodic part. By condition 3 of the definition of Φ-group, $C_G(i) = N_G(U)$.

Consider the group $\mathrm{gr}(c, j)$, where c is an element from the difference $U^h \setminus U$. The group $\mathrm{gr}(c, j)$ is finite in view of the layer-finiteness of U^h. By condition 4 in the definition of Φ-group, its complement must contain j and, simultaneously, the involution j must belong to the center of this group (j belongs to the center of U^h), but this is impossible. This implies that $U^h = U$. In view of the previous and $C_G(i) = N_G(U)$, the involutions i, j coincide. Thus, i is a single involution in its centralizer.

Let now S be an i-containing Sylow subgroup from G. If there is an involution t in S different from i, then, by the definition of Φ-group and by the property proved above, the group $\mathrm{gr}(i, t)$ is a Frobenius group. At the same time, it must be a 2-group. But this is impossible. Hence, i form a class of conjugate elements in S and S has a non-trivial center. By the preceding argument, $Z(S)$ can contain a single involution i and then, by what has been said above, the group S does not contain other involutions. It suffices to use Assertion 12 and note that S cannot be an infinite quaternions group because, in this case, it would not be layer-finite, contrary to condition 2 in the definition of Φ-group. The property of being conjugate is valid with regard to the same condition and 1).

Prove 3). The first part of the statement obviously follows from the infiniteness of the set of involutions proved above, which are conjugate with i. Assume that the second part of the statement is not valid, i.e., for some element c of finite order, which is strictly real with respect to i, all the groups of the form $\mathrm{gr}(c, c^s)$ are finite $(s \in \in C_G(i))$.

By condition 4 in the definition of Φ-group, all groups $\mathrm{gr}(i, c, c^s)$ are Frobenius groups with i-containing complements. The elements c, c^s, being strictly real with respect to involution i, are permutable by Statement 21. Then it follows from the assumption that $B = \mathrm{gr}(c^s \mid s \in C_G(i))$ is an Abelian group. Obviously $C_G(i) \leq \leq N_G(B)$. Then the group $C_G(i) \cdot B$ has a periodic part and strictly contains $C_G(i)$. Contradiction to condition 3 in the definition of Φ-group proves our statement. The theorem is proved.

Theorem 2 (M. N. Ivko, V. I. Senashov). *A periodic locally soluble group is layer-finite if and only if, for some of its finite subgroups B, the condition*

(**) *normalizer of any non-trivial B-invariant*
finite subgroup is layer-finite

is valid.

Proof. Consider first the case where $C_G(B)$ has an elementary Abelian subgroup R of the order p^2. Denote by P a Sylow p-subgroup which contains R. The centralizer of R in P, by Assertion 1 and condition (**), is a Chernikov group. Hence, by Assertion 19, all Sylow p-subgroups in G can be easily shown to be Chernikov.

If G is a Chernikov group, then there is a finite normal subgroup in it and, by condition (**), the group G is layer-finite. Hence, it is necessary to consider the case of non-Chernikov group. By Assertions 3 and 18, $O_{p'}(G) \leftthreetimes R$ is not a Chernikov group. By Assertion 14, $O_{p'}(G)$ is included in the subgroup $\mathrm{gr}(C_G(r_i) \mid r_i \in R^{\#})$. Every centralizer generating this subgroup is layer-finite by condition (**). In view of Assertion 9, primary Sylow subgroups in it are Chernikov groups and, being layer-finite groups, every centralizer has a finite index in others, so we can apply Assertion 2 to the pair of centralizers $C_G(r_1)$ and $C_G(r_2)$. Then there is a subgroup D_1 from the intersection $C_G(r_1) \cap C_G(r_2)$ which has finite indices in both $C_G(r_1)$, $C_G(r_2)$ and it is normal in each of them.

Let d_1 be an arbitrary non-unit element from D_1, let K_1 be a normal closure of D_1 in G. Denote by N_1 the normalizer of K_1 in the group $\mathrm{gr}(C_G(r_1), C_G(r_2))$ of the element $d_1^{b_1} d_1^{b_2} \ldots d_1^{b_n}$, where b_i – all elements of the group B, $i = 1, 2, \ldots, n$. Since K_1 is a finite B-invariant subgroup, by condition (**), N_1 is layer-finite and so the centralizer of the element $r_3 \in N_1$ has a finite index in N_1. The centralizer $C_G(r_1)$ is contained in N_1: hence, the intersection of N_1 and $C_G(r_3)$ is a subgroup of finite index. Applying again Assertion 2 to the pair $N_1, C_G(r_3)$, we shall receive the subgroup D_2 which is normal in both subgroups and again the normalizer N_2 of the finite B-invariant subgroup K_2 (chosen similarly to the subgroup K_1) is a layer-finite subgroup. In the same way, we construct the layer-finite subgroup N_{p^2-2} which contains all centralizers $C_G(r_i)$, $i = 1, 2, \ldots, p^2 - 1$. From this, we can deduce layer-finiteness of the subgroup $O_{p'}(G)$. It is normal in the group G, hence there is a finite subgroup in it which is normal in G. Again by condition (**), the group G is layer-finite.

Now we can assume that $C_G(B)$ has no non-cyclic elementary Abelian subgroups. Obviously, the rank of an arbitrary Abelian subgroup from $C_G(B)$ is equal to one. Hence, by Assertion 20, rank of $C_G(B)$ is finite.

If all B-invariant Abelian subgroups of G have finite ranks, then, by Zaitsev theorem (Assertion 20), the whole group has a finite rank but, by Kargapolov theorem (Assertion 4), such a group has a finite index subgroup whose commutant decomposes into a direct product of its Sylow subgroups of finite ranks. It is impossible to have infinitely many Sylow subgroups in this decomposition in view of condition (**). Then the lower of the commutant of the group G is a finite subgroup invariant in G. By condition (**), G is layer-finite. In this case, all is proved.

Hence, in G, there is a B-invariant Abelian subgroup L of infinite rank. Then it is apparent that in L there exists a finite B-invariant subgroup K. Its normalizer contains $T = L \leftthreetimes B$ and is a layer-finite group. By the properties of layer-finite groups, the centralizer $C_G(B)$ has a finite index in T, but its rank is finite, which ensures a finiteness of the rank of T. A contradiction. The theorem is proved.

2. Characterization of Φ-groups.

Theorem 3 (V. I. Senashov). *Let G be a group, let a be an involution of G, satisfying the conditions:*

1. *All subgroups of the form* $\mathrm{gr}(a, a^g)$, $g \in G$, *are finite;*
2. *Normalizer of every non-trivial (a)-invariant finite subgroup has a layer-finite periodic part.*

Then either the set of all elements of finite orders forms a layer-finite group or G is an Φ-group.

Proof. Let G have no layer-finite periodic part. By condition 2 of the theorem, $C_G(a)$ has a layer-finite periodic part U.

First, we consider the case where U is finite. If G has a non-trivial (a)-invariant finite subgroup K, then $N_G(K)$ has a layer-finite periodic part M by condition 2 of the theorem and $a \in M$. In view of the layer-finiteness of M, the element a is contained in the finite class of conjugate involutions from M. Hence, $|M : C_G(a) \cap M| < \infty$. But $C_G(a)$ has a finite periodic part by assumption. This implies the finiteness of M and, hence, the finiteness of the set of finite subgroups from $N_G(K)$, which contain the element a. This means that a is an involution-point of the group G. Then, by Assertion 13, G is an Φ-group and in this case the theorem is proved.

Let now U be an infinite group. Consider a chain of layer-finite subgroups

$$U < H_1 < H_2 < \ldots < H_n < \ldots$$

and their union V.

Assume that V is not a layer-finite group. Prove that all involutions of V are conjugate.

If the set of involutions from V is finite, then, by the Ditsman lemma, it generates a finite (a)-invariant subgroup which is normal in V. In this case, V, by condition 2, is layer-finite.

Let the set of involutions from V be infinite and

$$i_1 = a, i_2, i_3, \ldots$$

– some subset of it. Suppose that the involutions i_1, i_2 are not conjugate. Then $\mathrm{gr}(i_1, i_2)$ contains a central involution i_{12}. Therefore, in view of the layer-finiteness of periodic parts of $C_G(i_1)$, $C_G(i_{12})$, we conclude that the involution i_1 is not conjugate with only finitely many involutions from the list. Without loss of generality, we assume that i_1 is not conjugate with i_2 only. Then every group

$$\mathrm{gr}(i_1^{g_3}, i_2^{g_3}) = \mathrm{gr}(i_3, i_2^{g_3}), \ldots, \mathrm{gr}(i_1^{g_4}, i_2^{g_4}) = \mathrm{gr}(i_4, i_2^{g_4}), \ldots$$

has central involutions. It is impossible for the set of such involutions to be finite because, in this case, a centralizer of some involution contains an infinite set of involutions $i_1, i_2, i_3, \ldots$, contrary to condition 2 of the theorem. Then we can consider all of them to be different. There are no infinitely many equal involutions in the set $i_2, i_2^{g_3}, i_2^{g_4}, \ldots$ by the same reason. But then, as above, i_1 is conjugate with some involution from this set and this means that it is conjugate with i_2 too. This contradiction means conjugation of all involutions from V.

Let us show that the assumption of infiniteness of the set of involutions from V leads us to a contradiction and, in this way, we shall prove the layer-finiteness of V.

Let S be a Sylow 2-subgroup from V which contains a. Let us show that it is a Chernikov group. If S is finite, this is true. Let S be infinite. By condition 2 of the theorem, there are no infinite elementary Abelian subgroup in G. Then by Assertion 9, S is a Chernikov group.

By Assertion 7, there is an involution j in S from the center of S. Assume V to be not a layer-finite group. Then, by what was said above, the involutions a and j are conjugate in V. It is easy to see that all the conditions of the theorem hold for the involution j.

Consider the centralizers $U_1, U_2, \ldots, U_m$ of all involutions $j = t_1, t_2, \ldots, t_m$ from $U = C_V(j)$ in V. By the condition of the theorem, all these subgroups are layer-finite. The intersection $D_2 = U_1 \cap U_2$ has finite indices in U_1 and U_2. It is correct because the intersection of $C_V(t_1) = U_1$ and U_2 is a subgroup of finite index in view of layer-finiteness of U_2 and $t_1 \in U_2$. By the same reasons, the intersections of U_2 and U_1 form a subgroup of finite index. By Assertion 9, primary Sylow subgroup from U are Chernikov: hence, we can apply Assertion 2 for the subgroups U_1, U_2, D_2. By this assertion, there is a subgroup R_2 of finite indices in U_2 and U_1 and normal in both of them. The subgroup $H_2 = N_V(R_2)$ is layer-finite and $|H_2 : U_1| < \infty$. This follows from layer-finiteness of R_2 and from the condition of the theorem. Thus, $U_1, U_2 \le H_2$. By analogy, we get a layer-finite group $H_3 = N_V(R_3)$, where R_3 is a finite index subgroup in $D_3 = H_2 \cap U_3$. Now already $U_1, U_2, U_3 \le H_3$. Repeating this m times, we receive a layer-finite group N from V containing centralizers of all involutions from $C_V(j)$.

Denote by H a subgroup in N generated by the centralizers of all involutions from $C_V(j)$. Since we chose j from the center of the group S, it lies in H with the centralizer of j. Obviously, S will be a Sylow 2-subgroup in H. Let k be an arbitrary involution from H. By Assertion 8, Sylow 2-subgroups are conjugate in H, so there is an involution in S which is conjugate with k in H. Since the centralizer of this involution is contained in H by the construction of H, $C_V(k) \le H$.

Thus, H contains centralizers of all its involutions. Moreover, H is generated by the centralizers of the involutions contained in the centralizer of every involution l from H. Indeed, let the group M, generated by these centralizers, be strictly contained in H. Consider the group M^h for $h \in H$ such that $l^h = j$. Then M^h is generated by the centralizers of all involutions from $C_V(j)$ and is strictly contained in H. This contradicts the construction of the group H.

Let, for some element $g \in V \setminus H$, the intersection $H \cap H^g$ contain the involution k. Since centralizers of all involutions from H lie in H, H^g satisfy the same property. Hence, $C_V(k) \le H \cap H^g$. By the same reason, $H \cap H^g$ contains centralizers of all involutions from $C_V(k)$. But we have already showed before that H is generated by the centralizers of involutions from $C_V(k)$. The same is valid for H^g. Finally, $H = H^g$.

Consider now the group $T = N_V(H)$. If there is an involution k from the difference $T \setminus H$, then, in the dihedral group $F = \mathrm{gr}(a, k)$, the element ak must have an odd order in view of the properties of the group H, dihedral groups and since $a \in H$, $k \notin H$. By the properties of dihedral groups, in this case, there is an element $g \in (ak)$, for which $a^g = k$ but $g \in N_V(H)$, $a \in H$. Hence, $a^g = k \in H$. The contradiction means that there are no involutions in $T \setminus H$. Thus, for every element from $V \setminus T$, the intersection $T \cap T^g$ has no involutions. If there is an involution that belongs to the intersection, by applying the same argument as before, we see that this is an involution from $H \cap H^g$ and, hence, $H = H^g$. Therefore, $g \in T = N_G(H)$. This contradiction means that T is strictly embedded in V.

Suppose that V is not an almost locally soluble group. Then by Assertion 10, $\bar{V} = V / O_{2'}(V)$ either has a single involution or a normal subgroup of odd index in T, which is isomorphic to one of the group of the form $Sz(K)$, $SL(2, K)$, $PSU(3, K)$, where K is an infinite locally finite field of characteristic 2.

By Assertions 11, 12, in the first case, we can decompose the group V into the product $V = O_{2'}(V) \cdot C_V(j)$. In view of the Feit–Thompson theorem $O_{2'}(V)$ is locally soluble, so it suffices is to prove that $C_V(j)$ is almost locally soluble.

If $C_V(j)$ is finite, then V is almost locally soluble. Assume that $C_V(j)$ is infinite (by condition 2 of the theorem, it is layer-finite). If the Sylow 2-subgroup S_2 from $C_V(j)$ is infinite, then, by the properties of layer-finite groups, the complete part $\tilde{S}_2$ of S_2 is contained in the center of $C_V(j)$ and has a finite index in S_2. Consider a quotient-group $C_V(j) / \tilde{S}_2$. In this situation, it is necessary to prove that this layer-finite group with finite Sylow 2-subgroup is locally soluble. So, without loss of generality, assume that S_2 is finite. By induction on the order of Sylow 2-subgroup, prove that $C_V(j)$ is almost locally soluble.

Let S_3 be a normal closure of S_2 in $C_V(j)$. It is finite in view of the properties of layer-finite groups. Denote by C the centralizer of S_3 in $C_V(j)$. If $C \cap Z$ has no involutions, then C is locally soluble by Feit–Thompson theorem and since $| C_V(j) : C |$ is finite, in this case, all is proved. Then we can assume that $C \cap Z$ has an even order. Then, in the quotient-group C by Sylow 2-subgroup from $C \cap Z$, the order of Sylow 2-subgroup decreases and, by the inductive assumption, it is almost locally soluble. Hence, as we noted above, V is an almost locally soluble group. A contradiction to the assumption. In the second case, V has a non-Chernikov Sylow 2-subgroup, but this is impossible as we already showed. This means that our assumption is incorrect and V is almost locally soluble. By Theorem 2, V has a layer-finite subgroup of finite index. By the second condition of the theorem, in this case, V is layer-finite. A contradiction.

Consider a set $\mathfrak{M}$ of all subgroups with layer-finite periodic parts containing $C_G(a)$. This set is partially ordered by inclusion and, by what proved above, the union of every chain belongs to it. By Rzorn lemma, $\mathfrak{M}$ has a maximal element, i.e., a subgroup H from $\mathfrak{M}$ which is contained in no other subgroup from $\mathfrak{M}$. Let Y be the periodic part of the group H.

By the choice of the set $\mathfrak{M}$, the group Y is layer-finite. Obviously, Y is normal in H and it is a characteristic subgroup in H. Hence, in view of the maximality of H in $\mathfrak{M}$ and since $a \in Y$, we see that

$$N_G(Y) = H = N_G(H).$$

Let k be an arbitrary involution from Y. If

$$F = \mathrm{gr}(k^g \mid g \in G) \leq Y,$$

then, since F is a-invariant and Y is layer-finite, the group G has a layer-finite periodic part by condition 2. A contradiction to the assumption. Hence, for some $g \in G$, the involution $t = k^g \notin H$. Consider the dihedral group $L = \mathrm{gr}(a, t)$. Suppose that L is not a finite Frobenius group. In this case, the order of d is either infinite or even. The first possibility is not realized because the extension of the finite group $\mathrm{gr}(a, a^t)$ by (t) is a finite group $\mathrm{gr}(a, t)$.

Let $|d|$ be even. By the properties of dihedral groups, (d) has an involution $j \in C_G(a) \cap C_G(t) \leq H$. Since j belongs to a finite normal subgroup from H, $|H : C_G(j) \cap H| < \infty$.

In view of $a \in C_G(j)$, the centralizer $C_G(j)$ has a layer-finite periodic part R and a belong to a finite normal subgroup from $C_G(j)$. We conclude from it that $|C_G(j) : H \cap C_G(j)|$ is finite. Since, by Assertion 1, Sylow primary subgroups in R and in Y are Chernikov subgroups, we can apply Assertion 2 to the groups R and Y: there is a subgroup X in $R \cap Y$ of finite index in $R \cap Y$ such that $Y, R \leq N_G(X)$. Since $t \in R$, the involution t belongs to $N_G(X)$; on the other hand, $t \notin H$ and $H < N_G(X)$. Hence, $H \neq N_G(X)$ and, in view of the definition of H, the group $M = N_G(X)$ has no layer-finite periodic part. Further, $X \leq H$ and $|Y : X| < \infty$; moreover, X is layer-finite.

If $a \in X$, then a lies in a finite subgroup K which is characteristic in X. Then $N_G(K) \geq M$ and has a layer-finite periodic part. The contradiction means that $a \notin X$.

Since X is invariant in M and $a \in M$, in view of the layer-finiteness of X, we can find a finite non-trivial layer of elements of some order in it which generates (a)-invariant subgroup S characteristic in M. By condition 2, its normalizer has a layer-finite periodic part and contains M. This contradiction means that d is an element of odd order and the involutions a and t are conjugate in G by the properties of the dihedral groups. Then k, a are conjugate in G.

Let us prove that H is a strongly embedded subgroup in G. Suppose that this is not so. Then $H \neq H^g$ for some $g \in G$ and $H \cap H^g$ has some involution k. In view of the layer-finiteness of periodic parts of the groups H and H^g, the indices $|H : C_G(k) \cap H|$ and $|H^g : C_G(k) \cap H^g|$ are finite. Applying to H, H^g and $C_G(k)$ the considerations used for H and $C_G(j)$, we conclude that $C_G(k) \leq H \cap H^g$ and $H = H^g$. Hence, H is a subgroup strongly embedded in G.

Suppose that H has more than one involution. Then, by Assertion 5, there is a non-identity element c of finite order in H, which is strictly real with respect to some involution $j \in G \setminus H$. Consider the subgroup $M = C_G(k) \leftthreetimes (j)$. Since, by Assertion 5, a, j are conjugate in G, the group $M^g = C_G(c^g) \leftthreetimes (a)$ has a layer-finite periodic part and M also has a layer-finite periodic part. In view of the layer-finiteness of Y and since $c \in Y$, the index $|H : M \cap H|$ is finite. As above, we receive a contradiction with $j \notin H$. Hence, H has a single involution.

Now it suffices to show that $H \cap H^g$ is a torsion-free group. Indeed, if this is not so, then $H \cap H^g$ has an element h of finite order. In view of the layer-finiteness of Y, the centralizer $C_G(h)$ intersects H and H^g forming subgroups of finite indices. Again we arrive at a contradiction. Hence, $Y \cap Y^g = 1$. By Assertion 21 applied to any finite subgroup K from G, we conclude that if $K \cap Y = W$ and $K \leq Y$, then K is a Frobenius group with a complement W. Thus, if G has no layer-finite periodic part, then G is an Φ-group and the theorem is proved.

Corollary. *Let G be a group with involutions and let i be some involution from G satisfying the conditions:*

1) *G is generated by the involutions conjugate with i;*

2) *almost all groups* $\mathrm{gr}(i, i^g)$ *are finite $g \in G$;*

3) *normalizer of every (i)-invariant finite subgroup has a layer-finite periodic part.*

Then G is either a finite or an Φ-group.

1. *Adian S. I.* Burnsaid problem and identities in groups. – Moscow: Nauka, 1975. – 336 p. (in Russian).
2. *Ol'shanskii A. Yu.* Geometry of definite relations in groups. – Moscow: Nauka, 1989. – 448 p. (in Russian).
3. *Chernikov S. N.* On the theory of infinite p-groups // Dokl. Akad. Nauk USSR. – 1945. – **50**. – P. 71 – 74.
4. *Shunkov V. P.* On locally finite groups of a finite rank // Algebra i Logika. – 1971. – **10**, № 2. – P. 199 – 225.
5. *Chernikov S. N.* Groups with properties of subgroups system. – Moscow: Nauka, 1980. – 384 p. (in Russian).
6. *Kargapolov M. I.* Some questions of nilpotent and soluble group theory // Dokl. Akad. Nauk USSR. – 1959. – **127**, № 6. – P. 1164 – 1166.
7. *Shunkov V. P.* M_p-groups. – Moscow: Nauka, 1990. – 160 p. (in Russian).
8. *Busarkin V. M., Gorchakov Yu. M.* Finite decomposable groups. – Moscow: Nauka, 1968. – 111 p. (in Russian).
9. *Blackburn N.* Some remarks on Chernikov p-groups, III // J. Math. – 1962. – **6**. – P. 421 – 433.
10. *Kegel O. H., Wehrfritz B. A. F.* Locally finite groups. – Amsterdam; London: North-Holland Publ. Co., 1973. – 256 p.
11. *Brauer R., Suzuki M.* On finite groups with an Abelian Sylow subgroups // Can. J. Math. – 1962. – **14**. – P. 436 – 450.
12. *Shunkov V. P.* On one of the classes of p-groups // Algebra i Logika. – 1970. – **9**, № 4. – P. 484 – 496.
13. *Chernikov S. N.* On groups with minimality condition for subgroups // Subgroup characterization of groups. – Kiev: Inst. Math. AN Ukraine, 1982. – P. 3 – 12 (in Russian).
14. *Brauer R.* Some applications of theory of block of characters of finite groups II // J. Algebra. – 1964. – **1**. – P. 307 – 334.
15. *Kargapolov M. I., Merzliakov Yu. I.* Foundations of group theory (3rd ed.). – Moscow: Nauka, 1982. – 288 p. (in Russian).
16. *Zaitsev D. I.* On locally soluble groups of finite rank // Dokl. Akad. Nauk USSR. – 1978. – **240**, № 2. – P. 257 – 260.
17. *Hartley B.* Finite groups of automorphisms of locally soluble groups // J. Algebra. – 1979. – **57**, № 1. – P. 242 – 257.

Received 19.01.94